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Механика на флуидите

Determining the parameters of a boundary layer on the walls of an axisymmetric cavity in the case of filling with a viscous liquid

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In connection with solving the fluid-gasdynamic problem for filling a casting die of casting machines an investigation is being made of laminar filling of the vertical axisymmetric cavity with heavy incompressible viscous liquid. The method to determine the fluid dynamic pattern with such filling consists of solving the equation for unsteady fluid flow with a free surface in an axisymmetric cavity [1,2] together with the equations of the non-steady boundary layer [3], which develops on the inner surface of the cavity.

§ 1. Formulation of the Problem and Method of Solution

The formulation of the problem and the method of its solution are stated in detail in [4] for the case of filling of a vertical cylindrical pipe, that is why they shall be described here in outline and particular notice shall be given to the new results obtained. First, the movement of an ideal liquid in the cavity is to be determined and then its flow rate Q (t) is to be calculated. Second, the boundary layer equations should be solved by the one-parameter method of the integral relations, which proves to be the most convenient to solve some problems of the inner boundary layer theory with unknown velocity at its boundary. The problem reduces to solving the linear partial differential equation of the first order [5]

$$(1)\qquad \frac{\partial \varphi}{\partial t}+a\bar{U}\frac{\partial \varphi}{\partial \zeta}+\left[b\frac{\partial \bar{U}}{\partial \zeta}+\frac{c}{\bar{U}}\frac{\partial \bar{U}}{\partial t}+\bar{U}\frac{2a}{R}\frac{dR}{d\zeta}\right]\varphi=h,$$

obtained by means of integration of the equation of motion crosswise to the layer from the wall to its outer boundary determining its thickness δ. The function φ determines the displacement thickness $\delta^*=\sqrt{\nu\varphi}$ of the boundary layer, ν is the kinematic viscosity of the fluid. In (1) t is the time, the coordinate ζ is to be measured along the generant of the rotation surface from its lowest point (fig. 1), R is the radius of the circular section of the cavity with the plane of z=const., where z is the vertical coordinate, a, b, c and h are constants depending on the selected family of non-dimentional velocity profiles in the boundary layer. The fluid velocity $\bar{U}$ is determined by

(2) $$\overline{U}=\frac{Q}{F}\widetilde{U},$$

where F is the cavity section, $\widetilde{U}$ is the fluid non-dimentional velocity along the coordinate ζ.

Because the input data to solve equation (1), such ase the geometric parameters of the cavity, velocity profile of the outer flow against the layer, etc. are stated as functions of z, it is convenient to do a change of the variable

(3) $$\zeta=\int_0^z\left[1+\left[\frac{dR}{dc}\right]^2\right]^{1/2}dz$$

with the independent variable z. Then, the equation (1) for φ stated in the variables z and t shall be

(4) $$\frac{\partial\varphi}{\partial t}+a\frac{UQ}{F}\frac{\partial\varphi}{\partial z}+\left\{c\left[\frac{1}{U}\frac{\partial U}{\partial t}+\frac{1}{Q}\frac{dQ}{dt}\right]+\frac{UQ}{F}\left[b\left[\frac{1}{U}\frac{\partial U}{\partial z}\right.\right.\right.$$

$$\left.\left.\left.-\frac{1}{F}\frac{dF}{dz}+\frac{dR}{dz}\frac{d^2R}{dz^2}\left[1+\left[\frac{dR}{dz}\right]^2\right]^{-1}\right]+2a\frac{1}{R}\frac{dR}{dz}\right]\right\}\varphi=h,$$

where $\overline{U}$ is determined by (2), and $\widetilde{U}=U/\cos\alpha$, U is the non-dimentional velocity along the axis z. The angle α (z) between the axes ζ and z is to be determined by $\mathrm{tg}\,\alpha=dR/dz$.

