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Обща механика

Investigation of the practical stability of the solutions of systems with impulse effect by the help of the comparison method

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1. Introduction

One of the most important aspects in the theory of stability of the solutions of differential equations is the so-called practical stability. The main results in this respect are due to A. A. Martiniuk [1]—[4].

The main problem in the theory of practical stability consists in studying the solutions of systems of differential equations, given in advance the domain where the initial conditions change, and the domain, where the solutions should remain when the independent variable changes over a fixed interval (finite or infinite).

The consideration of the practical stability of the solutions of a system of the form

(1) $$\dot{x}=f(t, x)$$

can be done by studying the relations of this system and the system

(2) $$\dot{u}=F(t, u),$$

so that the practical stability of the solutions of system (2) should imply the practical stability of the solutions of system (1). These relations are obtained due to the employment of differential inequalities. System (2) is usually of lower order and its righthand side possesses a definite type of monotonicity which simplifies considerably the study of its solutions. Actually, this is the essense of the method of comparison in the theory of practical stability.

Along with the development of the theory of practical stability in recent years the mathematical theory of impulse systems has been intensively advancing.

Many problems of natural sciences and technology describe real processes which are subjected to short-time perturbations in the process of their evolution. When mathematical models of similar processes are built-up it is usually assumed that the perturbations have the nature of an instant "push" ("stroke", "impulse"). The mathematical description of such processes leads namely to systems of differential equations with impulse effect.

The first contribution devoted to systems of differential equations with impulses was due to V. D. Mil'man and A. D. Myshkis [5]. In [6]—[10] some problems of the qualitative theory of non-linear impulse systems have been considered. Here we will mention the results obtained in [11]—[14].

The present paper applies the method for **comparison** in order to study the practical stability of the trivial solution of a system of **differential** equations with impulse effect at fixed moments

$$\text{(3)}\qquad \begin{aligned} &\dot{x}=f(t, x), \quad t\neq t_i \\ &\Delta x|_{t=t_i}=x(t_i+0)-x(t_i-0)=I_i(x(t_i)), \quad i=1, 2, \ldots, \end{aligned}$$

where $x: I\to R^n$, $f: I\times\Omega\to R^n$, $I_i: \Omega\to R^n$, R^n is the n-dimensional Euclidean space with norm $\|x\|=\left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$, $I=[0, \infty)$, Ω is a domain in R^n, while the moments $\{t_i\}_{i=1}^{\infty}$ of impulse effect form a strictly increasing sequence

$$0<t_1<t_2<\ldots<t_i<\ldots, \quad \lim_{i\to\infty} t_i=\infty.$$

The system with impulse effect (3) is characterized by the fact that under the action of an instant "force" ("stroke", "impulse") at the moment $t=t_i$ the mapping point instantly jumps from the position $(t_i, x(t_i))$ into the position $(t_i, x(t_i)+I_i(x(t_i)))$. The solutions of system (3) are piecewise continuous functions with first order discontinuities $\{t_i\}_{i=1}^{\infty}$, where they are left continuous, i. e. the relations

$$x(t_i-0)=x(t_i); \quad x(t_i+0)=x(t_i)+\Delta x(t_i)=x(t_i)+I_i(x(t_i))$$

hold.

Along with system (3) consider the system

$$\text{(4)}\qquad \begin{aligned} &\dot{u}=F(t, u), \quad t\neq t_i \\ &\Delta u|_{t=t_i}=u(t_i+0)-u(t_i-0)=J_i(u(t_i)), \quad i-1, 2, \ldots, \end{aligned}$$

where $u: I\to R^m$, $F: I\times G\to R^m$, $J_i: G\to R^m$, G is a domain in R^m.

