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Dual Vector-Parameter in Manipulator Kinematics

The angular and linear displacements at each of the joints of a robot manipulator to attain a specified position and orientation of its hand are obtained by solving the closure equations of the manipulator. It is well known that for this purpose the Denavit-Hartenberg formalism as Rodrigues formula for describing the transformation matrices are used.

In the present paper the transformations mentioned above are expressed by the so-called "vector-parameter" used by the authoress in her previous works. The introducing of dual numbers or dual vector-parameter, the 6×6 matrices that arise collapse into 3×3 orthogonal matrices through dual numbers yield a dual form of the rotation matrix. A dual form of the Jacobian matrix is also discussed.

1. The Manipulator skeleton

A manipulator is a chain of rigid bodies joined in a series by either revolute or prismatic joints. A revolute joint, or hinge, allows only pure rotation about its axis, whereas the prismatic joint, or slider, allows only pure translation along its line of action. Revolute, prismatic joints may be considered as special cases of the two-degrees-of-freedom cylindric joint, where in the first case the cylindric joint is restrained from sliding and in the second the rotational freedom is restricted. This fact is the basis for the choice of particular coordinate frames in the links of the manipulators that enable the definition of a standard form of transformation matrix relating consecutive coordinate frames.

The system of linked rigid bodies B_i, $i=0, 1, 2, \ldots, n$ (B_0 is immovable one) that make up the manipulator can be reduced to a skeleton consisting of the lines forming the axes of the cylindric joints and the lines defined by the common normals to consecutive pairs of these axes forming the rigid links of the manipulator. We define the coordinate frames R_i and R_{i-1} for bodies B_i and B_{i-1} by choosing the z-axis of each reference frame along the axis of the joint and the x-axis of each frame along the common normals of succeeding axes (Fig. 1). The displacement of the body B_i relative to B_{i-1} is specified by the rotational angle θ_i about the line L_{i-1} coinciding with its z-axis, i. e. with axis z_{i-1} from the frame R_{i-1}, and the translation by the amount d_i along this line. The position of L_i in the reference frame R_i of B_i relative to the joint axis L_{i-1} is given by the angle α_i about the common normal between these axes (the x-axes of R_i, i. e. x_i) and the distance a_i along it.

The parameters d_i and a_i are dimensions of the link making up B_{i-1}, whereas the parameters θ_i and α_i are the joint variables that prescribe the position of B_i relative to B_{i-1}. The coordinate transformation relating the position of B_i to B_{i-1} is made up of a pair of transformations, each of which consists of a rotation about and translation along a given line. Such a transformation is called a screw displacement.

2. Vector-Parameter and Transformation Matrices

The relative position of two rigid bodies is represented by the point coordinate transformation between a reference frame R_d attached to the displaced body and a reference frame R_o attached to the base body, given by the matrix expression

$$r = O(c)\, r_d + p. \tag{1}$$

The vectors r and r_d are the coordinates of a generic point in the displaced body measured relative to the reference frames R_0 and R_d, respectively. The displacement vector p defines the position of R_d relative to R_o. The rotation matrix $O(c) \in SO$ (3) defines the orientation of R_d with respect to R_o. Here the special orthogonal group SO (3) is parametrized by the vector c — called vector-parameter (see [3, 5, 7]). This vector is directed along the rotation axis and its module is equal to the tangent of the half rotation angle α: $|c| = \operatorname{tg}\frac{\alpha}{2}$. These vectors make a Lie group with composition law

$$c = \langle c_1, c_2 \rangle = \frac{c_1 + c_2 + c_1 \times c_2}{1 - c_1 c_2}. \tag{2}$$

In our case:

$$c_1 = c_\theta = \left(0,\ 0,\ \operatorname{tg}\frac{\theta}{2}\right); \quad c_2 = c_\alpha = \left(\operatorname{tg}\frac{\alpha}{2},\ 0,\ 0\right). \tag{3}$$

Hence,

$$c = \langle c_\theta, c_\alpha \rangle = \left(\operatorname{tg}\frac{\alpha}{2},\ \operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\theta}{2},\ \operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\theta}{2}\right). \tag{4}$$