The inital and boundary conditions to solve the equation (4) are

(5) $$\begin{aligned}&\varphi(t,0)=0 && \text{at}\quad t\geq 0,\\ &\varphi(0,z)=0 && \text{at}\quad 0\leq z\leq z_p,\\ &\varphi(t,Z(t))=0 && \text{at}\quad t\geq 0,\end{aligned}$$

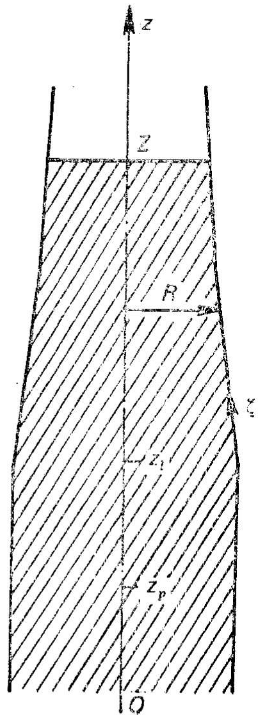

Fig. 1

where Z (t) is the coordinate of the free surface of fluid in the cavity, and $z_p=z(0)$

The fluid velocity in the section is determined by the non-dimentional function [1, 2, 4]:

(6) $$C(r,z,t)=\begin{cases}U(z,t) & \text{at}\quad 0\leq r\leq R-\delta\\ u(r,z,t) & \text{at}\quad R-\delta\leq r\leq R,\end{cases}$$

where r is the distance from the axis of the cavity, $u(r,z,t)$ is the non-dimentional longitudinal velocity in the boundary layer. Integrating the function $C(r,z,t)$ and using its normalized condition

(7) $$1=\frac{2}{R^2}\int_0^R C(r\,z,t)r\,dr,$$

the following formula is obtained for the velocity $U(z,t)$:

(8) $$U=\left[1-2\,\frac{\delta^*(z,t)}{R}\right]^{-1}.$$

The solution of the partial linear inhomogeneous equation of the first order (4) is usually reduced to the solution of the characteristic system of ordinary differential equations, which is of the following type as a rule:

$$\frac{dz}{dt}=g_1(\varphi, z, t)$$

(9)

$$\frac{d\varphi}{dt}=g_2(\varphi, z, t).$$

A substitution of $\frac{\partial U}{\partial t}$ and $\frac{\partial U}{\partial z}$ by (8) and a conversion of equation (4) yield the functions

$$g_1=U\frac{Q}{F}\cdot\frac{aR+bU\sqrt{\nu\varphi}}{R+cU\sqrt{\nu\varphi}},$$

(10)

$$g_2=\left\{h-\left[U\frac{Q}{F}\left[\frac{2a}{R}\frac{dR}{dz}-\frac{b}{F}\frac{dF}{dc}-2bU\frac{\sqrt{\nu\varphi}}{R^2}\frac{dR}{dz}\right.\right.\right.$$

$$\left.\left.\left.+b\frac{dR}{dz}\frac{d^2R}{dz^2}\left[1+\left[\frac{dR}{dz}\right]^2\right]^{-1}\right]+\frac{c}{Q}\frac{dQ}{dt}\right]\varphi\right\}\left[1+\frac{cU\sqrt{\nu\varphi}}{R}\right]^{-1}.$$

The domain of solution of the system (9), (10) in the plane (z, t) divides into three fields by means of two dividing lines L_{12} and L_{23}. At each moment the values of φ from the first and second field coincide on L_{12}, and those from the second and third field — on L_{23}. In case that $z=z_{12}(t)$ and $z=z_{23}(t)$ are the respective equations of the curves L_{12} and L_{23} then for a fixed moment of time the variable z changes in the first field within the interval of $[0, z_{12}]$, in the second field — within the interval of $[z_{12}, z_{23}]$, and in the third field — within the interval of $[z_{23}, Z]$.

In the first field the system (9), (10) is to be solved with the following initial conditions:

(11) $$z=0, \quad \varphi=0 \quad \text{at} \quad t=\tau_0\geq 0,$$

the obtained solution of $\varphi=\varphi^{I}(t, \tau_0)$, $z=z^{I}(t, \tau_0)$ depends on the parameter of τ_0.

