BULGARIAN ACADEMY OF SCIENCES
JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
Sofia, 1991, Year XXII, No 1

P. S. Simeonov
D. D. Bainov*

Theory of Lyapunov's Characteristic Exponents for Impulsive Differential Systems. I. General Properties of the Characteristic Exponents

1. Introduction

In the present work we shall expose the main results of the theory of Lyapunov's characteristic exponents for the solutions of the linear homogeneous impulsive system [1, 2, 3]

$$\begin{aligned} &\frac{dx}{dt}=A(t)x, \quad t\neq\tau_k, \\ &\Delta x=B_k x, \quad t=\tau_k, \quad k=1, 2,\dots \end{aligned} \tag{1}$$

The theory of Lyapunov's characteristic exponents for the systems of ordinary differential equations $dx/dt=f(t, x)$ is well developed [4, 5, 6). In the case when this system is linear ($f(t, x)=A(t)x$) the characteristic exponent is considered as a function defined on the linear space L of solutions $x(t)$. Then the elaboration of a great part of this theory is based just on some properties of the characteristic exponent and on the linearity of the space L without using the fact that $x(t)$ is a solution of the linear system $\dot{x}=A(t)x$ [6].

We shall develop the theory of the characteristic exponents under assumptions which provide linearity of the space of solutions of system (1). This allows the proofs of a great part of the theorems to be carried out analogously to the corresponding theorems for systems of ordinary differential equations in [5] or [6]. The proofs of such theorems will be omitted.

2. Characteristic exponent. Definition and properties

Introduce the following notation: $\mathbf{R}_+=[0, \infty)$; $\mathbf{C}^n$ is an n-dimensional complex Euclidean space with elements $x=\operatorname{col}(x_1,\dots,x_n)$, scalar product $(x\,|\,y)=\sum_{k=1}^{n} x_k y_k^*$ of

* The present investigation is financially supported by the Bulgarian Ministry of Science, Culture and Education under Grant 61.

the elements x, y and norm $|x|=(x|x)^{1/2}$; $C^{n\times n}$ is the space of $(n\times n)$-matrices $A=(a_{ij})$ with entries $a_{ij}\in C$; $\|A\|=\sup\{|Ax|: x\in C^n, |x|=1\}$ is the norm of the matrix $A\in C^{n\times n}$; E is the unit $(n\times n)$-matrix.

Definition 1. We shall call a characteristic exponent of the function $x: R_+\to C^n$ the number (or the symbol $+\infty$ or $-\infty$) defined by the formula

$$\chi[x]=\limsup_{t\to+\infty}\frac{1}{t}\ln|x(t)|.$$

If $x(t)=0$ for $t\geq t_0\geq 0$, then we assume that $\chi[x]=-\infty$.

We shal mention the following properties of the characteristic exponents.

Property 1. If $\chi[x]=\alpha\neq\pm\infty$, then for any $\varepsilon>0$ the following relations hold

$$\lim_{t\to+\infty}|x(t)|e^{-(\alpha+\varepsilon)t}=0,\quad \lim_{t\to+\infty}|x(t)|e^{-(\alpha-\varepsilon)t}=+\infty.$$

Conversely, if for some α and any $\varepsilon>0$ the above relations hold, then $\chi[x]=\alpha$.

Property 2. If $c\neq 0$, then $\chi[cx]=\chi[x]$.

Property 3. $\chi[x+y]\leq\max(\chi[x], \chi[y])$ and we have an equality if $\chi[x]\neq\chi[y]$.

Property 4. If $|x(t)|\leq|y(t)|$ for $t\geq t_0\geq 0$, then $\chi[x]\leq\chi[y]$.

Property 5. If $(x(t)|y(t))=1$ for $t\geq t_0\geq 0$, then $\chi[x]+\chi[y]\geq 0$.

