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A Laplace Transform Aided Numerical Approach to Dynamic Unilateral Contact Problems*

1. Introduction

Unilateral contact problems belong, as is well known, to inequality problems of mechanics, where the governing conditions are equalities as well as inequalities. A mathematical treatment of these nonlinear problems can be successfully done by the variational inequality concept [1,2]. On the other hand, the numerical treatment of inequality problems requires the use of nonlinear mathematical programming methods, see e. g. [1-6] and the references cited there.

The present paper deals with a numerical approach, based on the Laplace transform, for the dynamic case of the above problems. Besides of well-known linear applications in mechanics, Laplace transform has been recently used successfully to obtain two-sided solution estimates in unilateral viscoelastodynamics [7], and further to treat numerically nonlinear time-dependent problems [8]. Here the Laplace transform is used to derive a formally compact and computationally suitable approach to the dynamic unilateral contact problems of structural elastomechanics. So, it is shown that at every moment the problem is reduced to the solution of a convolutional linear complementarity problem, which, in comparison to the original problem, has a small number of unknowns. The latter concern the unilateral quantities only. Finally, a numerical example from civil engineering praxis is presented.

2. Problem conditions

For writing simplicity, first the problem is discretized in space by the finite element method using models of "natural" type [1,4-5]. The unilateral behaviour is simulated by unilateral constraints-elements located appropriately. Thus, following [9] and in a way similar to that used by Maier [4] in piece-wise linearized plasticity description, the constitutive laws for the unassembled discretized sructure are written in matrix notation:

(1) $$\mathbf{e}=\boldsymbol{\epsilon}+\boldsymbol{\mu}+\boldsymbol{\theta};$$

(2) $$\boldsymbol{\epsilon}=\mathbf{E}^{-1}\boldsymbol{\sigma} \quad \text{or} \quad \boldsymbol{\sigma}=\mathbf{E}\boldsymbol{\epsilon};$$

(3) $$\boldsymbol{\mu}=\mathbf{B}\boldsymbol{\upsilon};$$

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(4) $$\psi = B^T\sigma - A\upsilon - r, \quad (r \geqq 0);$$

(5), (6), (7) $$\psi \leqq 0, \quad \upsilon \geqq 0, \quad \psi^T\upsilon = 0.$$

Here e, ϵ, μ, θ are the total, pure elastic, unilateral and imposed (e. g. thermal) strain vectors, respectively; σ is the stress vector; υ, ψ and r are the deformation, stress-reaction and resistance (ultimate capacity) vectors of the unilateral constraints, respectively; B is a rectangular transformation matrix and B^T its transposed one; E is the symmetric and positive definite elasticity matrix and E^{-1} its inverted; and A is the unilateral interaction matrix, in general symmetric and positive (semi-) definite.

Further, the dynamic equilibrium and compatibility conditions for the assembled structure are, respectively,

(8) $$G\sigma = p - M\ddot{u} - C\dot{u};$$

(9) $$e = G^T u,$$

where G, G^T, M, C are the equilibrium, compatibility, mass and damping matrices, respectively, and u, p are the displacement and load vectors. As known, M and C are symmetric and in general positive (semi-) definite matrices, and dots over symbols denote derivatives with respect to time t.

Associated with the system (1)-(9) are the initial conditions

(10) $$u(t=0) = u_0;$$

(11) $$\dot{u}(t=0) = \dot{u}_0;$$

(12) $$\upsilon(t=0) = \upsilon_0.$$

Thus the problem consists in finding the response set $\{u(t),\ e(t),\ \sigma(t),\ \epsilon(t),\ \psi(t), \mu(t),\ \upsilon(t)\}$ satisfying (1)-(12) for a given excitation set $\{p(t),\ \theta(t),\ u_0,\ \dot{u}_0,\ \upsilon_0\}$. Questions concerning existence, uniqueness of solution etc. can be treated by the variational inequality concept [1,2].

Now, to reduce the number of unknowns of problem (1)-(12), we introduce the, positive definite stiffness matrix

(13) $$K = GEG^T$$

and the generalized load vector

(14) $$q = p + GE\theta.$$

So, after a suitable elimination of e, ϵ, σ and μ, we obtain from (1)-(4) and (8)-(9) the relations

(15) $$M\ddot{u} + C\dot{u} + Ku = q + GEB\upsilon;$$

(16) $$\psi = B^T E G^T u - (A + B^T E B)\upsilon - B^T E\theta - r.$$

Thus, the reduced problem in terms of $u(t)$, $\psi(t)$ and $\upsilon(t)$ only consists of (5)-(7) (10)-(12) and (15)-(16).

In the absence of unilateral constraints, (15) takes the usual form for linear (bilateral) problems in structural dynamics:

$$M\ddot{u} + C\dot{u} + Ku = q.$$

3. The Laplace transform approach

Now we use the Laplace transform for time-functions $f(t)$:

$$\tilde{f}(s)=\int_0^\infty f(t)\,e^{-st}\,dt, \quad (s\geqq s_0>0), \tag{17}$$

where s_0 is a suitable constant (convergence abscissa).