The solution of $\varphi=\varphi^{II}(t, \zeta_0)$, $z=z^{II}(t, \zeta_0)$ with initial conditions

(12) $$z=\zeta_0, \quad \varphi=0 \quad \text{at} \quad t=0$$

is seeked in the second field, where $\zeta_0>0$. The functions $\varphi=\varphi^{III}(t,T_0)$ and $z=z^{III}(t,T_0)$ satisfy the system and the initial conditions in the third field

(13) $$z=Z(t), \quad \varphi=0 \quad \text{at} \quad t=T_0\geq 0.$$

Solving the system of equations of (9), (10) with different positive values of the parameters of τ_0, ζ_0 and T_0 families of curves in the space (t, z, φ) are obtained which, over each of the fields in the region, depend on the respective parameter and form a surface representing the solution of the partial differential equation (4). The equation of the corresponding surface $\varphi=\varphi^{I}(t, \tau_0(z, t))$ is obtained by eliminating τ_0 from the solution in the first field. The equations of the surfaces over the other two fields are obtained by analogy.

It is necessary to determine also the boundaries of the fields, i. e. $z=z_{12}(t)$ and $z=z_{23}(t)$. Those dependences are determined by the conditions of coincidence of the solutions of the neighbouring fields over their dividing line. In order to find the line L_{12} the following equations are used

(14)

$$\varphi^{I}(t, \tau_0)=\varphi^{II}(t,\zeta_0),$$

$$z^{I}(t, \tau_0)=z^{II}(t, \zeta_0),$$

from which, with and arbitrary t, the functions $\tau_0=\tau_0\,(t)$ and $\zeta_0=\zeta_0\,(t)$ are obtained. Substituting $\tau_0=\tau_0\,(t)$ in the second equation (14) we determine $z_{12}=z^{I}(t,\ \tau_0\,(t))$. The equation of L_{23} is founded in the same way.

It should be pointed out that with $\tau_0=0$ and $\zeta_0=0$ conditions (11) and (12) coincide, therefore

$$\varphi^{I}(t,\ 0)=\varphi^{II}(t,\ 0), \qquad (15)$$
$$z^{I}(t,\ 0)=z^{II}(t,\ 0),$$

i. e. the solution of $\varphi=\varphi_I(t,\ 0)$ and $z=z^{I}(t,\ 0)$, being the integral curve of the surface over the first field, is at the same time a curve of the surface over the second field. In other words, that curve intersects the two surfaces and its projection on the plane $(z,\ t)$ determines the dividing line L_{12}. In this case we find out that $z_{12}=z^{I}(t,\ 0)$.

By analogy, with $T_0=0$ and $\zeta_0=Z(0)=z_p$, the conditions (12) and (13) coincide meaning also a coincidence of the respective solutions of the system and therefore the line L_{23} is described by the equation $z_{23}=z^{II}(t,\ z_p)$.

Searching a solution of the system numerically the step by time is selected to be equal to the parameter τ_0, and by z — equal to ζ_0. In this way the values of z and φ are calculated in equal-standing points by the time, which makes easier to search some derivatives during the solution.

After finding the solution $\varphi\ (z,\ t)$, following the scheme of solution of the non steady boundary layer equation, stated in detail in [4], the formparameter f is determined

$$f=\left(\frac{\partial \bar{U}}{\partial z}+\frac{1}{\bar{U}}\frac{\partial \bar{U}}{\partial t}\right)\varphi= \qquad (16)$$
$$\left\{\frac{QU}{F}\left[\frac{1}{U}\frac{\partial U}{\partial r}-\frac{1}{F}\frac{dF}{dz}+\frac{dR}{dz}\frac{d^2R}{d\,z^2}\left[1+\left[\frac{dR}{dz}\right]^2\right]^{-1}\right]+\frac{1}{Q}\frac{dQ}{dt}+\frac{1}{U}\frac{\partial U}{dt}\right\}\varphi.$$

According to the one-parameter method of the integral relations it provides the connection with the selected family of velocity profiles suggested by Watson [6] and depending on a parameter α that belongs to the interval $\left(-\frac{1}{2},\infty\right)$. Thus, the func tions C, C_1 and τ_0/ρ, which participate in the equation for non-steady flow with free surface in axisymmetric cavity can be determined as functions of α [4]. Differing from [4], where C_1 is taken to be 1 for filling the cylindrical pipe, in the case considered here

$$C_1=\frac{2}{R^2}\int_0^R C^2\ (r,\ z,\ t)\ r\,d\,r=U^2\left[1-\frac{2\sqrt{\nu\varphi}}{R}\ \frac{D\ (\alpha)}{A\ (\alpha)}\right], \qquad (17)$$

where

$$A(\alpha)=\frac{\Gamma(\alpha+1)}{2\Gamma\left(\alpha+\frac{3}{2}\right)},\quad D(\alpha)=2A(\alpha)\left[1-\frac{1}{2}S\left(\alpha-\frac{1}{2}\right)\right], \qquad (18)$$