Lemma 1. Let the functions $x_k: R_+\to C^n$ $(k=1,\dots,m)$ have different characteristic exponents, i. e. $\chi[x_i]\neq\chi[x_j]$ $(i\neq j)$.

Then the functions $x_1,\dots,x_m$ are linearly independent in R_+.

The proofs of properties 1-5 and Lemma 1 are given in [5].

3. Spectrum of a linear homogeneous impulsive system

Introduce the following conditions (A):

A1. $B_k\in C^{n\times n}$, $\det(E+B_k)\neq 0$ $(k=1, 2,\dots)$.

A2. $\tau_0=0<\tau_1<\tau_2<\dots$, $\lim_{k\to+\infty}\tau_k=+\infty$.

A3. The matrix-valued function $A: R_+\to C^{n\times n}$ is continuous for $t\in R_+$, $t\neq\tau_k$ and for $t=\tau_k$ it has a discontinuity of the first kind and is continuous from the left.

We shall note that if conditions (A) hold, then for any $t_0\in R_+$ and $x_0\in C^n$ there exists a unique solution $x(t; t_0, x_0)$ of system (1) which is defined for $t\in R_+$ and is such that $x(t_0+0; t_0, x_0)=x_0$. Moreover, the set L of all solutions of system (1) defined in R_+ forms an n-dimensional linear space. By Lemma 1 for $x\in L$, $x\neq 0$ the function $\chi[x]$ can take not more than n different values

$$-\infty\leq\alpha_1<\alpha_2<\dots<\alpha_m\leq+\infty\quad(m\leq n),$$

where the least one may be $-\infty$ and the largest $+\infty$.

Definition 2. The set ω of all different finite characteristic exponents of the nonzero solutions of system (1) is said to be the spectrum of system (1).

Definition 3. The set $\bar{\omega}$ of all different characteristic exponents of the nonzero solutions of system (1) is said to be the extended spectrum of system (1).

Remark 1. The spectrum may be empty.

Indeed, for the system

$$\frac{dx}{dt}=0,\quad \frac{dy}{dt}=0,\quad t\neq k,$$

$$\Delta x=(k-1)x, \quad \Delta y=\frac{1-k}{k}y, \quad t=k, \quad k=1, 2, \ldots$$

we have $\omega=\emptyset$, $\bar{\omega}=\{-\infty, +\infty\}$.

Let $U_k(t, s)$ be the Cauchy matrix for the system

$$\frac{dx}{dt}=A(t)x \quad (\tau_{k-1}<t\leq\tau_k). \tag{2}$$

Then the normalized at $t=0$ fundamental matrix of solutions of system (1) has the form

$$V(t)=U_{i+1}(t, \tau_i)\prod_{j=i}^{1}(E+B_j)U_j(\tau_j, \tau_{j-1}) \quad (\tau_i<t\leq\tau_{i+1}) \tag{3}$$

and the Cauchy matrix has the form

$$W(t, s)=U_{i+1}(t, \tau_i)\prod_{j=i}^{k+1}(E+B_j)U_j(\tau_j, \tau_{j-1})(E+B_k)U_k(\tau_k, s) \tag{4}$$

$(\tau_{k-1}<s\leq\tau_k<\tau_i\leq t\leq\tau_{i+1})$.

Theorem 1. Let conditions (A) hold as well as the inequalities

$$\gamma_k\leq\|E+B_k\|\leq\Gamma_k \quad (k=1, 2, \ldots), \tag{5}$$

where the constants γ_k and Γ_k are such that $0\leq\gamma_k\leq\Gamma_k$.

Then for any solution $x(t)\neq 0$ of system (1) we have

$$\chi[m]\leq\chi[x]\leq\chi[M], \tag{6}$$

where $m(t)=\prod\limits_{\tau_k<t}\gamma_k\exp\left(-\int\limits_0^t\|A(s)\|\,ds\right)$, $M(t)=\prod\limits_{\tau_k<t}\Gamma_k\exp\left(\int\limits_0^t\|A(s)\|\,ds\right)$.