The transformation $O(c)$ has the form

$$O(c) = \frac{(1-c^2)\, I + 2c \cdot c + 2c^x}{1+c^2} \tag{5}$$

with inverse one

$$c^x = \frac{O - O^T}{1 + trO}, \tag{6}$$

where: $c \cdot c$ means diadic, c^x is the screw 3×3 matrix, I is 3×3 identity matrix and trO — the trace of matrix O. Using the notations $\bar{a}(a, 0, 0)$ and $\bar{d}(0, 0, d)$ the Euclidean group of motion in R^3 is defined as a set of pairs with the following composition law

$$(O(c_\theta), \bar{d})(O(c_\alpha), \bar{a}) = (O(\langle c_\theta, c_\alpha \rangle),\ O(c_\theta)\bar{a} + \bar{d}) = (O(c),\ p), \tag{7}$$

in which the action of $O(c_\theta)$ over the vector $\bar{a}$ is given by

(8) $$O(c_\theta)\bar{a}=\frac{(1-c_\theta^2)\,\bar{a}+2c_\theta\,(c_\theta\cdot\bar{a})+2\,(c_\theta\times\bar{a})}{1+c_\theta^2}.$$

Hence, in terms of 4×4 homogenious transformation equation, for Eq. (1) we have

(9) $$\begin{bmatrix} r_1\\ r_2\\ r_3\\ 1\end{bmatrix}=\left[\begin{array}{ccc|c} & O(c_\theta) & & \begin{matrix}0\\0\\d\end{matrix}\\ \hline 0 & 0 & 0 & 1\end{array}\right]\left[\begin{array}{ccc|c} & O(c_\alpha) & & \begin{matrix}a\\0\\0\end{matrix}\\ \hline 0 & 0 & 0 & 1\end{array}\right]\begin{bmatrix} r_{d_1}\\ r_{d_2}\\ r_{d_3}\\ 1\end{bmatrix}$$

Or:

(10) $$\begin{bmatrix} r_1\\ r_2\\ r_3\\ 1\end{bmatrix}=\left[\begin{array}{c|c} O(\langle c_\theta,\ c_\alpha\rangle) & O(c_\theta)\bar{a}+\bar{d}\\ \hline \bar{0} & 1\end{array}\right]\begin{bmatrix} r_{d_1}\\ r_{d_2}\\ r_{d_3}\\ 1\end{bmatrix},$$

where $\bar{0}$ is a zero vector $\bar{0}=(0,\ 0,\ 0)$. The matrix in the right hand side of Eq. (10) is exactly the matrix

$$\left[\begin{array}{c|c} O(c) & p\\ \hline \bar{0} & 1\end{array}\right],$$

that is equivalent to the well-known Denavit-Hartenberg matrix.

3. Line Geometry

To obtain the analog of 4×4 matrix in Eq. (10) for the transformations of line coordinates, we first derive the general form of the line coordinate transformation matrix. A line L in space is represented by the parametric equation

(11) $$L(t)=X+t(Y-X),$$

where X and Y are any two points on the line. If we define $E=Y-X$, then we have six dimensional Plücker coordinate vector (see [1]) of line L, where

(12) $$L=(E,\ X\times E).$$

The rigid transformation equation relating the line coordinates l of L measured in the reference frame R_d to its coordinates L measured in R_o is obtained from transformation equations relating point coordinate vectors x, y and X, Y of points on L measured in R_d and R_o, respectively. From Eq. (1) we get

(13) $$Y=O(c)y+p \text{ and } X=O(c)x+p.$$

Hence,

(14) $$E=Y-X=O(c)\,(x-y)-O(c)\,e.$$

We now compute

(15) $$X\times E=(O(c)x+p)\times O(c)e,$$

which yields the relation

(16) $$X\times E=O(c)(x\times e)+p^x O(c)e,$$

where p^x is a skew symmetric matrix obtained from the components of p

(17) $$p^x=\begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}.$$

Finally, Eqs (14) and (16) are assembled into 6×6 matrix equation relating the coordinates l and L

(18) $$\begin{bmatrix} E \\ \cdots \\ X\times E \end{bmatrix}=\begin{bmatrix} O(c) & \vdots & \varnothing \\ \cdots & \cdots & \cdots \\ p^xO(c) & \vdots & O(c) \end{bmatrix}\begin{bmatrix} e \\ \cdots \\ x\times e \end{bmatrix},$$

where $\varnothing$ denotes 3×3 zero matrix. The transformation equation (18) relates ine coordinates measured in the reference frames R_d and R_o.

4. Dual Vector-Parameter

Dual Orthogonal Matrices. The 6×6 transformation matrix in Eq. (18) has a special form that leads to the introduction of a special set of complex numbers. It is possible to show that the set of matrices of the form $\begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$ have the same algebraic properties as the set of the complex number $a+\mu b$, where a and b are real numbers and the symbol μ is manipulated by the rule $\mu^2=0$ (Clifford's operator). Numbers of the form $a+\mu b$ are called dual numbers, and are denoted by a caret ($\widehat{\ }$), $\widehat{a}=a+\mu b$. Vectors and matrices with dual number elements are called dual vectors and dual matrices, respectively.