Γ is the gamma function, $S\left(\alpha-\frac{1}{2}\right)={}_1F_1\left(2\alpha+\frac{3}{2},\ 1,\ 2\alpha+2,\ -1\right)$ is the hypergeometric function.

By means of the calculated functions of C_1 and τ_0/ρ determination is made of new values of the coefficients in the equation for the position of the fluid free surface in the axisymmetric cavity [1, 2, 4]. The new distribution of the quantity $Q\,(t)$ as well as the liquid level $Z\,(t)$, necessary to calculate the corresponding approximation of the solution of the boundary layer problem, is obtained in this way.

The suggested method for calculation the filling of the axisymmetric cavity, together with taking into account the viscosity influence through the parameters of the non-steady boundary layer, could be applied also in the case when a rotational body is located in the cavity and the flow past it [2]. In this case to the system of equations an addition should be made of an equation to calculate the boundary layer on the surface of that body, which shall be analogous to equation (1). The system of equations (9), (10) alters. Both, in it and in the new system for the boundary layer along the rotation body the profile of the flow velocity, with a fixed z, is determined by

$$C\ (r,\ z,\ t)=\begin{cases} u_2(r,\ z,\ t) & \text{при } R_2\leq r\leq R_2+\delta_2, \\ U(z,\ t) & \text{при } R_2+\delta_2\leq r\leq R_1-\delta_1, \\ u_1(r,\ z,\ t) & \text{при } R_1-\delta^1\leq r\leq R_1, \end{cases} \tag{19}$$

where R_1 and R_2 are the radii of the cavity and of the body located along its axis; δ_1 and δ_2 are the respective thicknesses of the boundary layers. Obtained is the following for the velocity $U(z,\ t)$:

$$U=\left[1-2\frac{R_1\delta_1^*+R_2\delta_2^*}{R_1^2-R_2^2}\right]^{-1}, \tag{20}$$

where δ_1^* and δ_2^* are the respective thicknesses of displacement. The remaining quantities participating in the solution of the problem are to be changed further by analogy.

The scheme to solve the complete problem for filling the cavity with rotational symmetry of heavy incompressible viscous fluid is performed numerically by a FORTRAN IV program module.

§ 2. Numerical Results

In their basic part the axisymmetric cavities consist of straight cylindrical pipes, confusor and diffusor cavities and such ones where axisymmetric cores are located. In [4] there are some numerical results for the case of filling of a straight cylindrical pipe. It is proved theoretically and experimentally [7] that when the cavity is a diffusor, even at small angles of widening, the boundary layer separates at a distance near the entrance. That is why provision is made here for numerical experiments with the following two cases of alteration of the cavity section: confusor and cylindrical pipe with a cone core.

The viscosity of the smelts at the liquidus temperature (when they could be considered as newtonian fluids) is near that of water at room temperature. For example [8] for aluminium $\nu=0.52.10^{-6}$ m^2/s with $T=660°C$; for iron $\nu=0{,}86\,.10^{-6}$ m^2/s with $T=1536^0C$; for nickel $\nu=0{,}69\,.10^{-6}$ m^2/s with $T=1452°C$; for cobalt $\nu=0{,}61\,.10^{-6}\,m^2/s$ with $T=1495°C$; for cast irons ν changes from $0{,}6:10^{-6}$ up to $1{,}3.10^{-6}$ with $T=1500°C$ depending on their composition, and so on. That is why the numerical experiments are performed with fluid with the parameters of water at indoor temperature — kinematic viscosity of $\nu=1{,}01\,.10^{-6}$ m^2/s and density of $\rho=1000$ kg/m^3.