Let $t_0\in I$, $x_0\in\Omega$, $u_0\in G$. Introduce the denotation $x(t)=x(t; t_0, x_0)$ for the solution of (3) satisfying the initial condition $x(t_0+0)=x_0$, and by $J=J(t_0, x_0)$ denote the maximal interval of the form $[t_0, \tilde{t})$, where the solution $x(t)$ is defined. Introduce the denotation $u(t)=u(t; t_0, u_0)$ for the solution of system (4) fulfilling the initial condition $u(t_0+0)=u_0$, and by $J^+=J^+(t_0, u_0)$ denote the maximal interval where the solution $u(t)$ is defined.

II. Preliminary Notes and Definitions

Following A. A. Martiniuk [1] we will introduce definitions for practical stability of the trivial solution of (3).

Let λ, A, B be positive constants ($\lambda<A$, $B<A$), let K be the class of all continuous and monotonically increasing functions $\varphi(\cdot): I\to I$ such that $\varphi(0)=0$ and $\varphi(r)\to\infty$ as $r\to\infty$, while K^* is the class of all continuous functions $\varphi(.,.): I\times I\to I$, monotonically increasing by their second argument and such that $\varphi(t, 0)=0$ anb $\varphi(t, r)\to\infty$ as $r\to\infty$.

Definition 1. The trivial solution of system (3) is called:

1. Practically stable with respect to λ, A, I if

$$(\forall t_0\in I)(\forall x_0\in\Omega, \|x_0\|<\lambda)(\exists\varphi\in K^*)(\forall t\in J(t_0, x_0)):$$
$$\|x(t;t_0, x_0)\|\leq\varphi(t_0, \|x_0\|) \text{ and } \varphi(t_0, \lambda)<A.$$

2. Uniformly practically stable with respect to λ, A, I if $(\forall x_0\in\Omega, \|x_0\|<\lambda)(\exists\varphi\in K)(\forall t_0\in I)(\forall t\in J(t_0, x_0)):$

$$\|x(t;\ t_0,\ x_0)\|\leq\varphi(\|x_0\|)\ \text{ and }\ \varphi(\lambda)<A.$$

3. Contractively practically stable with respect to λ, A, B, I if $(\forall t_0\in I)(\forall x_0\in\Omega,$ $\|x_0\|<\lambda)(\exists\ \varphi\in K^*)(\exists\ \psi: I\to I)$

$$(\forall t\in J(t_0,\ x_0)):\ \|x(t;\ t_0,\ x_0)\|\leq\varphi(t_0,\ \|x_0\|)\psi(t),\ \ \varphi(t_0,\ \lambda)\psi(t)<A$$

and $\varphi(t_0,\ \lambda)\psi(t_0+\tau)<B$ for some $\tau>0$.

4. Contractively uniformly practically stable with respect to λ, A, B, I if $(\forall x_0\in\Omega,$ $\|x_0\|<\lambda)(\exists\varphi\in K)$

$$(\exists\psi:\ I\to I)(\forall t_0\in I)(\forall t\in J(t_0,\ x_0)):\ \|x(t;\ t_0,\ x_0)\|\leq\varphi(\|x_0\|)\psi(t),$$

$\varphi(\lambda)\psi(t)<A$ and $\varphi(\lambda)\psi(t_0+\tau)<B$ for some $\tau>0$.

Introduce in R^m a partial ordering defined in the following natural way: for u, $v\in R^m$ we will write $u\geqq v$ $(u>v)$ if and only if $u_j\geqq v_j$ $(u_j>V_j)$ for any $j=1,\ 2,\dots,\ m$.

Definition 2. The function $\psi:\ G\to R^m$ is called monotonically increasing in G if $\psi(u)>\psi(v)$ for $u>v$ and $\psi(u)\geqq\psi(v)$ if $u\geqq v$.

Definition 3. The function $F:\ I\times G\to R^m$ is called quasi-monotonically increasing in $I\times G$ if for each pair of points $(t,\ u)$, $(t,\ v)$ from $I\times G$ and for any $j=1,\ 2,\dots,m$ $F_j(t,\ u)\geqq F_j(t,\ v)$ holds always when $u_j=v_j$ and $u\geqq v$.