Equation (13) can now be written in the compact form

(19) $$\widehat{E}=\widehat{O}(c)\,\widehat{e},$$

where

(20) $$\widehat{E}=E+\mu X\times E,\ \widehat{e}=e+\mu x\times e$$

and

(21) $$\widehat{O}(c)=O(c)+\mu p^x O(c).$$

We show that $\widehat{O}(c)$ is orthogonal by direct computation of $\widehat{O}(c)\widehat{O}(c)^T$

(22) $$(O(c)+\mu p^xO(c))\,(O^T(c)+\mu O^T(c)p^{xT})=I+\mu(p^x+p^{xT})+\mu^2p^xp^{xT}=I.$$

$\widehat{O}(c)$ is called a dual orthogonal matrix; it is 3×3 matrix with dual number elements representing a general line coordinate transformation.

We now introduce the dual angle between two lines as the dual number $\widehat{\varphi}=\varphi+\mu\varphi_0$, where φ_0 is the distance between the two lines along their common normal and φ is the angle between them about the common normal. According to this definition the dual angles $\widehat{\theta}$ and $\widehat{\alpha}$ are expressed in the following way:

(23) $$\widehat{\theta}=\theta+\mu d \text{ and } \widehat{\alpha}=\alpha+\mu a.$$

As in fact the matrix $O(c_\theta)$ in terms of the angle θ looks like:

$$O(c_\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{24}$$

then its dual matrix is

$$\hat{O}(c_\theta)=\begin{bmatrix}\cos\hat{\theta} & -\sin\hat{\theta} & 0\\ \sin\hat{\theta} & \cos\hat{\theta} & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{25}$$

It may be immediately seen that the corresponding dual vector-parameter $\hat{c}_\theta$ is

$$\hat{c}_\theta=\left(0,\ 0,\ \operatorname{tg}\frac{\hat{\theta}}{2}\right). \tag{26}$$

Analiogically:

$$\hat{c}_\alpha=\left(\operatorname{tg}\frac{\hat{\alpha}}{2},\ 0,\ 0\right). \tag{27}$$

We will not mention here how the operations between dual vectors (addition, subtraction, dot and screw multiplication, sine, cosine of dual, etc.) seem because they come from the transition principle of Kotel'nikov's study [2].

The resultant dual vector $\hat{c}$, because of the analogy to (2) group law, is expressed by

$$\hat{c}=(\hat{c}_\theta,\ \hat{c}_\alpha)=\frac{\hat{c}_\theta+\hat{c}_\alpha+\hat{c}_\theta\times\hat{c}_\alpha}{1-\hat{c}_\theta\,\hat{c}_\alpha}. \tag{28}$$

Hence,

$$\hat{c}=\left(\operatorname{tg}\frac{\hat{\alpha}}{2},\ \operatorname{tg}\frac{\hat{\theta}}{2}\operatorname{tg}\frac{\hat{\alpha}}{2},\ \operatorname{tg}\frac{\hat{\theta}}{2}\operatorname{tg}\frac{\hat{\alpha}}{2}\right). \tag{29}$$

Equation (28) gives the composition law of the dual vector-parameter group. Every vector $\hat{c}$ may be presented also as

$$\hat{c}=\hat{E}\operatorname{tg}\frac{\hat{\varphi}}{2}, \tag{30}$$

where $\hat{\varphi}$ is a dual angle and $\hat{E}$ is a unit screw along the screw axis.
Thus we obtain the dual orthogonal transformation relating the coordinate frames in two consecutive links B_{i-1} and B_i of the manipulator chain, given by

$$\begin{bmatrix}\hat{E}_1\\ \hat{E}_2\\ \hat{E}_3\end{bmatrix}=O(\hat{c})\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix}, \tag{31}$$

where: $\hat{c}=(\hat{c}_{\theta_i},\ \hat{c}_{\alpha_i})$, $\hat{E}=(\hat{E}_1,\ \hat{E}_2,\ \hat{E}_3)$ and $\hat{e}=(\hat{e}_1,\ \hat{e}_2,\ \hat{e}_3)$ are the coordinates of a generic line measured in the reference frames of B_{i-1} and B_i resp.

5. The Dual Closure Equations

As we already mentioned $B_0, B_1, B_2, \ldots, B_n$ are the links of the considered manipulator system. B_0 is fixed one, B_n may be treated as a gripper (end-effector). With every link a movable frame is built. R_0 is the base one (fixed) and with a characteristic point C of the gripper (usually it coincides with its gravity centre) a frame R_{n+1} is connected.

