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Механика на твърдото деформируемо тяло

Model of an elastic-visco-plastic body with damage

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1. Introduction

A number of metals exibit damage (residual volume change) over a mean stress tension limit, characteristic for the material. This is a well-known phenomenon called microfracture, accompanied by microvoid nucleation (see [2], [4], [8], and others). We consider the case, when microfracture is mainly isotropic and the voids can be taken as spherical. The microdamage doesn't introduce an orientation into the material structure and it does not develop a deformation anisotropy.

We consider bodies, submitted to external dynamic impacts, causing elastic-visco-plastic deformations with damage. For such a deformable processes a number of mechanomathematical models are proposed, that describe this phenomenon at a different degree cf approximation (see [2], [4], [5], etc.). In the present work we approach the same problem together for both the inelastic deformable processes (visco-plastic deformation and microfracture) employing specially introduced energetical measures and starting from the fact that for both processes the prevailing micromechanism is thermofluctuational and on micro level the statistical Bolzmann's law is valid. This implies to include a corresponding deformation energy in the constitutive equations for inelastic strain rates however the deformation energy excceeds a characteristic barrier [6]. On such a basis we propose a physically motivated mechano-mathematical model, which in a reasonable way describes the inelastic deformation of the bodies under consideration. The model is convenient to investigate the dynamic bearing capacity of structural elements.

2. Mechano-mathematical model

The proposed model for inelastic deformation of the material subjected to dynamical processes is based on the following assumptions:

I. The strain rate consists of elastic and inelastic part, i. e.

$$\dot{\varepsilon}_{ij}=\dot{\varepsilon}^{e}_{ij}+\dot{\varepsilon}^{a}_{j} \quad (i,\ j=1,\ 2,\ 3), \tag{1}$$

where by point are marked the material derivatives with respect to time; $\dot{\varepsilon}^{e}_{ij}$ is the rate of the elastic strain part; $\dot{\varepsilon}^{a}_{ij}$ — of the corresponding inelastic part.

II. The elastic strain rate part is small and is connected with the stress rates $\dot{\sigma}_{ij}$ (σ_{ij} is the Cauchy's stress tensor) by the Hook's law:

$$(2) \qquad \dot{\varepsilon}^e_{ij} = H_{ijkl}\dot{\sigma}_{kl} \qquad (i,\ j,\ k,\ l=1,\ 2,\ 3),$$

where H_{ijkl} is the elastic resistance tensor.

III. The inelastic deviatoric part of the strain rate is describing the visco-plastic deformation process of the material in accordance with the model from [1].

$$(3) \qquad \dot{e}^a_{ij} = \Lambda \exp\{D\zeta\} \,.\, S_{ij}/\tau_0\,; \quad \tau_0 > \tau_s(\gamma^a) \,.\, \mathrm{sign}\,(\tau_s)$$

$$\dot{e}^a_{ij} = \dot{\varepsilon}^a_{ij} - \frac{1}{3}\delta_{ij}\dot{\varepsilon}^a_{kk}, \quad S_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}$$

$$\gamma^a = \sqrt{\frac{1}{2} e^a_{ij} e_{ij}}, \quad \tau_0 = \sqrt{\frac{1}{2} S_{ij} S_{ij}}, \quad \beta = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}},$$

where

$$(4) \qquad \tau_0^2 = \tau_p^2(\gamma^a,\ \beta)$$

is a yield condition for plasticity, expressing the plastic and dynamic hardening. τ_p—the yield limit at pure shear; $\tau_s = \tau_p(\gamma^a,\ \beta_s)$ — the yield limit at pure shear and static strain rate β_s (of the order of 10^{-5} sec. $^{-1}$); Λ and D are material functions depending on the process factors: the inelastic shear strain intensity γ^a and the strain rate intensity β; ζ is a nondimensional energetic parameter, expressing the exeed of certain energetic level, over which visco-plastic deformation takes place:

$$(5) \qquad \zeta = \sqrt{\frac{\tau_0^2 + 2\mu S_{ij} e^a_{ij}}{\tau_s^2}} - 1 \qquad \mu = \mathrm{const}$$

(elastic shear modulus)

IV. The inelastic (residual) increase of the volume is described by the spherical part of the inelastic strain tensor $\varepsilon^a_{kk} = \xi$ and it consists of two parts: ξ_N — describing the void nucleation and ξ_G — describing the growth of the already nucleated voids.

$$(6) \qquad \xi = \xi_N + \xi_G$$

the whole volume change as a result of void nucleation and growth is treated as residual.