Figure 1 shows the boundary layer structure on the wall of a pipe with a constant radius of $R=0{,}025\,m$ at $0\leq z<z_e$ (where $z_e=0{,}5$ m) and $R=0{,}025-(z=z_e)$ tgε at $z\geq z_e$ in different moments of time. At $z=z_e$ the pipe passes to a confusor with a small angle of narrowing $\varepsilon=1°$. The cylindrical reservoir radius, where the pipe is submerged at depth $z_p=0{,}1\,m$, is $R_0=0{,}25\,m$. A research is done for filling under the action of a non-steady difference of gas pressures $\Delta p=98(\exp\frac{t}{2}-1)kPa$. As shown in Figure 2, until the free surface of liquid does not enter in the confusor part of the pipe (the curves with $t=0{,}195$ s and $t=0{,}22\,s$), the form of the boundary layer does not differ from that for the case of filling of a cylindric pipe [4[. When the fluid sur-

face enters the confusor part ($Z>z_e$). the boundary layer structure in that area becomes deformed considerably. Figure 3 shows the boundary layer change ($\delta^*/\Delta R$) along the wall of a cone core with an angle at the top $\varepsilon=1°$. The core is located along the axis of a cylindrical pipe with a radius of $R_1=0{,}025\,m$ and ΔR_1-R_2, where