In the case when the function $F:\ I\times G\to R^m$ is continuous and quasi-monotonically increasing all solutions of system (4) starting from the point $(t_0,\ u_0)\in I\times G$ lie between two singular solutions — the maximal, and the minimal one.

Definition 4. The solution $u^+:\ [t_0,\ \tilde{t})\to R^m$ of system (4) for which $u^+(t_0+0)=u_0$, is called a maximal solution if for any other solution $u:[t_0,\ \tilde{t})\to R^m$ for which $u(t_0+0)=u_0$ the inequality $u(t)\leqq u^+(t)$ holds for $t\in[t_0,\ \tilde{t})\cap[t_0,\ \tilde{t}_1)$. Analogously, the minimal solution of system (4) is defined.

Further we will use the following lemmas:

Lemma 1 ([7], Lemma 2). Let the following conditions be fulfilled:

1. The function $F:\ I\times G\to R^m$ is continuous and quasi-monotonically increasing and $u^+:\ J^+(t_0,\ u_0)\to R^m$ is the maximal solution of (4) for which $u^+(t_0+0)=u_0$, $(t_0,\ u_0)\in I\times G$.

2. The functions $\psi_i:\ G\to R^m$, $\psi_i(u)=u+J_i(u)$, $i=1,\ 2,\dots$ are monotonically increasing in G.

3. The function $v:\ J^+(t_0,\ u_0)\to G$ is piecewise continuous with first order discontinuities $t_i\in J^+(t_0,\ u_0)$ where it is left continuous and such that:

$$v(t_0+0)\leqq u_0 \tag{5}$$

$$Dv(t)\leqq F(t,\ v(t))\ \text{ for }\ t\in J^+(t_0,\ u_0)\ \text{ and }\ t\neq t_i, \tag{6}$$

where $Dv(t)$ is any Dini derivative of the function $v(t)$.

$$v(t_i+0)\leq\psi_i(v(t_i))\ \text{ for }\ t_i\in J^+(t_0,\ u_0). \tag{7}$$

Then for $t\in J^+(t_0,\ u_0)$ the inequality

$$v(t)\leqq u^+(t) \tag{8}$$

holds.

Lemma 2 ([7], Lemma 3). Let the following conditions be fulfilled:

1. The function $F:\ I\times G\to R^m$ is continuous and quasi-monotonically increasing in $I\times G$ and $u^-:\ J^+(t_0,\ u_0)\to R^m$ is the minimal solution of system (4) for which $u^-(t_0+0)=u_0$, $(t_0,\ u_0)\in I\times G$.

2. Condition 2 of Lemma 1 holds.

3. The function $v:\ J^+(t_0,\ u_0)\to G$ is piecewise continuous with first order discontinuities $t_i\in J^+(t_0,\ u_0)$ where it is left continuous and such that:

$$v(t_0+0)\geqq u_0$$

$$Dv(t)\geqq F(t,\ v(t))\ \text{for}\ t\in J^+(t_0,\ u_0)\ \text{and}\ t\neq t_i$$

$$v(t_i+0)\geqq \psi_i(v(t_i))\ \text{for}\ t_i\in J^+(t_0,\ u_0).$$

Then for $t\in J^+$ the inequality

$$v(t)\geqq u^-(t)$$

holds.