V. The void nucleation rate $\dot{\xi}_N$ is expressed by a dependence that reflects on macrolevel the action of two micro mechanisms: a thermofluctuational one and a mechanism of force bond destruction [6]. The experiments show that the dependence $\dot{\xi}_N \leftrightarrow \sigma_M$ changes its character, when the deformation energy of the residual volume change $\sigma_M\xi$ reaches a certain energetic level $\frac{\alpha}{K}(\sigma_M^2 - \sigma_{N_0}^2)$ (see [6]) (σ_{N_0} is the mean stress over which the void nucleation starts, K-elastic volume modulus, α-material constant). This change can be explained by the more significant influence of the mechanism of force bond destruction under this energetic limit. With the increase of the void volume ξ the void nucleation rate $\dot{\xi}_N$ decreases, i. e. a "saturation" takes place. It can be described introducing the function $(1-\zeta^2)$ if the complete saturation is at $\xi \to 1$. But it can not be reached because of the damage fracture at $\xi = \xi_F < 1$ (ξ_F is the damage fracture limit of the material). The effect of the thermofluctuational mechanism on macrolevel is described by the nondimensional energetical parameter ξ_N. This parameter gives the excess of the deformation energy over a certain energetical limit. Therefore we have good reason to propose the following equation:

$$(7) \qquad \dot{\xi}_N=\begin{cases} L_N^*(1-\xi)^2\exp[R_N^*\zeta_N]; & K\sigma_M\xi\leq\alpha(\sigma_M^2-\sigma_{N_0}^2) \\ L_N^{**}(1-\xi)^2\exp[R_N^{**}\zeta_N^2]; & K\sigma_M\xi>\alpha(\sigma_M^2-\sigma_{N_0}^2), \end{cases}$$

where

$$(8) \qquad \zeta_N=\sqrt{\frac{\sigma_M^2+K\sigma_M\xi}{\sigma_{N_0}^2}-1}; \quad L_N^*=C_1.v_{\min}; \; L_N^{**}=C_2.v_{\min}$$

$$R_N^{**}=C_3/\xi, \quad \sigma_M\geq\sigma_{N_0}>0$$

C_1, C_2, C_3, R_N^* are material constants; $v_{\min}$ is the minimum void volume, characteristic for the material that nucleates practically immediately when $\sigma_M=\sigma_{N_0}>0$.

VI. The rate of the void growth $\dot{\xi}_G$ is expressed by a relation, which reflects the following effects:

— Micromechanism, presented by nondimensional energetic parameter ζ_G.

— "Saturation" when $\xi\to0.5$, presented by the function $\zeta(1-\xi)$.

— Development of the process, when $\sigma_M\geq\sigma_{G_0}>0$ (σ_{G_0} is the mean stress over which the void growth starts).

$$(9) \qquad \dot{\xi}_G=L_G\xi(1-\xi)R_G.\zeta_G^2; \quad \sigma_M\geq\sigma_{G_0}>0$$

at

$$(10) \qquad \zeta_G=\sqrt{\frac{\sigma_M^2+K\sigma_M\xi}{\sigma_M^2+K\sigma_{G_0}\xi_{G_0}}-1}; \quad R_G=\sqrt{\sigma_{G_0}^2+K\sigma_{G_0}\xi_{G_0}}$$

$$L_G=C_4.v_{\min}$$

C_4-material constant; ξ_{G_0} is the residual volume change, corresponding to the mean stress $\sigma_M=\sigma_{G_0}$.

The material constants in relations (7) and (9) could be experimentally determined by the methods, proposed in the next paragraph. The constitutive equations of the model are formed by equations (1), (2), (3), (6), (7), and (9). They are valid for $0\leq\xi\leq\xi_F$.

