On the Remark by M. M. Konstantinov Concerning the Solution of the Stokes' Problem for Circular Cylinder

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The author of this remark awkwardly suggests that the function p, obtained in [1], could not fit for a solution of the stated Stokes' problem.

A direct verification is made, confirming that p satisfies the Stokes' differential equations for each θ (a slight erratum made out on p. 63, where the $\frac{r}{a}$ must be in power $2s$). The function p is to satisfy the Stokes' Eqns and the condition $p \to p_0$ for $r \to \infty$. Any other requirement as in [2] for $\theta = 0, \frac{\pi}{2}$ goes beyond the correct statement of this Stokes' problem. Indeed there is no condition for the variable θ.

The Stokes' equations, as being approximate, are unable to represent adequate picture of the flow. One shouldn't expect more from its solution. In the paper [1] a mathematical solution of the Stokes' equations subject to boundary conditions is obtained.

The remark [2] is absolutely unreasonable in the light of the test of the solution, as well as the mentioned before.

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On the Incorrectness of a Mathematical Proof about Unsolvability of the Stokes' Paradox

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Introduction

As it is well known, the Stokes' paradox is the impossibility to obtain an exact solution of the boundary value problem for the Stokes' differential equations

$$\operatorname{grad} p = \mu \Delta \bar{v}, \quad \operatorname{div} \bar{v} = 0, \tag{1}$$

for a long circular cylinder subject to boundary conditions appropriate of viscous incompressible fluid. For the flow past a motionless cylinder the boundary conditions are

$$\bar{v}_s = \bar{0}, \quad \bar{v}_\infty = -\bar{U}, \quad p_\infty = p_0, \tag{2}$$

where s is the cylinder surface, $\bar{U}$ is a constant vector, and p_0 is a scalar. First the stated problem had been considered by G. Stokes (1851) with a solution that the velocity field obtained cannot satisfy all the boundary conditions (2).

There is a mathematical proof concerning the impossibility to exist a solution for the velocity field exactly satisfying all the boundary conditions of this Stokes' problem [1]. The final result in this proof is that the stream function ψ which satisfies the biharmonic equation $\Delta\Delta\psi = 0$, at the same time must be a harmonic one, i. e. it must be satisfying $\Delta\psi = 0$.

A solution of this Stokes' boundary value problem which exactly satisfy all the boundary conditions (2) in [2] is given.

In the present note the incorrectness of the mathematical proof about unsolvability of the Stokes' paradox in the Birkhoff's book [1] has shown.

Formulation of the Stokes' problem

The boundary value problem (1), (2) in cylindrical co-ordinates is reduced to the system differential equations

$$\frac{\partial p}{\partial r}=\frac{\mu}{r}\frac{\partial}{\partial \theta}\frac{1}{r}\left(\frac{\partial v_r}{\partial \theta}-\frac{\partial r v_\theta}{\partial r}\right),$$

$$(3)\qquad \frac{\partial p}{\partial \theta}=-\mu r\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{\partial v_r}{\partial \theta}-\frac{\partial r v_\theta}{\partial r}\right),$$

$$\frac{\partial r v_r}{\partial r}+\frac{\partial v_\theta}{\partial \theta}=0$$

subject to the following boundary conditions

$$(4)\qquad v_r\big|_{r=\infty}=-U\cos\theta,\; v_\theta\big|_{r=\infty}=U\sin\theta,\; v_r\big|_{r=a}=0,\; v_\theta\big|_{r=a}=0,\; p\big|_{r=\infty}=p_0.$$

Introducing the stream function ψ through

$$(5)\qquad v_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta}\qquad v_\theta=-\frac{\partial\psi}{\partial r}$$

and eliminating the pressure terms of (3) by cross differentiation, then the system (3) leads to biharmonic equation

$$(6)\qquad \frac{\partial^2\Delta\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Delta\psi}{\partial\theta^2}=0,$$

where Δ is the Laplace's operator. The boundary conditions for v_r and v_θ, through (5) are to be reduced to

$$(7)\qquad \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)_{r=\infty}=-U\cos\theta,\; \left(\frac{\partial\psi}{\partial r}\right)_{r=\infty}=U\sin\theta,\; \left(\frac{\partial\psi}{\partial\theta}\right)_{r=a}=0,\; \left(\frac{\partial\psi}{\partial r}\right)_{r=a}=0.$$

In this way the stated boundary problem (3), (4) is reduced to the problem to find a solution of equation (6) for the stream function ψ subject to the boundary conditions (7).