Proof. Let $t, \tau\in(\tau_{k-1}, \tau_k]$ and $y\in\mathbf{C}^n$. Then for the solution $U_k(t, \tau)y$ of (2) we have

$$U_k(t, \tau)y=y+\int\limits_\tau^t A(s)U_k(s, \tau)y\,ds,$$

which implies that

$$|U_k(t, \tau)y|\leq|y|+\left|\int\limits_\tau^t\|A(s)\|\,|U_k(s, \tau)y|\,ds\right|. \tag{7}$$

From (7) in virtue of the general inequality of Gronwall-Bellman [5] there follows the two-sided estimate

$$|y|\,e^{-\int\limits_\tau^t\|A(s)\|ds}\leq|U_k(t, \tau)y|\leq|y|\,e^{\int\limits_\tau^t\|A(s)\|ds} \tag{8}$$

For the solution $x(t)\neq 0$ of system (1) in view of (4), (8) and (5) we obtain

(9) $$|x(0)|\prod_{\tau_k<t}\gamma_k e^{-\int_0^t \|A(s)\|ds} \leq |x(t)| \leq |x(0)|\prod_{\tau_k<t}\Gamma_k e^{\int_0^t \|A(s)\|ds}$$

In virtue of properties 2 and 4 from (9) there follows (6).

Theorem 1 is proved.

Remark 2. If $\chi[M]<+\infty$, the extended spectrum does not contain $+\infty$.

Remark 3. If $\chi[m]>-\infty$ and $x\in L$, then $\chi[x]=-\infty$ only for $x=0$.

Theorem 2. Let conditions (A) hold and let the maximal characteristic exponent of the solutions of system (1) be negative.

Then system (1) is asymptotically stable.

Proof. Let $\alpha<0$ be the maximal characteristic exponent of the solutions of system (1). Choose $\varepsilon>0$ so that $\alpha+\varepsilon<0$. Since for any solution $x(t)$ of (1) we have $\chi[x]\leq\alpha$, then in view of condition 1 we obtain

$$\lim_{t\to+\infty}|x(t)|e^{-(\alpha+\varepsilon)t}=0$$

from which we conclude that $\lim_{t\to+\infty}|x(t)|=0$. Hence, the linear system (1) is asymptotically stable. Theorem 2 is proved.

4. Normal Basis

Let the solutions of system (1)

(10) $$x_1(t),\ldots,x_n(t)$$

form a basis of the space L and to them let there correspond the fundamental matrix

(11) $$X(t)=\{x_1(t),\ldots,x_n(t)\}$$

whose columns are the vector-valued functions (10).

Let the spectrum of system (1) consist of the numbers

$$\alpha_1<\alpha_2<\cdots<\alpha_m \quad (m\leq n).$$

Introduce the sets

$$L_s=\{x\in L:\ \chi[x]\leq\alpha_s\}\quad(s=1,\ldots,m),$$
$$L_0=\{x\in L:\ \chi[x]=-\infty\},\quad L_{m+1}=L.$$

Let N_s be the maximal number of linearly independent solutions of system (1) with characteristic exponent equal to α_s, $N_0=\dim L_0$, $N_{m+1}=\dim L=n$.

Then from properties 2 and 3 it follows that

$$\{0\}\subset L_0\subset L_1\subset\cdots\subset L_m\subset L_{m+1}=L$$

and that $L_s(s=0,1,\ldots,m)$ are linear subspaces of the space L.

Lemma 2. Let conditions (A) hold.

Then $N_s=\dim L_s$ $(s=0,1,\ldots,m+1)$.

Lemma 2 is proved as in [5].

Corrollary 1. $0 \leq N_0 < N_1 < \cdots < N_m \leq N_{m+1} = n$.