Describe the class V_0 of piecewise continuous auxiliary functions employed in our further considerations, [7]. Introduce the denotation

$$\Omega_i=\{(t,\ x)\in I\times\Omega:\ t_{i-1}<t<t_i\},\quad i=1,\ 2,\ldots,\ t_0=0.$$

Consider the class V_0 of all functions $V\colon I\times\Omega\to R^n$, $(t,\ x)\to V(t,\ x)$ for which the following conditions are satisfied:

1. The functions V are continuous over any of the sets Ω_i and $V(t,\ 0)=0$ as $t\in I$.
2. The functions V are locally Lipschitzian by x.
3. For each $i=1,\ 2,\ldots$ the limits

$$V(t,-0,\ x)=\lim_{\substack{t\to t_i\\ (t,x)\in\Omega_i}} V(t,\ x);\quad V(t_i+0,\ x)=\lim_{\substack{t\to t_i\\ (t,x)\in\Omega_{i+1}}} V(t,\ x)$$

exist and are finite and the equality

$$V(t_i-0,\ x)=V(t,\ x),\quad x\in\Omega.$$

holds.

Lemma 3. Let the following conditions be fulfilled:

1. Functions $V\in U_0$ and $a\in K$ exist such that the inequalities

(9) $$a(\|x\|)\leqq\max_{1\leqq j\leqq m} V_j(t,\ x),\quad (t,\ x)\in I\times\Omega$$

(10) $$\dot V_{(3)}(t,x)=\limsup_{h\to+0}\frac{1}{h}\,[V(t+h,\ x+hf(t,\ x))-V(t,\ x)]\leqq F(t,\ V(t,\ x)),\quad t\neq t_i$$

(11) $$V(t_i+0,\ x+I_i(x))\geqq\psi_i(V(t_i,\ x)),\quad i=1,\ 2,\ldots$$

hold.

2. Condition 2 of Lemma 1 holds.
3. The function $F\colon I\times G\to R^m$ is continuous and quasi-monotonically increasing in $I\times G$ and $u^+\colon J^+(t_0,\ u_0)\to R^m$ is the maximal solution of (4) for which $u^+(t_0+0)=u_0$, and $u_0\geqq V(t_0,\ x_0)$, $(t_0,\ x_0)\in I\times\Omega$.

Then for $t\in J(t_0,\ x_0)\cap J^+(t_0,\ u_0)$ the inequality

(12) $$\|x(t;\ t_0,\ x_0)\|\leqq a^{-1}[\max_j u_j^+(t;\ t_0,\ u_0)]$$

holds.

Proof. The conditions of Lemma 3 imply that the function $v(t)=V(t,\ x(t;\ t_0,\ x_0))$ satisfies the conditions of Lemma 1. Using this and (9) as well, we obtain

$$a(\|x(t;\ t_0,\ x_0)\|)\leqq\max_j V_j(t,x(t;\ t_0,\ x_0))=\max_j v_j(t)\leqq$$

$$\leqq\max_j u^+(t;\ t_0,\ u_0),\quad t\in J(t_0,\ x_0)\cap J^+(t_0,\ u_0),$$

whence follows inequality (12).

Let $e\in R^m$, $e=(1,\ 1,\ldots,\ 1)$ and $G\supset\{u\colon 0\leqq u\leqq e\}$. Further we will consider such solutions of (4) only for which $u(t)\geqq 0$. Hence, the following modification of definition 1 seems most appropriate.

Definition 5. The trivial solution of system (4) is called:

1. Practically u-stable with respect to λ, A, I if

$$(\forall t_0 \in I)(\forall u_0 \in G,\ 0 \leq u_0 \leq \lambda e)(\exists\ \varphi \in K^*)$$

$$(\exists a \in K)(\forall t \in J^+(t_0,\ u_0)):\ u^+(t;\ t_0,\ u_0) \leq \varphi(t_0,\ \|u_0\|)e \quad \text{and} \quad \varphi(t,\ \lambda) < a(A).$$

2. Uniformly practically u-stable with respect to λ, A, I if

$$(\forall u_0 \in G,\ 0 \leq u_0 \leq \lambda e)(\exists \varphi \in K)(\exists a \in K)$$

$$(\forall t_0 \in I)(\forall t \in J^+(t_0,\ u_0)):\ u^+(t;\ t_0,\ u_0) \leq \varphi(\|u_0\|)$$

and $\varphi(\lambda) < a(A)$.

