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A Solution of the Stokes' Problem for a Circular Cylinder

K. Shulev

Introduction. The well known classical Stokes' problem is to find a solution of the Stokes' differential equations

$$\text{grad}\, p = \mu \Delta \bar{u}, \quad \text{div}\, \bar{u} = 0, \tag{1}$$

subject to boundary conditions of viscous and incompressible fluid for a long circular cylinder. These boundary conditions for a transverse flow past a motionless circular cylinder and uniforme with constant speed $-\bar{U}$ at infinity are

$$\bar{u}_s = \bar{0}, \quad \bar{u}_\infty = -\bar{U}, \quad p_\infty = v_0, \tag{2}$$

where s denotes the cylinder surface. In the case of transverse motion of the cylinder with constant speed $\bar{U}$ through viscous fluid which is at rest at infinity these boundary conditions are

$$\bar{u}_s = \bar{U}, \quad \bar{u}_\infty = 0, \quad p_\infty = p_0. \tag{3}$$

This problem had been considered by G. Stokes [1] with a solution that cannot satisfy all above-mentioned boundary conditions for the velocity $\bar{u}$. Unfortunately the tries to solve the same problem later on had not been successful, in fact this case is widely known as so-called Stokes' paradox. Nevertheless the Stokes' solution is considered by many authors, e. g. [2], [3]. As certain advancement is the solution by means of Oseen's differential equations with partial account of inertia terms, but the boundary conditions for the velocity still cannot satisfy exactly.

In the present paper a solution of Stokes' differential equations (1) exactly satisfying all the boundary conditions either (2) or (3) for a circular cylinder is given, as well as the pressure distribution. This solution has a different form in respect of the classical Stokes' solution.

1. Boundary problem for motionless cylinder

Following the suggestion to study the flow past a motionless cylinder by means of the steady Stokes' differential equations, the problem is then to find a solution of equations (1) subject to boundary conditions (2). Let cylinder axis coincides with axis z of an absolute frame of reference, cylinder radius is a and the flow with constant speed $-\bar{U}$ at infinity is directed in opposite to axis x. Considering in cylindrical co-ordinates r, θ, z then the velocity component u_z also all partial derivatives with respect to variable z vanish in the governing equations (1). The boundary conditions (2) for velocity in cylindrical co-ordinates are

$$u_r\big|_{r=a}=0,\quad u_\theta\big|_{r=a}=0,\quad u_r\big|_{r=\infty}=-U\cos\theta,\quad u_\theta\big|_{r=\infty}=U\sin\theta, \tag{4}$$

where $U=|\bar{U}|$. Substituting

$$u_r=U\cos\theta+v_r,\quad u_\theta=-U\sin\theta+v_\theta \tag{5}$$

into (1) it can be readily checked the invariance of the system (1) relative to transformation (5). Thus the problem I is reduced to a transverse motion of the cylinder with constant speed $\bar{U}$ as the differential equations remain the same.

II. Stokes' boundary problem for moving cylinder

The corresponding system (1) in cylindrical co-ordinates is

$$\begin{aligned}
&\frac{\partial p}{\partial r}=\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{2}{r}\frac{\partial v_\theta}{\partial\theta}\right]\\
&\frac{\partial p}{\partial\theta}=\mu r\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{2}{r}\frac{\partial v_r}{\partial\theta}\right]\\
&\frac{\partial r v_r}{\partial r}+\frac{\partial v_\theta}{\partial\theta}=0,
\end{aligned} \tag{6}$$

as the boundary conditions following from (3) are

$$v_r\big|_{r=a}=U\cos\theta,\quad v_\theta\big|_{r=a}=-U\sin\theta,\quad v_r\big|_{r=\infty}=0,\quad v_\theta\big|_{r=\infty}=0, \tag{7}$$

as the condition $p_\infty=p_0$ remains the same. The problem is to find a solution for the velocity components v_r and v_θ and the pressure p of system (6) subject to boundary conditions (7), also $p_\infty=p_0$.