It is worth emphasizing that the boundary conditions (7) concern only the partial derivatives of the stream function ψ and do not contain any condition for the stream function itself. Obviously if ψ is a solution of equation (6) and satisfying the boundary conditions (7), then $\psi+C$ (where C is an arbitrary constant) is a solution too. Therefore, the solution of the problem for the stream function depends on an arbitrary additive constant C which cannot be determined from the boundary conditions (7).

The result of the proof

To make clear the incorrectness of the proof in [1], some details of the proof need to be given. Making use the theorem of divergence in the proof

$$(8)\qquad \iint_G(\varphi\Delta\psi-\psi\Delta\varphi)d\sigma=\int_C\left(\varphi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\varphi}{\partial n}\right)ds$$

for the plane domain G, bounded by circumference of radius a and by another of large radius R, then the relationship (8) for $\varphi=\Delta\psi$ and $\frac{\partial\psi}{\partial n}=\frac{\partial\psi}{\partial r}$ in cylindrical co-ordinates is considered. Since $\Delta\varphi=\Delta\Delta\psi=0$, it follows, from (8), that

$$(9)\qquad \iint_G(\Delta\psi)^2d\sigma=\int_{C_1}\left(\Delta\psi\frac{\partial\psi}{\partial r}-\psi\frac{\partial\Delta\psi}{\partial r}\right)ds+\int_{C_2}\left(\Delta\psi\frac{\partial\psi}{\partial r}-\psi\frac{\partial\Delta\psi}{\partial r}\right)ds,$$

where C_1 is the circumference with radius a and C_2 is with radius R. Because the asymptotic estimates over C_2 $\Delta\psi=O(R^{-2})$; $\psi=O(R)$; $\frac{\partial\psi}{\partial r}=O(1)$; $\frac{\partial\Delta\psi}{\partial r}=O(R^{-3})$ it follows when $R\to\infty$, that

$$\int\limits_{C_2}\left(\Delta\psi\frac{\partial\psi}{\partial r}-\psi\frac{\partial\Delta\psi}{\partial r}\right)ds=O(R^{-2})O(R)\to 0.$$

In order to estimate the integral over C_2, then the following boundary in the proof conditions

$$\left(\frac{\partial\psi}{\partial r}\right)_{r=a}=0,\quad \psi|_{r=a}=0,$$

are used to yield a conclusion that this integral becomes zero. Since $(\Delta\psi)^2\geq 0$ it follows that $\Delta\psi=0$, and this being so, the function ψ must be harmonic one. As it is well known, a harmonic function is not be able to satisfy all four boundary conditions of (7), hence the boundary value problem (6), (7) cannot be solved in general. Therefore, the Stokes' boundary value problem (3), (4) for the circular cylinder cannot be solved too, and the proof is finished.

The incorrectness of the proof

The incorrectness of the proof is that the condition $\psi|_{r=a}=0$ does not follow from the boundary conditions (7). In general from the conditions (7) it follows that $\psi|_{r=a}=C$, where C is an arbitrary constant, consequently the linear integral over C_2 is not zero in general, as well as the stream function ψ is not a harmonic one. The condition $\psi|_{r=a}=0$ is artificial and its application together with the conditions (7) makes the stated boundary value problem overdetermined, consequently not possessing of solution. The incorrectness of the proof in [1] is shown.

A conditional assumption of the proof in [1], that the zeroth stream line ($C=0$) is the circumference with radius a is made. Such an assumption is arbitrary because, as it is known from [3], p. 185, the zeroth stream line ($C=0$) may be arbitrary chosen, since the stream function ψ depends of the arbitrary additive constant C.

References

1. Birkhoff, G. Hydrodynamics. Princeton Univ. Press, 1960, & 30.
2. Shulev, K. A Solution of the Stokes' Problem for a Circular Cylinder. — Bulg. Acad. Sci., Theor. Appl. Mech., 1987, 2.
3. Loytzianski, L. Mechanics of liquid and gas. Fismatgis, Moscow, 1973 (in Russian).

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