Let n_s of the solutions from basis (10) have a characteristic exponent equal to α_s $(s=1, \ldots, m)$, n_0 of them have a characteristic exponent equal to $-\infty$ and n_{m+1}— equal to $+\infty$.

Then the following corollary is valid:

Corollary 2. $n_0 + n_1 + \cdots + n_s \leq N_s$ $(s=0, 1, \ldots, m)$,

$$n_0 + n_1 + \cdots + n_{m+1} = n.$$

Definition 4. The basis (10) of solutions of system (1) is said to be normal if

$$n_0 + n_1 + \cdots + n_s = N_s \quad (s=0, 1, \ldots, m+1).$$

In this case the fundamental matrix (11) of solutions of system (1) is said to be normal.

Before we formulate necessary and sufficient conditions for normality of the basis, we shall give the following definition:

Definition 5. The system of nonzero vector-valued functions x_k: $\mathbf{R}_+ \to \mathbf{C}^n$ $(k=1, \ldots, m)$ is said to have a nondecreasing property if for any nontrivial linear combination

$$y = \sum_{k=1}^{m} c_k x_k \quad \text{(with at least one } c_k \neq 0)$$

the following equality holds

$$\chi[y] = \max_{k:\, c_k \neq 0} \chi[x_k].$$

Theorem 3. Let conditions (A) hold.

Then the basis (10) of solutions of system (1) is normal iff it has a nondecreasing property.

Theorem 3 is proved as in [5].

Theorem 4. Let conditions (A) hold.

Let $x_1, \ldots, x_n$ be a normal basis with characteristic exponents

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

and $y_1, \ldots, y_n$ be an arbitrary basis with characteristic exponents

$$\lambda'_1 \leq \lambda'_2 \leq \cdots \leq \lambda'_n.$$

Then $\lambda'_k \geq \lambda_k$ $(k=1, \ldots, n-1)$, $\lambda'_n = \lambda_n$ and, moreover, the basis $y_1, \ldots, y_n$ is normal iff

$$\lambda'_k = \lambda_k \quad (k=1, \ldots, n).$$

Theorem 4 is proved as in [6].

We shall note that in all normal bases the number of solutions with characteristic exponent equal to α_s $(s=1, \ldots, m)$ is equal to $N_n - N_{s-1}$ and the number of solutions with characteristic exponent equal to $-\infty$ or $+\infty$ is equal respectively to N_0 or $n - N_m$.

Definition 6. The numbers

$$\nu_s = N_s - N_{s-1} \quad (s=1,\dots,m),$$
$$\nu_0 = N_0, \qquad \nu_{m+1} = n - N_m$$

are said to be respectively multiplicities of the characteristic exponents α_s, $-\infty$, $+\infty$.

Definition 7. The spectrum of system (1) is said to be finite if $\bar{\omega}=\omega$.

Let the spectrum of system (1) be finite and let it consist of the numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ $(m \leq n)$.

Let the basis $x_1, \dots, x_n$ have n_s solutions with characteristic exponent equal to α_s.

Definition 8. The number $\sigma_x = \sum_{s=1}^{m} n_s \alpha_s$ is said to be the sum of the characteristic exponents of the basis $x_1, \dots, x_n$.

Theorem 5. Let conditions (A) hold and let the spectrum of system (1) be finite. Then the basis $x_1, \dots, x_n$ is normal iff

$$\sigma_x = \sum_{s=1}^{m} \nu_s \alpha_s,$$

where $\nu_s = N_s - N_{s-1}$ $(N_0 = 0)$ are the multiplicities of the characteristic exponents α_s $(s=1,\dots,m)$.

Theorem 5 is an immediate consequence of Theorem 4.

Theorem 6. Let conditions (A) hold and $Z(t)=\{z_1(t),\dots,z_n(t)\}$ be a fundamental matrix of system (1).