Solution of problem II

A necessary condition for compatibility of system (1) is the vector $\mu\Delta\bar{v}$ be a potential one, as we consider $\bar{v}$ in the place of $\bar{u}$. Applying the operator rot of both sides of first equation (1) follows the system for the velocity $\bar{v}$ only.

$$\operatorname{rot}\Delta\bar{v}=\bar{0},\quad \operatorname{div}\bar{v}=0. \tag{8}$$

It is worth noticing that the first equation of (8) follows easily from the two first equations of (6) by cross differentiation and eliminating the pressure terms.

Introducing the stream function $\psi(r,\theta)$ through

$$v_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta},\quad v_\theta=-\frac{\partial\psi}{\partial r} \tag{9}$$

the system (6) is reduced then to the following system

$$\frac{\partial p}{\partial r}=\frac{\mu}{r}\frac{\partial\Delta\psi}{\partial\theta},\quad \frac{\partial p}{\partial\theta}=-\mu r\frac{\partial\Delta\psi}{\partial r}, \tag{10}$$

where Δ is the Laplace's operator

(11) $$\Delta\psi=\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}.$$

Differentiating first equation of (10) with respect to θ and the second to r and eliminating the pressure terms yields the biharmonic equation

(12) $$\frac{\partial^2\Delta\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Delta\psi}{\partial\theta^2}=0,$$

as $\Delta\psi$ by (11) is given.

Remark. The function ψ of the Stokes' solution is sought as

(13) $$\psi=f(r)\sin\theta.$$

After substituting (13) into (12) follows the Euler's equation

(14) $$f^{(4)}+\frac{2}{r}f'''-\frac{3}{r^2}f''+\frac{3}{r^3}f'-\frac{3}{r^4}f=0$$

with general solution

$$f(r)=Ar^3+Br\ln r+Cr+\frac{D}{r}.$$

It is not possible the constants of integration A, B, C, D be determined to be satisfied for all the boundary conditions (7).

We try to find the solution choosing ψ of a different from (13) form. For this purpose we consider the variable θ to vary over the symmetric interval $[-\pi, \pi]$. Then as is well known the $\sin\theta$ can be represented in the half-interval $[0, \pi]$ by the following uniformly convergent Fourier's series [5]

(15) $$\sin\theta=\frac{2}{\pi}+\frac{4}{\pi}\sum_{s=1}^{\infty}\frac{\cos 2s\theta}{1-4s^2},$$

which follows from the Fourier's series of $|\sin\theta|$ over $[-\pi, \pi]$ (since $\sin\theta=|\sin\theta|$ when θ is over $[0, \pi]$). Some reasons of integration require the components v_r and v_θ together with their first and second derivatives be expressed by uniformly convergent Fourier's series. Thus, it is necessary four time integration of (15), which is reasonable because of a uniformly convergence. After integrating four time, determining the constants for $\theta=0$ and summing up the numerical convergent series one obtains

(16) $$\cos\theta=1-\left(\frac{2}{\pi}-\frac{\pi}{6}\right)\theta-\frac{\theta^2}{2}+\frac{\theta^3}{3\pi}-\frac{4}{\pi}\sum_{s=1}^{\infty}\frac{\sin 2s\theta}{(2s)^3(1-4s^2)},$$

(17) $$\sin\theta=\frac{2}{\pi}-\frac{\pi}{6}-\frac{\pi^3}{360}+\theta-\left(\frac{2}{\pi}-\frac{\pi}{6}\right)\frac{\theta^2}{2}-\frac{\theta^3}{6}+\frac{\theta^4}{12\pi}+\frac{4}{\pi}\sum_{s=1}^{\infty}\frac{\cos 2s\theta}{(2s)^4(1-4s^2)}.$$

In the same way when θ is over $[-\pi, 0]$ it follows (because of in this case $\sin\theta=-|\sin\theta|$) that

(18) $$\sin\theta=-\frac{2}{\pi}-\frac{4}{\pi}\sum_{s=1}^{\infty}\frac{\cos 2s\theta}{1-4s^2},$$