3. Methods for determining the material characteristics of the inelastic volume change model

The relations $\sigma_{11}\leftrightarrow\varepsilon_{11}$ for different $\dot{\varepsilon}_{11}=\text{const}$ and $\sigma_M\leftrightarrow\vartheta$ (resp. $\sigma_M\leftrightarrow\xi$ for $\sigma_M\geq\sigma_{N_0}$, $\vartheta=\varepsilon_{kk}=\varepsilon_{kk}^e+\varepsilon_{kk}^a=\frac{1}{K}\sigma_M+\xi$, $\xi=\vartheta-\frac{1}{K}\sigma_M$) can be determined by using various experimental equipment (e. g. split Hopkinson pressure or tension bar, where the specimen cross section can be measured). Analysing specimen, sections, the parts ξ_N and ξ_G, as well as the limit stresses σ_{N_0} and σ_{G_0}, can be determined for different values of σ_M (see [3], [4] and others). The relations $\sigma_{11}\leftrightarrow\varepsilon_{11}$ for $\dot{\varepsilon}_{11}=\text{const}$ are used for determining the material characteristics of the model for visco-plastic deformation of the body (constitutive relations (3)), by the method given in [1]. The material characteristics of the damage model (constitutive relations (7) and (9)) are determined from the relation $\sigma_M\leftrightarrow\xi$. Moreover, following the change of the letter in the interval $[\sigma_{N_0}, \sigma_{G_0}]$, we determine the parameter α from the energy limit that changes the effect of the micro mechanism of force bond destruction. The parameter L_N^* is determined as initial rate $\dot{\xi}_N$ for $\xi=0$ at $\sigma_M=\sigma_{N0}$ using $\dot{\vartheta}=\Delta\vartheta/\Delta t$. For determining $\dot{\vartheta}$ we assume that the increments $\Delta\sigma_M \leftrightarrow \Delta\vartheta$ are in the same relationship as the derivatives $\Delta\sigma_M/\Delta t \leftrightarrow \Delta\vartheta/\Delta t$ or $\dot\sigma_M \leftrightarrow \dot\vartheta$, i. e. from $\sigma_M=\varphi(\vartheta)$, $\Delta\sigma_M=\varphi'(\vartheta)\Delta\vartheta$, $\dot\sigma_M=\varphi'(\vartheta)\dot\vartheta$. By analogy, we obtain from the relation $\sigma_M=f(\xi)$: $\Delta\sigma_M=f'(\xi)\Delta\xi$ and $\dot\sigma_M=f'(\xi)\dot\xi$ (resp. $\dot\sigma_M=f'(\xi_N)\dot\xi_N$ if $\sigma_M \in [\sigma_{N_0}; \sigma_{G_0}]$). $v_{\min}$ is determined by a microstructural analysis of the void nucleation for the respective material. Then $c_1=L_N^*/v_{\min}$.

The relation (7) takes the form:

$$\eta_N=\ln\frac{\dot\xi_N}{L_N^*(1-\xi)^2}=R_N^*\zeta_N \text{ or } R_N^*=\frac{\eta_N}{\zeta_N} \tag{11}$$

$$\zeta_N=\sqrt{\frac{\sigma_M^2+K\sigma_M\xi_N}{\sigma_{N_0}^2}-1}$$

We determine η_N and ζ_N from data for σ_M, ξ_N and $\dot\xi_N$, and by them, using (11) — the material constant R_N^* as well. For the values $\bar\sigma_M$ and $\bar\xi_N$, where $\bar\sigma_M\bar\xi_N=\frac{a}{K}(\sigma_M^2-\sigma_{N_0}^2)$, a smooth transition of ξ_N is to be ensured. So the following relation is obtained:

$$L_N^*\exp[R_N^*\bar\zeta_N]=L_N^{**}\exp[R_N^{**}\bar\zeta_N^2]. \tag{12}$$

We determine $\dot{\bar{\bar\zeta}}_N$ for another value of $\bar{\bar\sigma}_M \in [\bar\sigma_M; \sigma_{G_0}]$ and $\bar{\bar\zeta}$ and we obtain a new relation:

$$\dot{\bar{\bar\xi}}=L_N^{**}(1-\bar{\bar\xi})^2\exp[R_N^{**}\bar{\bar\zeta}_N^2]. \tag{13}$$

Equations (12) and (13) form a system of two transcedental equations for the material constants C_2 and C_3 ($R_N^{**}=C_N/\zeta_N$). The constants are determined using a computer program.