3. Contractively practically u-stable with respect to λ, A, B, I if $(\forall t_0 \in I)(\forall u_0 \in G,$ $0 \leq u_0 \leq \lambda e)(\exists \varphi \in K^*)$

$$(\exists a \in K)(\exists \sigma:\ I \to I)(\forall t \in J^+(t_0,\ u_0)):$$

$u^+(t;\ t_0,\ u_0) \leq (t_0,\ \|u_0\|)\sigma(t)e$, $\varphi(t_0,\ \lambda)\sigma(t) < a(A)$ and $\varphi(t_0,\ \lambda)\sigma(t_0+\tau) < a(B)$ for some $\tau > 0$.

4. Contractively uniformly practically u-stable with respect to λ, A, B, I if $(\forall u_0 \in G,$ $0 \leq u_0 \leq \lambda e)$

$$(\exists \varphi \in K)(\mathrm{E}\, a \in K)(\exists \sigma:\ I \to I)(\forall t_0 \in I)(\forall t \in J^+(t_0,\ u_0)):$$

$$u^+(t;\ t_0,\ u_0) \leq \varphi(\|u_0\|)\sigma(t)e,\ \varphi(\lambda)\sigma(t) < a(A)$$

and $\varphi(\lambda)\sigma(t_0+\tau) < a(B)$ for some $\tau > 0$.

III. Main Results

Theorem 1. Let the following conditions be fulfilled:

1. The conditions of Lemma 3 hold.
2. A function $b \in K^*$ exists such that

$$V(t_0,\ x) \leq b(t_0,\ \|x\|)e,\quad t_0 \in I,\ x \in \Omega. \tag{13}$$

3. $f(t,\ 0)=0$ for $t \in I$, $I_i(0)=0$, $F(t,\ 0)=0$ for $t \in I$, $J_i(0)=0$, $i=1,\ 2, \ldots$ and the solutions of system (4) are defined over the interval $[t_0,\ \infty)$.

Then the following assertions hold:

a) If the trivial solution of (4) is practically u-stable then the trivial solution of (3) is practically stable.

b) If the trivial solution of (4) is uniformly practically u-stable then the trivial solution of (3) is uniformly practically stable.

c) If the trivial solution of (4) is contractively practically u-stable then the trivial solution of (3) is contractively practically stable.

d) If the trivial solution of (4) is contractively uniformly practically u-stable then the trivial solution of (3) is contractively uniformly practically stable.

Proof. Inequality (13) implies the inequality $V(t_0,\ x_0) \leq b(t_0,\ \|x_0\|)e$, and, applying Lemma 3 for $u_0 = b(t_0,\ \|x_0\|)e$ we get the estimate

$$\|x(t;\ t_0,\ x_0)\| \leq a^{-1}[\max_j u_j^+(t;\ t_0,\ b(t_0, \|x_0\|))]. \tag{14}$$

a) If the solution $u(t) \equiv 0$ of (4) is practically u-stable then a function $\varphi_1 \in K^*$ exists so that

$$u^+(t;\ t_0,\ b(t_0,\ \|x_0\|)e) \leq \varphi_1(t_0,\ \|x_0\|)e \quad \text{and} \quad \varphi_1(t_0,\ \lambda) < a(A).$$

Then (14) yields $\|x(t;\ t_0,\ x_0)\| \leq a^{-1}[\varphi_1(t_0, \|x_0\|)] = \varphi(t_0,\ \|x_0\|)$ and $\varphi(t_0,\ \lambda) = a^{-1}[\varphi_1(t_0,\ \lambda)]$ $< a^{-1}(a(A)) = A$. Since $\varphi_1 \in K^*$ and $a \in K$ then $\varphi \in K^*$.