(19) $$\cos\theta=1+\left(\frac{2}{\pi}-\frac{\pi}{6}\right)\theta-\frac{\theta^2}{2}-\frac{\theta^3}{3\pi}+\frac{4}{\pi}\sum_{s=1}^{\infty}\frac{\sin 2s\theta}{(2s)^3(1-4s^2)}$$

$$(20)\qquad \sin\theta=-\left(\frac{2}{\pi}-\frac{\pi}{6}-\frac{\pi^3}{360}\right)+\theta+\left(\frac{2}{\pi}-\frac{\pi}{6}\right)\frac{\theta^2}{2}-\frac{\theta^3}{6}-\frac{\theta^4}{12\pi}-\frac{4}{\pi}\sum_{s=1}^{\infty}\frac{\cos 2s\theta}{(2s)^4(1-4s^2)}\,.$$

First we seek the solution for v_r and v_θ when θ is over $[0, \pi]$. In view of representations (16) and (17), then the two first conditions of (7) lead to

$$(21)\qquad v_r(a,\theta)=U\left[1-\left(\frac{2}{\pi}-\frac{\pi}{6}\right)\theta-\frac{\theta^2}{2}+\frac{\theta^3}{3\pi}-\frac{4}{\pi}\sum_{s=1}^{\infty}\frac{\sin 2s\theta}{(2s)^3(1-4s^2)}\right],$$

$$(22)\qquad v_\theta(a,\theta)=-U\left[\frac{2}{\pi}-\frac{\pi}{6}-\frac{\pi^3}{360}+\theta-\left(\frac{2}{\pi}-\frac{\pi}{6}\right)\frac{\theta^2}{2}-\frac{\theta^3}{3}+\frac{\theta^4}{12\pi}+\frac{4}{\pi}\sum_{s=1}^{\infty}\frac{\cos 2s\theta}{(2s)^4(1-4s^2)}\right].$$

The stream function ψ is sought then of the form

$$(23)\qquad \psi(r,\theta)=Ua\left[\alpha_0\left(\frac{r}{a}\right)+\alpha_1\left(\frac{r}{a}\right)\theta+\alpha_2\left(\frac{r}{a}\right)\theta^2+\alpha_3\left(\frac{r}{a}\right)\theta^3+\alpha_4\left(\frac{r}{a}\right)\theta^4 + \sum_{s=1}^{\infty}\beta_{2s}\left(\frac{r}{a}\right)\cos 2s\,\theta\right],$$

where $\frac{r}{a}$ is dimensionless variable. Denoting $\xi=\frac{r}{a}$ then in accordance with (5) follows

$$(24)\qquad v_r(\xi,\theta)=\frac{U}{\xi}\left[\alpha_1(\xi)+2\alpha_2(\xi)\theta+3\alpha_3(\xi)\theta^2+4\alpha_4(\xi)\theta^3-\sum_{s=1}^{\infty}(2s)\,\beta_{2s}(\xi)\sin 2s\theta\right],$$

$$(25)\qquad v_\theta(\xi,\theta)=-U\left[\alpha_0'(\xi)+\alpha_1'(\xi)\theta+\alpha_2'(\xi)\theta^2+\alpha_3'(\xi)\theta^3+\alpha_4'(\xi)\theta^4+\sum_{s=1}^{\infty}\beta_{2s}'(\xi)\cos 2s\theta\right],$$

where the prime denotes differentiation with respect to ξ. The boundary conditions (21) and (22) are to be reduced to (24) and (25) when $\xi=1$, while the two last conditions of (7) follow from (24) and (25) when $\xi\to\infty$. After differentiating ψ from (23), substituting the corresponding partial derivatives into (12) and annulating the coefficients of θ^k ($k=0$, 1, 2, 3, 4) as well as of the $\cos 2s\theta$ it follows the system Euler type differential equations for the unknown functions $\alpha_k(\xi)$ ($k=0$, 1, 2, 3, 4) and $\beta_{2s}(\xi)$

$$(26)\qquad \alpha_4^{(4)}+\frac{2}{\xi}\alpha_4'''-\frac{1}{\xi^2}\alpha_4''+\frac{1}{\xi^3}\alpha_4'=0$$

$$(27)\qquad \alpha_3^{(4)}+\frac{2}{\xi}\alpha_3'''-\frac{1}{\xi^2}\alpha_3''+\frac{1}{\xi^3}\alpha_3'=0$$