The material parameter L_G (or its corresponding constant G_4) is determined similarly. $\dot\xi_N$ is calculated from the already specified equation, also we have $\dot\xi=\dot\sigma_M/f'(\xi)$ and we obtain $\dot\xi_G=\dot\xi-\dot\xi_N$. Then for a chosen value of $\sigma_M \in [\sigma_{G_0}, \sigma_F]$ and corresponding ξ_G ($\xi_G=\xi-\xi_N$) and $\dot\xi_G$ using (9) we calculate L_G and from there $C_4=L_G/v_{\min}$. σ_F is the mean stress corresponding to $\xi=\xi_F$. σ_F and ξ_F are determined also from the relation $\sigma_M \leftrightarrow \xi$ for the moment when the specimen is breaking.

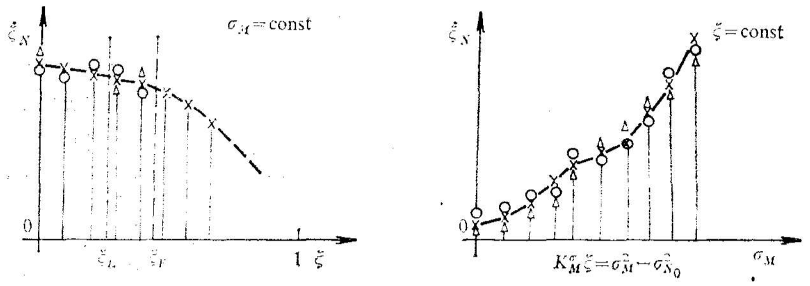

Fig. 1. Comparison of the proposed model for void nucleation with experimental data for aluminium (× — numerical results, △ — data after Krzeminski, ○ — data after Davison, Stivens and Kipp)

The methods are applied for experimental data of aluminium and copper [6]. The approximation that was made, together with the proposed relations of the model (7) and (9) leads to adequate results from statistical point of view. The maximum relative error is of order 3—5% for $\xi_N \leftrightarrow \sigma_M$ and 4—8% for $\xi_G \leftrightarrow \sigma_M$. In Fig. 1 and Fig. 2 are given the comparisons of theoretical and experimental results for these relations. It is seen that the proposed model can describe rationally and physically motivated the volume damage of metals.

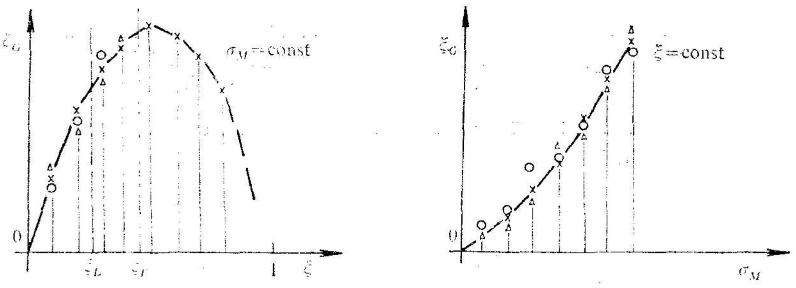

Fig. 2. Comparison of the proposed model for void growth with experimental data for aluminium (× — numerical results, △ — data after Krzeminski, ○ — data after Davison, Stivens and Kipp)

4. Application of the model for determining the dynamic bearing capacity of flexible inelastic rectangular plates

We take into account thin flexible rectangular plates, made of metals, which deform visco-plastically under high intensity external loadings and have the possibility for volume damage. The plates are submitted to impulsive loading $P(t)$, evenly distributed in the middle plane of the plate (before deformation). They undergo large deflections and not so large deformations. This fact gives us the reason to apply the Fon Karmann's theory [6] with precised nonlinear relations between the deformations of the middle plane and the deflections derivatives. The material is described by the mechano-mathematical model given in § 2.