Let c_i be the vector parameter giving the orientation of R_i referred to R_{i-1} and $\hat{c}_i$ — its dual one. The corresponding orthogonal and dual orthogonal matrices are $O(c_i)$ and $\hat{O}(c_i)= O(\hat{c}_i)$. Remark: $c_i=\langle c_{\theta_i}, c_{a_i}\rangle$ and $\hat{c}_i=\langle \hat{c}_{\theta_i}, \hat{c}_{a_i}\rangle$.

We shall denote further on $O(c_i)$ as $O_{i-1,i}$. Given the structure of the manipulator, we can determine the sequence of dual matrices $\hat{O}_{01}, \hat{O}_{12}, \hat{O}_{23} \ldots \hat{O}_{n-1,n}$. Thus we have the pair of transformation equations relating line coordinates measured in R_n to those measured in R_o:

(32A) $$\hat{E}=\hat{O}_{0n}\hat{e};$$

(32B) $$(\hat{E}=\hat{O}(c)\,\hat{e})$$

and

(33A) $$\hat{E}=\hat{O}_{01}\hat{O}_{12}\ldots\hat{O}_{n-1,n}\,\hat{e}$$

(33B) $$(\hat{E}=\hat{O}(c_1)\,\hat{O}(c_2)\ldots\hat{O}(c_n)\,\hat{e}).$$

From these relations we obtain the matrix identity

(34) $$\hat{O}_{on}=\hat{O}_{01}\hat{O}_{12}\ldots\hat{O}_{n-1,n}.$$

The corresponding relation in dual vector-parameter is

(35) $$\hat{c}=\langle\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n\rangle.$$

Equating the real and dual parts of these matrices (34) or vectors (35) yields a set of nonlinear relations defining the joint parameters θ_i and d_i required to position the hand as desired. Equation (34) or (35) is the dual closure equation of the manipulator

6. The Dual Angular Velocity Vector

The matrix equation (34) or the vector one (31) both relate the position and orientation of the end effector of a manipulator to its joint parameters. By considering the dual angular velocity vector associated with the dual matrix $\hat{O}(c)$ from Eq.(31), we obtain a transformation relating the velocity of the motion of these joints to the velocity of the end effector. This transformation is called the Jacobian of the manipulator.

In general a skew-symmetric matrix Ω is associated with any orthogonal matrix $O(t)$. It is defined by the relation

(36) $$dO/dt\,O^T=\Omega.$$

In R^3 the matrix Ω has three independent components that may be assembled into a vector ω called the angular velocity vector associated with $O(t)$, i. e.

(37) $$\Omega=\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.$$

As it is obtained in [5], the angular velocity vector associated with the matrix $O(c)$ (Eq. (5)) is

(38) $$\omega=\frac{2}{1+c^2}[c^x+I]\dot{c}.$$

ts dual form is

(39) $$\hat{\omega}=\frac{2}{1+\hat{c}^2}[\hat{c}^x+I]\dot{\hat{c}}.$$

Having in mind (26) and following the above prescription, we obtain the dual matrix $\hat{\Omega}$ associated with the dual orthogonal matrix $\hat{O}(c)$.

(40) $$\hat{\Omega}=\begin{bmatrix} 0 & -\hat{\omega}_3 & 0 \\ \hat{\omega}_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\hat{\omega}_3$ is the third component ot the dual vector $\hat{\omega}$ and

(41) $$\hat{\omega}_3=\frac{2}{1+\hat{c}_{\theta_3}^2}\dot{\hat{c}}_{\theta_3}$$

$$\hat{c}_\theta=(\hat{c}_{\theta_1},\ \hat{c}_{\theta_2},\ \hat{c}_{\theta_3}).$$

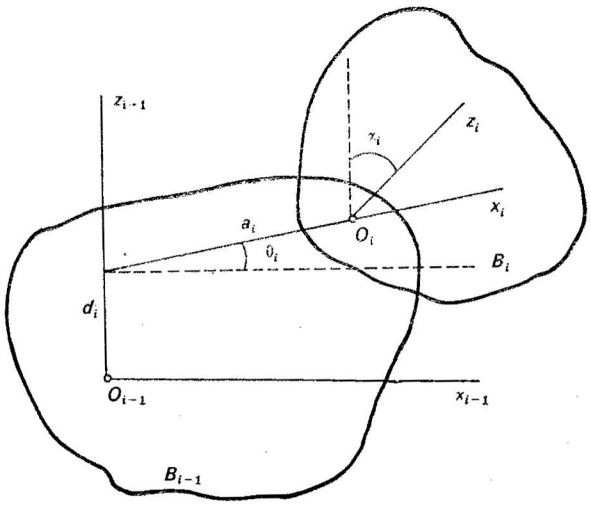

Fig. The standard reference frames associated with a consecutive pair of cylindric joints