$$(28)\qquad \alpha_2^{(4)}+\frac{2}{\xi}\alpha_2'''-\frac{1}{\xi^2}\alpha_2''+\frac{1}{\xi^3}\alpha_2'=-3.4\left(\frac{2}{\xi^2}\alpha_4''-\frac{2}{\xi^3}\alpha_4'+\frac{4}{\xi^4}\alpha_4\right)$$

$$(29)\qquad \alpha_1^{(4)}+\frac{2}{\xi}\alpha_1'''-\frac{1}{\xi^2}\alpha_1''+\frac{1}{\xi^3}\alpha_1'=-2.3\left(\frac{2}{\xi^2}\alpha_3''-\frac{2}{\xi^3}\alpha_3'+\frac{4}{\xi^4}\alpha_3\right)$$

$$(30)\qquad \alpha_0^{(4)}+\frac{2}{\xi}\alpha_0'''-\frac{1}{\xi^2}\alpha_0''+\frac{1}{\xi^3}\alpha_0'=-2\left(\frac{2}{\xi^2}\alpha_2''-\frac{2}{\xi^3}\alpha_2'+\frac{4}{\xi^4}\alpha_2+\frac{3.4}{\xi^4}\alpha_4\right)$$

$$(31)\qquad \beta_{2s}^{(4)}+\frac{2}{\xi}\beta_{2s}'''-\frac{2(2s)^2-1}{\xi^2}\beta_{2s}''+\frac{2(2s)^2+1}{\xi^3}\beta_{2s}'+\frac{(2s)^4-4(2s)^2}{\xi^4}\beta_{2s}=0.$$

The equations (28), (29) and (30) are non-homogeneous ones whose right-band sides are to be determined from the solutions of the preceding equations.

The general solution of the equation (26) is

$$\alpha_4(\xi) = a_{40} + a_{41} \ln \xi + a_{42}\xi^2 + a_{43}\xi^2 \ln \xi,$$

as the constants of integration a_{4i} $(i=0,\ 1,\ 2,\ 3)$ are to be determined subject to the following boundary conditions

$$4\alpha_4(1) = \frac{1}{3\pi}, \quad \alpha_4'(1) = \frac{1}{12\pi}, \quad \lim_{\xi\to\infty} \frac{a^4(\xi)}{\xi} = 0, \quad \lim_{\xi\to\infty} \alpha_4'(\xi) = 0,$$

whereupon $a_{40} = \frac{1}{12\pi}$, $a_{41} = \frac{1}{12\pi}$, $a_{42} = a_{43} = 0$. The function $\alpha_4(\xi)$ is

$$\alpha_4(\xi) = \frac{1}{12\pi}(1 + \ln \xi). \tag{32}$$

Likewise the general solution of (27) is

$$\alpha_3(\xi) = \alpha_{30} + a_{31} \ln \xi + a_{32}\xi^2 + a_{33}\xi' \ln \xi,$$

and the constants a_{3i} one determines from the boundary conditions

$$3\alpha_3(1) = -\frac{1}{2}, \quad \alpha_3'(1) = -\frac{1}{6}, \quad \lim_{\xi\to\infty} \frac{\alpha_3(\xi)}{\xi} = 0, \quad \lim_{\xi\to\infty} \alpha_3'(\xi) = 0,$$

as $a_{30} = a_{31} = -\frac{1}{6}$, $a_{32} = a_{33} = 0$. The function $\alpha_3(\xi)$ is

$$\alpha_3(\xi) = -\frac{1}{6}(1 + \ln \xi). \tag{33}$$

Substituting α_4 from (32) into the right-hand side of the equation (28), after some simplification follows the non-homogeneous differential equation

$$\alpha_2^{(4)} + \frac{2}{\xi}\alpha_2''' - \frac{1}{\xi^2}\alpha_2'' + \frac{1}{\xi^3}\alpha_2' = -\frac{4}{\pi}\frac{\ln \xi}{\xi^4}, \tag{34}$$

whose general solution is

$$\alpha_2(\xi) = a_{20} + a_{21} \ln \xi + a_{22}\xi^2 + a_{23}\xi^2 \ln \xi - \frac{1}{6}(3 \ln^2 \xi + \ln^3 \xi).$$