A material description is adopted. A Cartesian coordinate system $OXYZ$ is introduced. The basic equations for a rectangular plate with dimensions a-length, b-width and h-thickness have the form:

I. Geometry equations (in rates):

(14)
$$\frac{\partial e^0_{xx}}{\partial t}=\frac{\partial V_x}{\partial x}+\frac{\partial V_z}{\partial x}\cdot\frac{\partial w}{\partial x}$$
$$\frac{\partial e^0_{yy}}{\partial t}=\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial y}\cdot\frac{\partial w}{\partial y}$$
$$\frac{\partial e^0_{xy}}{\partial t}=\frac{1}{2}\left(\frac{\partial V_x}{\partial y}+\frac{\partial V_y}{\partial y}\right)+\frac{\partial V_z}{\partial x}\cdot\frac{\partial w}{\partial y}+\frac{\partial V_z}{\partial y}\cdot\frac{\partial w}{\partial x},$$

where $(e^0_{xx}, e^0_{yy}, e^0_{xy})$ are the deformations of the middle plane; (u, v, w) are the displacements; (V_x, V_y, V_z) — the velocities of the points in this plane;

(15)
$$\frac{\partial \varkappa_x}{\partial t}=\frac{-\dfrac{\partial^2 V_z}{\partial x^2}}{\left[1-\left(\dfrac{\partial w}{\partial x}\right)^2\right]^{3/2}}+\frac{3\dfrac{\partial^2 w}{\partial x^2}\cdot\dfrac{\partial w}{\partial x}\cdot\dfrac{\partial V_z}{\partial x}}{\left[1+\left(\dfrac{\partial w}{\partial x}\right)^2\right]^{5/2}}$$

$$\frac{\partial\varkappa_y}{\partial t}=\frac{-\frac{\partial^2 V_z}{\partial y^2}}{\left[1-\left(\frac{\partial w}{\partial y}\right)^2\right]^{3/2}}+\frac{3\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial w}{\partial y}\cdot\frac{\partial V_z}{\partial y}}{\left[1-\left(\frac{\partial w}{\partial y}\right)^2\right]^{5/2}}$$

$$\frac{\partial\varkappa_{xv}}{\partial t}=\frac{-\frac{\partial^2 V_z}{\partial x\partial y}}{\left[1+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right]^{3/2}}+\frac{3}{2}\cdot\frac{\frac{\partial^2 w}{\partial x\partial y}\cdot\left(\frac{\partial w}{\partial x}\cdot\frac{\partial V_z}{\partial y}+\frac{\partial w}{\partial y}\cdot\frac{\partial V_z}{\partial x}\right)}{\left[1+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right]^{5/2}},$$

where $(\varkappa_x, \varkappa_y, \varkappa_{xy})$ are the curvatures of the middle plane.

II. Physical equations:

$$N_x=K[(e^0_{xx}+\nu e^0_{yy})-(e^{0(a)}_{xx}+\nu e^{(a)}_{yy})]$$

(16)
$$N_y=K[(e^0_{yy}+\nu e^0_{xx})-(e^{0(a)}_{yy}+\nu e^{0(a)}_{xx})]$$

$$N_{xy}=K(1-\nu)(e^0_{xy}-e^{0(a)}_{xy})$$

$$M_x=D[(\varkappa_x+\nu\vartheta_y)-(\varkappa^{(a)}_x+\nu\varkappa^{(a)}_y)]$$

$$M_y=D[(\varkappa_y+\nu\varkappa_x)-(\varkappa^{(a)}_y+\nu\varkappa^{(a)}_x)]$$

$$M_{xy}=D(1-\nu)(\varkappa_{xy}-\varkappa^{(a)}_{xy}),$$

where (N_x, N_y, N_{xv}) are the tractions and (M_x, M_y, M_{xy}) are the moments in the plate; $K=\frac{Eh}{(1-\nu)^2}$, $D=\frac{Eh^3}{12(1-\nu)^2}$.