Then there exists a constant matrix of the form

$$C = \begin{bmatrix} 1 & c_{12} & c_{13} \cdots c_{1n} \\ 0 & 1 & c_{23} \cdots c_{2n} \\ \cdot & \cdot & \cdot \;\cdots\; \cdot \\ 0 & 0 & 0 \quad\;\; 1 \end{bmatrix}$$

such that the matrix

(12) $$X(t) = Z(t)\,C$$

is a normal fundamental matrix of system (1).

Proof. Without loss of generality we assume that $\chi[z_i] \leq \chi[z_j]$ for $i<j$. Set

(13) $$x_k = c_{1k} z_1 + \cdots + c_{k-1,k} z_{k-1} + z_k \quad (k=1,\dots,n),$$

where the coefficients $c_{1k}, \dots, c_{k-1,k}$ are chosen so that

$$\chi[x_k] = \min_{b_1,\dots,b_{k-1}} \chi[b_1 z_1 + \cdots b_{k-1} z_{k-1} + z_k],$$

hence

$$\chi[z_k] \geq \chi[x_k] \quad (k=1,\dots,n).$$

From (13) it follows that $X(t)$ has the form (12) and $\det X(t) = \det Z(t) \neq 0$, i. e. $X(t)$ is a fundamental matrix of system (1).

For $i<j<\cdots<k$ let $\chi[x_i]=\chi[x_j]=\cdots=\chi[x_k]=\alpha$ and $y=a_i x_i+a_j x_j+\cdots a_k x_k$ ($a_k\neq 0$). We shall prove that $\chi[y]=\alpha$. In fact, from (13) it follows that y has the form

$$y=a_k\left(z_k+\sum_{s=1}^{k-1} b_s z_s\right).$$

From the way of choice of x_k it follows that

$$\chi[y]=\chi\left[z_k+\sum_{s=1}^{k-1} b_s z_s\right]\geq\chi[x_k]=\alpha$$

and from properties 2 and 3 we have that $\chi[y]\leq\alpha$. Hence, $\chi[y]=\alpha$.

We shall prove that the basis $x_1,\ldots,x_n$ has a nondecreasing property. Then Theorem 3 implies that the fundamental matrix $X(t)$ is normal.

Consider an arbitrary nontrivial linear combination

$$x=\sum_{k=1}^{n} a_k x_k \quad \text{(with at least one } a_k\neq 0). \tag{14}$$

Separate these solutions of the linear combination (14) having the lagrest characteristic exponent. Let these be

$$x_i,\ x_j,\ldots,x_k\ (i<j<\cdots<k) \text{ and } \chi[x_i]=\chi[x_j]=\cdots=\chi[x_k]=\alpha.$$

Then we can represent x in the form

$$x=a_i x_i+a_j x_j+\cdots+a_k x_k+z,$$

where $\chi[z]<\alpha$ and obtain that

$$\chi[x]=\chi[a_i x_i+a_j x_j+\cdots+a_k x_k]=\alpha=\max_{s\,:\,a_s\neq 0}\chi[x_s].$$

Theorem 6 is proved.

References

1. Мильман, В. Д., А. Д. Мышкис. Об устойчивости движения при наличии толчков. — Сиб. мат. ж., 1, 1960, № 2, 233—237.
2. Мышкис, А. Д., А. М. Самойленко. Системы с толчками в заданные моменты времени — В: Мат. сб., 74, 1967, № 2, 202—208.
3. Самойленко А. М., Н. А. Перестюк. Устойчивость решений дифференциальных уравнений с импульсным воздействием. — Дифф. уравн., 1977, № 11, 1981—1992.
4. Ляпунов, А. М. Общая задача об устойчивости движения. М. — Л., Гостехиздат, 1950.
5. Демидович, Б. П., Лекции по математической теории устойчивости. Москва, Наука, 1967.
6. Былов, Б. Ф., Р. Э. Виноград, М. Д. Гробман, В. В. Немыцкий. Теория показателей Ляпунова. Москва, Наука, 1966.

Received on June 23, 1990