The boundary conditions determining the constants a_{2i} are

$$2\alpha_2(1) = -\left(\frac{2}{\pi} - \frac{\pi}{6}\right), \quad \alpha_2'(1) = -\frac{1}{2}\left(\frac{2}{\pi} - \frac{\pi}{6}\right), \quad \lim_{\xi\to\infty} \frac{\alpha_2(\xi)}{\xi} = 0, \quad \lim_{\xi\to\infty} \alpha_2'(\xi) = 0,$$

whence it follows that $a_{20} = a_{21} = \frac{\pi}{12} - \frac{1}{\pi}$, $a_{22} = a_{23} = 0$. The function $\alpha_2(\xi)$ is

$$\alpha_2(\xi) = \left(\frac{\pi}{12} - \frac{1}{\pi}\right)(1 + \ln \xi) - \frac{1}{6\pi}(3 \ln^2 \xi + \ln^3 \xi). \tag{35}$$

Similarly from (29) and (33) it follows the equation

$$\alpha_1^{(4)} + \frac{2}{\xi}\alpha_1''' - \frac{1}{\xi^2}\alpha_1'' + \frac{1}{\xi^3}\alpha_1' = 4\frac{\ln \xi}{\xi^4} \tag{36}$$

with general solution

$$\alpha_1(\xi) = a_{10} + a_{11} \ln \xi + a_{12}\xi^2 + a_{13}\xi^2 \ln \xi + \frac{1}{6}(3 \ln^2 \xi + \ln^3 \xi).$$

The constants a_{1i} one determines subject to the boundary conditions

$$\alpha_1(1)=1,\quad \alpha_1'(1)=1,\quad \lim_{\xi\to\infty}\frac{\alpha_1(\xi)}{\xi}=0,\quad \lim_{\xi\to\infty}\alpha_1'(\xi)=0,$$

as $a_{10}=a_{11}=1$, $a_{12}=a_{13}=0$. The function $\alpha_1(\xi)$ is

$$\alpha_1(\xi)=1+\ln\xi+\frac{1}{6}(3\ln^2\xi+\ln^3\xi). \tag{37}$$

In the same way from (30), (32) and (34) follows the nonhomogeneous equation

$$\alpha_0^{(4)}+\frac{2}{\xi}\alpha_0'''-\frac{1}{\xi^2}\alpha_0''+\frac{1}{\xi^3}\alpha_0'=\frac{2}{\pi\xi^4}-\left(8a_{20}+\frac{6}{\pi}\right)\frac{\ln}{\xi^4}+\frac{4}{3\pi}\frac{\ln^3\xi}{\xi^4}, \tag{38}$$

whose general solution is

$$\alpha_0(\xi)=a_{00}+a_{01}\ln\xi+a_{02}\xi^2+a_{03}\xi^2\ln\xi-a_{20}\ln^2\xi+\frac{a_{20}}{3}\ln^3\xi+\frac{1}{12\pi}\ln^4\xi+\frac{1}{60\pi}\ln^5\xi.$$

The corresponding boundary conditions are

$$\alpha_0'(1)=\frac{2}{\pi}-\frac{\pi}{6}-\frac{\pi^3}{360},\quad \lim_{\xi\to\infty}\alpha_0'(\xi)=0,$$

that determine

$$a_{01}=\frac{2}{\pi}-\frac{\pi}{6}-\frac{\pi^3}{360},\quad a_{02}=a_{03}=0.$$

Here the constant a_{00} remains indetermined, but it is no essentially since $\alpha_0(\xi)$ is out of the presence into (24) and (25) respectively. The function $\alpha_0(\xi)$ is expressed as

$$\alpha_0(\xi)=a_{00}+\left(\frac{2}{\pi}-\frac{\pi}{6}-\frac{\pi^3}{360}\right)\ln\xi-\left(\frac{\pi}{12}-\frac{1}{\pi}\right)\left(\ln^2\xi+\frac{\ln^3\xi}{3}\right)+\frac{1}{12\pi}\ln^4\xi+\frac{1}{60\pi}\ln^5\xi. \tag{39}$$