(17)
$$\varkappa^{(a)}_x=\frac{12}{h^3}\int_{-h/2}^{h/2}e^{(a)}_{xx}zdz;\ \varkappa^{(a)}_y=\frac{12}{h^3}\int_{-h/2}^{h/2}e^{(a)}_{y}zdz;\ \varkappa^{(a)}_{xy}=\frac{12}{h^3}\int_{-h/2}^{h/2}e^{(a)}_{xy}zdz.$$

The Kirchoff-Love hypothesis is taken into account.

III. Equations of motion

$$m_\rho\frac{\partial V_x}{\partial t}=\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}$$

18)
$$m_\rho\frac{\partial V_y}{\partial t}=\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}$$

$$m_\rho\frac{\partial V_z}{\partial t}=\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+P_z(t)$$

$$-(N_x\varkappa_x+2N_{xv}\varkappa_{xy}+N_y\varkappa_y),$$

where $m_\rho=\bar{\rho}h$, $\bar{\rho}$ is the density of the plate per unit thickness.

The Finite Systems Method [1] is applied for determining the tractions, the moments, the deformations and the curvatures of the middle plane. It combines FDM discretisation with specially introduced iterrative procedure, that simulates the action of the plate as cybernetic system, consisting of a finite number of interconnected subsystems. The method is well applicable for solving nonlinear problems and also initial — boundary value problems, where the form of the constitutive equations can be subsequently changed. In the case before reaching the yield limit the material is taken to be elastic, over it — visco-plastic, and when $\sigma_M\geq\sigma_{N_0}$ the volume damage model is added.

During calculation we check whether the fracture criterion $\xi=\xi_F$ is satisfied and hence we determine the strength dinamic bearing capacity regarding deformability (max $|w| \leq U_{zg}$ — limit admissible deflection). The results thus obtained make possible the determination of the residual thickness change of the plate using the calculated residual volume change (in case of micro damage) or the condition for inelastic incompressibility (in case of visco-plastic deformation). To calculate rectangular plates at

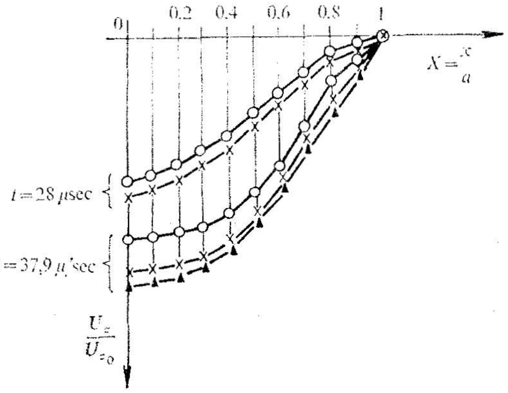

Fig. 3. Deflections of a square plate at two fixed time moments when different mechano-mathematical models are used (○ — elastic model, × — elasto-visco-plastic model, ▲ — elasto-visco-plastic model with inelastic damage; $U_{z_0}=(U_z)$ elastic, $t=28$ [μsec.]).

various boundary conditions a computer program was created. The program was applied for square plate made of aluminium. The thickness of the plate is $h=0.002\,[m]$, the length $2l=0.8\,[m]$, the Young modulus $E=7.10^{10}$ [N/m²], the yield limit $\sigma_s^0=2.10^7$[N/m²], the density per unit thickness of the plate $\bar{\rho}=2700$ [sec² N/m⁴], the limit stress for void nucleation $\sigma_{N_0}=30$ [MPa], the limit stress for void growth $\sigma_{G_0}=50$ [MPa]. In Fig. 3 is shown a comparison between the deflections of the plate at fixed time moments when an elastic model, an elasto-visco-plastic model, an inelastic model with damage are used. It is seen that to account for the microdamage seems reasonable, when conditions for microdamage development are created in the plate. The example shows also the possibility for an effective application of the proposed mechano-mathematical model for the description of the inelastic deformation process of such a typical structure element.

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