Since $d\hat{\theta}/dt = \dot{\theta} + \mu\dot{d}$ (the dot denotes differentiation with respect to t), substituting (26) in (41) we obtain that $\hat{\omega}_3 = d\hat{\theta}/dt$ and the dual angular velocity vector is

$$\hat{\omega} = d\hat{\theta}/dt \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = d\hat{\theta}/dt\hat{k}. \tag{42}$$

$\hat{k}$ is the dual vector representing the z-axis (its dual part is 0). The standard reference frames used are such that the dual angular velocity of link B_i of the manipulator relative to another B_{i-1} consists of an angular rotation $\hat{\theta}_i$ about and translation d_i along the z-axis $\hat{k}_{i-1}$ of R_i. The dual angular velocity vector $\hat{\omega}_{i-1,i}$ of B_i relating to B_{i-1} is transformed to the base reference frame R_o by the equation

$$\hat{\omega}^o_{i-1,i} = \hat{O}_{01}\hat{O}_{12}\ldots\hat{O}_{i-1,i}\,\hat{\omega}_{i-1,i}. \tag{43}$$

Using the notation $\hat{K}_{i-1}$ as the image of $\hat{k}_{i-1}$ in the base frame R_o, we have:

$$\hat{\omega}^o_{i-1,i} = d\hat{\theta}_i/dt\, O_{01}\, O_{12}\ldots O_{i-2,i-1}\,\hat{k}_{i-1} = d\hat{\theta}_i/dt\hat{K}_{i-1}; \tag{44}$$

$$\hat{\omega}^o_{on} = d\hat{\theta}_1/dt\hat{K}_1 + d\hat{\theta}_2/dt\hat{K}_2 + \cdots + d\hat{\theta}_n/dt\,\hat{K}_n \tag{45}$$

which in turn may be written as the matrix equation

$$\hat{\omega}^o_{on} = [\hat{K}_1\,\hat{K}_2\ldots\hat{K}_n]\,[d\hat{\theta}_1/dt\; d\hat{\theta}_2/dt\ldots d\hat{\theta}_n/dt^T]. \tag{46}$$

For a typical manipulator, $n=6$ and $[\hat{K}_1\hat{K}_2\ldots\hat{K}_6]$ is a 3×6 matrix with dual number elements called the dual Jacobian matrix of the manipulator [4]. It is easy to see that the columns of this matrix are simply the dual vectors of the lines corresponding to the manipulator's joint axes measured in the base reference frame R_o.

Equating the real and dual parts of Eq. (46) and introducing the notation ω^* and K_i^* for the dual parts of $\hat{\omega}$ and $\hat{K}_i$, we form the matrix equation

$$\begin{bmatrix} \omega \\ \omega^* \end{bmatrix} = \begin{bmatrix} [K_1\cdots K_6] & [0\cdots0] \\ [K_1^*\cdots K_6^*] & [K_1\cdots K_6] \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_6 \\ d_1 \\ \vdots \\ d_6 \end{bmatrix}. \tag{47}$$

Twelve unknowns, the real and dual parts of the six dual angular velocities, appear in Eq. (47) because single-degree-of-freedom revolute and prismatic joints are being modelled in this general case by two-degrees-of-freedom cylindric joints. Actually, only six of these unknowns are non-zero valued for a six-degrees-of-freedom manipulator. The extra parameters introduced in this way are constants of the manipulator, so their time derivatives are zero. Thus, the matrix in Eq. (47) becomes 6×6 as we eliminate the columns associated with nonexistent joint velocities. This matrix is called the Jacobian of the manipulator. It relates the velocities of the manipulator's joints to the velocity of its end-effector for any given configuration.

8. Conclusion

Using a special dual number algebra, we obtain the transformation equations relating line coordinates measured in the two reference frames in a form identical to a sequence of simple rotations. These transformations are used to define the closure equations of a manipulator. The special structure of a manipulator as a series of linked bodies allows the specialization of the general line coordinate transformation matrix in a form called dual orthogonal matrix, as defining of the notion dual vector-parameter. The closure equations are expressed through dual orthogonal matrices as well using dual vector-parameters. On the base of the defined dual orthogonal matrix, the angular velocity vector of the end effector takes a particularly simple form leading to a simple geometric interpretation of the Jacobian of the manipulator. This work presents the interesting interplay of special geometrical considerations and special algebraic structures in an important area of application of linear algebra.

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