Also the general solution of equation (31) is

$$\beta_{2s}(\xi)=A_{2s}\xi^{2s+2}+B_{2s}\xi^{2s}+C_{2s}\xi^{-2s}+D_{2s}\xi^{2-2s},$$

as the constants of integrations A_{2s}, B_{2s}, C_{2s} and D_{2s} are to be determined subject to the following boundary conditions:

$$-2s\,\beta_{2s}(1)=-\frac{4}{\pi(2s)^3(1-4s^2)},\quad \beta_{2s}'(1)=\frac{4}{\pi(2s)^4(1-4s^2)},$$

$$\lim_{\xi\to\infty}\frac{\beta_{2s}(\xi)}{\xi}=0,\quad \lim_{\xi\to\infty}\beta_{2s}'(\xi)=0.$$

From the two last conditions it follows that $A_{2s}=B_{2s}=0$. The two first conditions lead to the system

$$2sC_{2s}+2sD_{2s}=\frac{4}{\pi(2s)^3(1-4s^2)},\quad -2sC_{2s}+(2-2s)D_{2s}=\frac{4}{\pi(2s)^4(1-4s^2)},$$

whose solution is

$$C_{2s}=\frac{2}{\pi(2s)^4(2s+1)},\quad D_{2s}=\frac{-2}{\pi(2s)^4(2s-1)}.$$

Finally the functions β_{2s} are $\left(\xi=\frac{r}{a}\right)$

$$\beta_{2s}\left(\frac{r}{a}\right)=\frac{2}{\pi(2s)^4}\left[\frac{1}{2s+1}\left(\frac{a}{r}\right)^{2s}-\frac{1}{2s-1}\left(\frac{a}{r}\right)^{2s-2}\right]. \tag{40}$$

Thus the functions α_i and β_{2s} are already determined. Substituting them into (24) and (25) one obtains the velocity components v_r and v_θ for θ over $[0, \pi]$.

The solution for v_r and v_θ when θ is over $[-\pi, 0]$ follows in the same way making use $\cos\theta$ from (19) and $\sin\theta$ from (20) into the two first boundary conditions of (7) respectively. We give the final results as follows

$$\alpha_4(\xi) = -\frac{1}{12\pi}(1+\ln\xi), \quad \alpha_3(\xi) = -\frac{1}{6}(1+\ln\xi),$$

$$\alpha_2(\xi) = -\left(\frac{\pi}{12} - \frac{1}{\pi}\right)(1+\ln\xi) + \frac{1}{6\pi}(3\ln^2\xi + \ln^3\xi), \quad \alpha_1(\xi) = 1 + \ln\xi + \frac{1}{6}(3\ln^2\xi + \ln^3\xi),$$

$$\alpha_0(\xi) = -a_{00} - \left(\frac{2}{\pi} - \frac{\pi}{6} - \frac{\pi^3}{360}\right)\ln\xi + \left(\frac{\pi}{12} - \frac{1}{\pi}\right)\left(\ln^2\xi + \frac{\ln^3\xi}{3}\right) - \frac{1}{12\pi}\ln^4\xi - \frac{1}{60\pi}\ln^5\xi,$$

$$\beta_{2s}\left(\frac{r}{a}\right) = -\frac{2}{\pi(2s)^4}\left[\frac{1}{2s+1}\left(\frac{r}{a}\right)^{2s} - \frac{1}{2s-1}\left(\frac{r}{a}\right)^{2s-2}\right].$$

Let us denote $\theta = \theta_+$, $\alpha_i = \alpha_{i+}$, $\beta_{2s} = \beta_{2s+}$ when $\theta \in [0, \pi]$ and $\theta = \theta_-$, $\alpha_i = \alpha_{i-}$, $\beta_{2s} = \beta_{2s-}$ when $\theta \in [-\pi, 0]$. As is obvious

$$\theta_- = -\theta_+; \quad \alpha_{0-} = -\alpha_{0+}; \quad \alpha_{1-} = \alpha_{1+}; \quad \alpha_{2-} = -\alpha_{2+}; \quad \alpha_{3-} = \alpha_{3+}, \quad \alpha_{4-} = -\alpha_{4+}, \quad \beta_{2s-} = -\beta_{2s+}.$$

Accounting these properties through the expression (24) for v_r and (25) for v_θ it can be readily shown that

$$v_r(r, \theta_-) = v_r(r, \theta_+), \quad v_\theta(r, \theta_-) = -v_\theta(r, \theta_+).$$

These relationships show the symmetry of the velocity field relatively the plane $\theta = 0$.