The practical stability of the solution $x(t)\equiv 0$ of (3) is proved.

b) If the solution $u(t)\equiv 0$ of (4) is uniformly practically u-stable then a function $\varphi_2 \in K$ exists so that

$$u^+(t;\ t_0,\ b(t_0,\ \|x_0\|)e)\leqq \varphi_2(\|x_0\|)e \quad \text{and} \quad \varphi_2(\lambda)<a(A).$$

Then (14) yields $\|x(t;\ t_0,\ x_0)\|\leqq a^{-1}(\varphi_2(\|x_0\|))=\varphi(\|x_0\|)$ and $\varphi(\lambda)=a^{-1}(\varphi_2(\lambda)) <a^{-1}(a(A))=A$. Since $\varphi_2,\ a\in K$ then $\varphi\in K$. Therefore, the solution $x(t)\equiv 0$ of (3) is uniformly practically stable.

c) If the solution $u(t)\equiv 0$ of (4) is contractively practically u-stable then a func- i on $\varphi_3\in K^*$ and a function $\sigma\colon I\to I$ exist so that $u^+(t;\ t_0,\ b(t_0,\|x_0\|)e)\leqq\varphi_3(t_0,\ \|x_0\|)\sigma(t)e$, $\varphi_3(t_0,\ \lambda)\sigma(t)<a(A)$ and $\varphi_3(t_0,\ \lambda)\sigma(t_0+\tau)<a(B)$ for some $\tau>0$. Then (14) yields initial conditions, the inequality $v(t)\leqq v^+(t)$ holds for all t for which both solutions are defined and $x(t)\in\Omega$.

Here $x(t)$ is viewed as a given function for $[t_0,\ \infty)$ v-minimal solution of system (15) is defined analogously.

In this case the method of comparison is based on three lemmas whose proof does not vary from the respective proofs of Lemmas 1, 2 and 3.

Lemma 4. Let the following conditions be fulfilled:

1. The function $g\colon I\times G\times\Omega\to R^m$, $(t,\ v,\ x)\to g(t,\ v,\ x)$ is continuous and quasimonotonically increasing along v and $(x(t),\ v^+(t;\ t_0,\ v_0,\ x_0))$ is the v-maximal solution of (15) defined in $[t_0,\tilde{t})$.

2. Condition 2 of Lemma 1 holds.

3. The function $w\colon [t_0,\ \tilde{t})\to G$ is piecewise continuous with first order discontinuities $t_i\in[t_0,\ \tilde{t})$, where it is left continuous and satisfies the conditions:

$$w(t_0+0)\leqq v_0$$

$$Dw(t)\leqq g(t,\ w(t),\ x),\quad t\in[t_0,\ \tilde{t}),\ t\neq t_i,\ x\in\Omega$$

$$w(t_i+0)\leqq\psi_i(w(t_i))\quad \text{for } t_i\in[t_0,\ \tilde{t}).$$

Then $w(t)\leqq v^+(t;\ t_0,\ v_0,\ x_0)$ for $t\in[t_0,\ \tilde{t})$.

Lemma 5. Let the following conditions be fulfilled:

1. The function $g\colon I\times G\times\Omega\to R^m$, $(t,\ v,\ x)\to g(t,\ v,\ x)$ is continuons and quasimonotonically increasing by v and $(x(t),\ v^-(t;\ t_0,\ v_0,\ x_0))$ is the v-minimal solution of (15) defined over the interval $[t_0,\ \tilde{t})$.

$$\|x(t;\ t_0,\ x_0)\|\leqq a^{-1}[\varphi_3(t_0,\ \|x_0\|)\sigma(t)]=\varphi(t_0,\ \|x_0\|)\psi(t),$$

where $\varphi\in K^*$ and $\psi\colon I\to I$. Besides, $\varphi(t_0,\ \lambda)\psi(t)=a^{-1}[\varphi_3(t_0,\ \lambda)\sigma(t)]<a^{-1}(a(A))=A$ and

$$\varphi(t_0,\ \lambda)\psi(t_0+\tau)=a^{-1}[\varphi_3(t_0,\ \lambda)\sigma(t_0+\tau)]<a^{-1}(a(B))=B$$

for some $\tau>0$. Hence, the trivial solution of (3) is contractively practically stable.