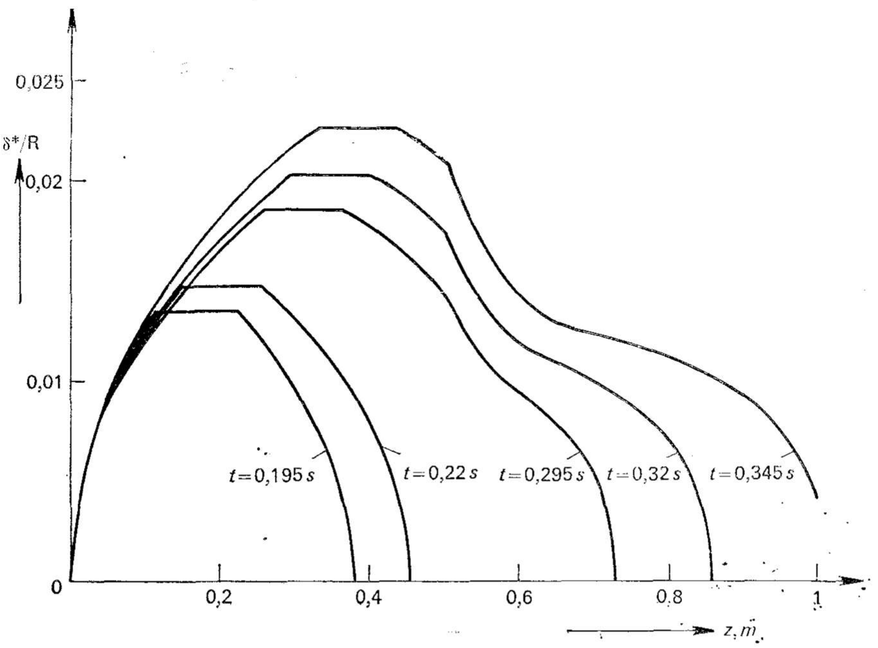

Fig. 2

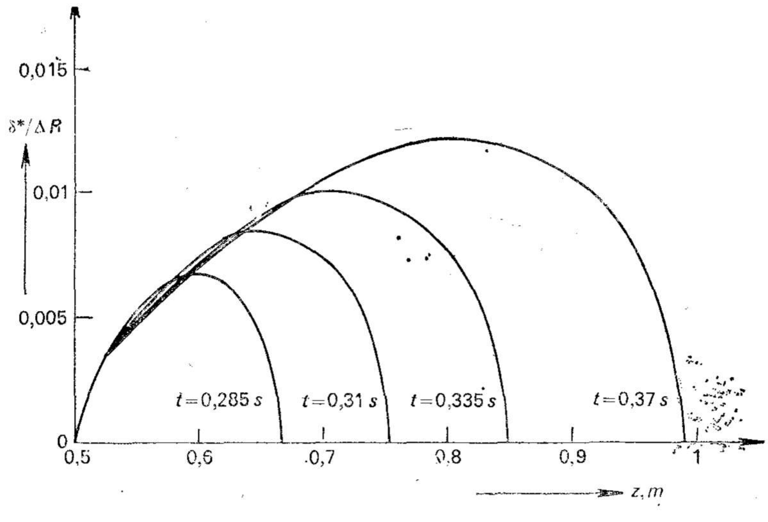

Fig. 3

$R_2(z)$ is the core radius. The coordinate of the core top is $z_i=0{,}5\,m$. At the initial time the pipe is submerged at a depth $z_p=0{,}1\,m$ in the cylindrical reservoir with a radius of $R_0=0{,}25\,m$. Calculation is performed under the action on the same non-steady difference of pressures. As regards their structure the curves in Figure 3 are close to those shown in [4]. The difference consists in the lack of a medium area (the so called second field) since at the initial moment $z_i > z_p$.

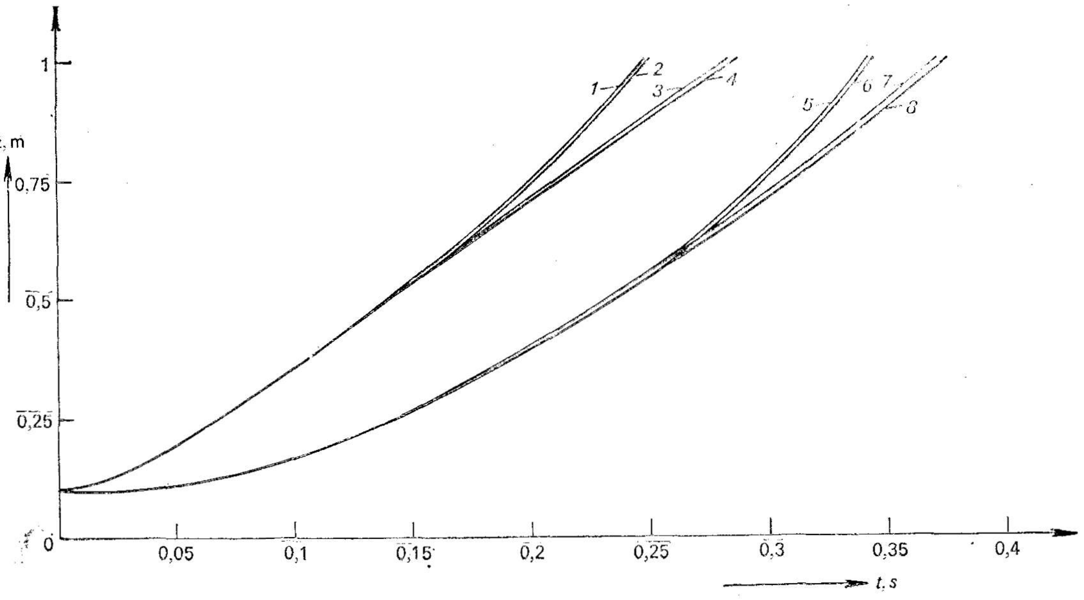

Fig. 4

Alteration of the fluid free surface coordinates for two here considered cases of filling is represented by the curves in Figure 4. Curves 5 and 6 give the fluid level with filling with ideal fluid and from the first iteration of viscous filling of a pipe and confusor cavity. Curves 7 and 8 represent the respective dependences at a cylindric pipe filling with a cone core. Curves 1—4 are plotted for the same two cases of filling but with constant gas pressure difference $\Delta p = 9{,}8\,kPa$. The differences of the curves with ideal filling from those with including the boundary layer influence for both cases at the end of the process of filling do not exceed 2-2,5 %.

§ 3. Conclusion

From the carried out theoretical researches for filling of a straight cylindrical pipe and confusor cavity with a fluid having viscosity close and higher than that of smelts at the liquidus temperature, it can be realized that the fluid viscosity does not cause considerable influence on the filling process. That conclusion is also confirmed by the experimental results [9] for filling a cavity consisting of a cylindrical part and a cone-shaped diffusor. Therefore it could be claimed with confidence that in the cases not only of confusor but of diffusor cavities too, what in fact the casting dies are in their basic form, the filling is performed according to the ideal filling model. Consequently, the liquid friction on the cavity walls does not cause considerable influence, and in most cases it could be eliminated when solving the complete fluid-gasdynamic problem.

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