Since θ^k $(k = 0, 1, 2, 3, 4)$ of (24) and (25) can be expressed by convergent Fourier series, then v_r and v_θ can be wholly expressed by Fourier series as ξ is a parameter. Then the generalised Dirichlé's theorem [6] holds concerning the sum of Fourier series at the end of intervals. So that for $\theta = 0$, and $\theta = \pm\pi$ one obtains

$$v_r(\xi, \theta)_{\,\theta=0} = \frac{1}{2}\lim_{\theta\pm\to 0}[v_r(\xi, \theta_+) + v_r(\xi, \theta_-)] = v_r(\xi, 0),$$

$$v_\theta(\xi, 0)|_{\theta=0} = \frac{1}{2}\lim_{\theta\pm\to 0}[v_\theta(\xi, \theta_+) + v_\theta(\xi, \theta_-)] = \frac{1}{2}\lim_{\theta\pm\to 0}[v_\theta(\xi, \theta_+) - v_\theta(\xi, \theta_+)] = 0,$$

$$v_r(\xi, \pm\pi) = v_r(\xi, \pi), \quad v_\theta(\xi, \pm\pi) = 0.$$

Thus the stated boundary problem II for the velocity field satisfying all the boundary conditions (7) is completely solved.

Replacing v_r and v_θ into (5) one obtains the solution for the velocity field of the boundary problem I.

Solution for the pressure distribution

Substituting now the stream function ψ from (23) into the system (10) also accounting the fact that the function

$$\psi_1\left(\frac{r}{a}, \theta\right) = Ua\left[\alpha_0\left(\frac{r}{a}\right) + \alpha_1\left(\frac{r}{a}\right)\theta + \alpha_2\left(\frac{r}{a}\right)\theta^2 + \alpha_3\left(\frac{r}{a}\right)\theta^3 + \alpha_4\left(\frac{r}{a}\right)\theta^4\right.$$

$$\left. + \frac{2}{\pi}\sum_{s=1}^{\infty}\frac{2\cos 2s\theta}{(2s)^4(2s+1)}\left(\frac{a}{r}\right)^{2s}\right]$$

is harmonic one ($\Delta\psi_1=0$) and after integrating we obtain

$$p=C+\frac{8\mu U}{\pi a}\sum_{s=1}^{\infty}\frac{\sin 2s\theta}{(2s)^4}\left(\frac{a}{r}\right)^{2s+1},$$

when θ is over $[0, \pi]$. Because of $p|_{r=\infty}=p_0$ it follows $C=p_0$ so that

$$p=p_0+\frac{8\mu U}{\pi a}\sum_{s=1}^{\infty}\frac{\sin 2s\theta}{(2s)^4}\left(\frac{a}{r}\right)^{2s+1}.$$

Since $\beta_{2s-}\left(\frac{r}{a}\right)=-\beta_{2s+}\left(\frac{r}{a}\right)$ then the function p when θ is over $[-\pi, 0]$ is expressed as

$$p=p_0-\frac{8\mu U}{\pi a}\sum_{s=1}^{\infty}\frac{\sin 2s\theta}{(2s)^4}\left(\frac{a}{r}\right)^{2s+1}.$$

Obviously there is no difference since $\theta_-=-\theta_+$ and $\sin 2s\theta_-=-\sin 2s\theta_+$. This fact shows the symmetry of the pressure field relatively the plane $\theta=0$.

Thus the stated Stokes' problem is completely solved.

Conclusion

The solution of the Stokes' problem presented here confirms that the so-called Stokes' paradox does not exist. There is only a case when the Stokes' problem of a cylinder cannot be solved by means of a stream function of the form (13) like the Stokes' solution.

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