Assertion d) of Theorem 1 is proved in the same way as assertion c) of the same theorem.

In some cases for the study of the practical stability of the solutions of system (3) it is suitable to represent the differential inequalities used in Lemma 1 and Lemma 2 in the form

$$Dv(t)\leqq g(t,\ v(t),\ x),$$

where x is viewed as a parameter ([1]—[4]). Here we consider the auxuliary system

$$\dot{x}=f(t,\ x),\ t\neq t_i$$

$$\dot{v}=g(t,\ v,\ x),\quad t\neq t_i.$$

$$(15)\qquad \Delta x|_{t=t_i}=I_i(x(t_i)),\quad \Delta v|_{t=t_i}=J_i(v(t_i)),\quad i=1,2,\ldots$$

where $v: I\to R^m$, $g: I\times G\times\Omega\to R^m$.

Introduce the denotation $(x(t;\, t_0, v_0, x_0), v(t;\, t_0, v_0, x_0))=(x(t), v(t))$ for the solution of system (15) fulfilling the initial conditions $x(t_0+0)=x_0$, $v(t_0+0)=v_0$, $(t_0, v_0, x_0)\in I\times G\times\Omega$.

Definition 6. The solution $(x(t), v^+(t))=(x(t;\, t_0, v_0, x_0), v^+(t;\, t_0, v_0, x_0))$ of system (15) is called v-maximal solution if for any other solution $(x(t), v(t))$, satisfying the same.

2. Condition 2 of Lemma 1 is fulfilled.

3. The function $w: [t_0, \tilde{t})\to G$ is piecewise continuous with discontinuities $t_i\in[t_0, \tilde{t})$, where it is left continuous and such that

$$w(t_0+0)\geqq v_0$$

$$Dw(t)\geqq g(t, w(t), x),\ t\in[t_0, \tilde{\tau}),\ t\pm t_i,\ x\in\Omega$$

$$w(t_i+0)\geqq\psi_i(w(t_i))\quad\text{for } t_i\in[t_0, \tilde{t}).$$

Then $w(t)\geqq v^-(t;\, t_0, v_0, x_0)$ for $t\in[t_0, \tilde{t})$.

Lemma 6. Let the following conditions be fulfilled:

1. A function $V\in U_0$ exists so that for some function $a\in K$ and for any point $(t, x)\in I\times\Omega$

$$a(\|x\|)\leqq\max_j V_j(t, x)$$

$$\dot{V}_{(3)}(t, x)\leqq g(t, V(t, x), x)\quad\text{for } t\neq t_i$$

$$V(t_i+0, x+I_i(x))\leqq\psi_i(V(t_i, x)),\quad i=1, 2,\ldots$$

holds.

2. Condition 2 of Lemma 1 holds.

3. The function $g: I\times G\times\Omega\to R^m$, $(t, v, x)\to g(t, v, x)$ is continuous and quasi-monotonically increasing by v and $(x(t), v^+(t;\, t_0, v_0, x_0))$ is the v-maximal solution of (15) for which $x(t_0+0)=x_0$, $v(t_0+0)=v_0$ and $v_0\geqq V(t_0, x_0)$.

Then for $t\in J(t_0, x_0)\cap J^+(t_0, v_0, x_0)$ the inequality

$$(16)\qquad \|x(t;\, t_0, x_0)\|\leqq a^{-1}[\max_j v_j^+(t;\, t_0, v_0, x_0)]$$

holds.

Let P be the class of all continuous and monotonically increasing functions $\varphi: R^m\to[0, \infty)$ such that $\varphi(0)=0$ and $\varphi(x)\to\infty$ for $\|x\|\to\infty$, while P^* is the class of all continuous functions $\varphi: I\times R^m\to[0, \infty)$, $(t, x)\to\varphi(t, x)$, monotonically increasing along x, $\varphi(t, 0)=0$ and $\varphi(t, x)\to\infty$ as $\|x\|\to\infty$.

Definition 7. The trivial solution of system (15) is called:

1. Practically v-stable with respect to λ, A, I if

$$(\forall t_0\in I)(\forall v_0\in G,\ 0\leqq v_0\leqq\lambda e)(\exists\varphi\in P^*)(\exists a\in K)$$

$$(\forall x_0\in\Omega)(\forall t\in J^+(t_0, v_0, x_0)):\ v^+(t;\, t_0, v_0, x_0)\leqq\varphi(t_0, v_0)e$$

and $\varphi(t_0, \lambda e)<a(A)$.

2. Uniformly practically v-stable with respect to λ, A, I if $(\forall v_0\in G,\ 0\leqq v_0\leqq\lambda e)\times(\exists\varphi\in P)(\exists a\in K)(\forall t_0\in I)$

$$(\forall x_0\,\Omega)(\forall t\in J^+(t_0, v_0, x_0)):\ v^+(t;\, t_0, v_0, x_0)\leqq\varphi(v_0)e$$

and $\varphi(\lambda e)<a(A)$.

3. Contractively practically v-stable with respect to λ, A, B, I if $(\forall t_0\in I)(\forall v_0\in G,$ $0\leqq v_0\leqq\lambda e)(\exists\varphi\in P^*)$

$$(\exists a\in K)(\exists\sigma: I\to I)(\forall x_0\in\Omega)(\forall t\in J^+(t_0, v_0, x_0)):$$

$$v^+(t; t_0, v_0, x_0)\leq\varphi(t_0, v_0)\sigma(t)e, \quad \varphi(t_0, \lambda e)\sigma(t)\leq a(A)$$

and $\varphi(t_0, \lambda e)\sigma(t_0+\tau)<a(B)$ for some $\tau>0$.

4. Contractively uniformly practically v-stable with respect to λ, A, B, I if $(\forall v_0\in G, 0\leq v_0\leq\lambda e)$

$$(\exists\varphi\in P)(\exists a\in K)(\exists\sigma: I\to I)(\forall t_0\in I)(\forall x_0\in\Omega)(\forall t\in J^+(t_0, v_0, x_0)):$$

$$v^+(t; t_0, v_0, x_0)\leq\varphi(v_0)\sigma(t)e, \quad \varphi(\lambda e)\sigma(t)<a(A)$$

and $\varphi(\lambda e)\sigma(t_0+\tau)<a(B)$ for some $t>0$.

Theorem 2. Let the following conditions be fulfilled:

1. The conditions of Lemma 6 are fulfilled.

2. A function $b\in K^*$ exists such that

$$V(t_0, x)\leq b(t_0, \|x\|)e, \quad t_0\in I, \ x\in\Omega.$$

3. $f(t, 0)=0$ for $t\in I$, $I_i(0)=0$, $g(t, 0, 0)=0$ for $t\in I$, $J_i(0)=0$ and the solutions of system (15) are defined over the interval $[t_0, \infty)$.

Then the following assertions hold:

a) If the trivial solution of (15) is practically v-stable then the trivial solution of (3) is practically stable.

b) If the trivial solution of (15) is uniformly practically v-stable then the trivial solution of (3) is uniformly practically stable.

c) If the trivial solution of (15) is contractively practically v-stable then the trivial solution of (3) is contractively practically stable.

d) If the trivial solution of (15) is contractively uniformly practically v-stable then the trivial solution of (3) is contractively uniformly practically stable.

The proof of Theorem 2 is analogous to that of Theorem 1, however, definition 7 is used instead of definition 5 and inequality (16) of Lemma 6 is applied instead of inequality (12) of Lemma 3.

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