Taking into account (10), (11), first we obtain from (15), (16):

$$\tilde{\mathbf{D}}\,\tilde{\mathbf{u}}=\tilde{\mathbf{d}}+\mathbf{GEB}\tilde{\mathbf{v}}; \tag{18}$$

$$\tilde{\boldsymbol{\psi}}=\mathbf{B}^T\mathbf{EG}^T\tilde{\mathbf{u}}-(\mathbf{A}+\mathbf{B}^T\mathbf{EB})\,\tilde{\mathbf{v}}-\mathbf{B}^T\mathbf{E}\,\tilde{\boldsymbol{\theta}}-\mathbf{r}/s, \tag{19}$$

where

$$\tilde{\mathbf{D}}=\tilde{\mathbf{D}}(s)=\mathbf{K}+s\mathbf{C}+s^2\mathbf{M}; \tag{20}$$

$$\tilde{\mathbf{d}}=\tilde{\mathbf{d}}(s)=\tilde{\mathbf{q}}+\mathbf{M}\dot{\mathbf{u}}_0+(s\mathbf{M}+\mathbf{C})\,\mathbf{u}_0. \tag{21}$$

Matrix $\tilde{\mathbf{D}}$ is symmetric and positive definite, and therefore $\tilde{\mathbf{D}}^{-1}$ exists. Solving (18) for $\tilde{\mathbf{u}}$ and defining

$$\tilde{\boldsymbol{\psi}}^L=\tilde{\boldsymbol{\psi}}^L(s)=\mathbf{B}^T\mathbf{EG}^T\tilde{\mathbf{D}}^{-1}\tilde{\mathbf{d}}-\mathbf{B}^T\mathbf{E}\tilde{\boldsymbol{\theta}}-\mathbf{r}/s; \tag{22}$$

$$\tilde{\boldsymbol{\Lambda}}=\tilde{\boldsymbol{\Lambda}}(s)=\mathbf{A}+\mathbf{B}^T\{\mathbf{E}+\mathbf{EG}^T\tilde{\mathbf{D}}^{-1}\mathbf{GE}\}\mathbf{B}, \tag{23}$$

(19) is written as follows:

$$\tilde{\boldsymbol{\psi}}(s)=\tilde{\boldsymbol{\psi}}^L(s)-\tilde{\boldsymbol{\Lambda}}(s)\,.\,\tilde{\mathbf{v}}(s); \tag{24a}$$

or

$$\tilde{\boldsymbol{\psi}}=\tilde{\boldsymbol{\psi}}^L-\tilde{\boldsymbol{\Lambda}}\,\tilde{\mathbf{v}}. \tag{24b}$$

Thus, in the Laplace domain the transformed reactions $\tilde{\boldsymbol{\psi}}$ of the unilateral constraints are the sum of two constituent parts. The first part, $\tilde{\boldsymbol{\psi}}^L$, concerns the response of the structure, considered as a usual linear one with bilateral constraints ($\tilde{\mathbf{v}}=\mathbf{0}$), to the external actions. The second part is caused exclusively by the unilateral deformations, $\tilde{\mathbf{v}}(s)$, considered as imposed dislocations to the same linear structure. The involved influence matrix $\tilde{\boldsymbol{\Lambda}}(s)$ is a symmetric positive definite matrix and plays the role of a Green matrix-function.

4. Equivalent problem formulation in time-domain

Returning back to the time-domain and using the well-known concept of convolution, see e. g. [10-12], for two time-functions $f_1(t)$ and $f_2(t)$;

$$f_1(t)*f_2(t)=\int_0^t f_1(\tau)\,f_2(t-\tau)\,d\tau, \tag{25}$$

we obtain from (24) by Laplace inversion the relation

(26) $$\psi(t)=\psi^L(t)-\Lambda(t)*\upsilon(t).$$

Thus, at every time-moment we have to solve the problem of relations (5)-(7) and (26). This problem is called a Convolutional Linear Complementarity Problem (CLCP) and its numerical solution is obtained by time-discretization as explained consequently.

The linear constituent term $\psi^L(t)$ is computed by considering the unilateral constraints as usual bilateral ones. For this purpose, anyone of the well-known methods in linear structural dynamics can be used directly in time-domain. Similarly, matrix $\Lambda(t)$ can be computed in time-domain by various well-known techniques used for dynamic stiffness matrices [10-12]. Alternatively, both $\psi^L(t)$ and $\Lambda(t)$ can be computed by Laplace inverting (22) and (23), respectively, using available computer codes [10]. Further, a time discretization is used for the step-by-step solution of the CLCP. To save computationall work, various formulae for the recursive evaluation of convolution integrals in (26) can be used in a way similar to that for elastodynamic analysis by the boundary element method [10, 12]. So, the CLCP for the time-moment $t_n=n\,.\Delta t$, where Δt is the time-step and $n=0, 1, \ldots, N$, takes the form

(27a) $$\psi_n=\mathbf{a}_n+\Lambda_n\upsilon_n;$$

(27b, c, d) $$\psi_n\leqq 0,\quad \upsilon_n\geqq 0,\quad \psi_n^T\upsilon_n=0,$$

where $\mathbf{a}_n$ is a known term from previous time-steps. Problem (27) is now a usual linear complementarity problem, which is solved by available mathematical programming algorithms [1, 4, 6].

The above formulated CLCP of rels. (26) and (5)-(7) expresses an equivalence principle in unilateral elastodynamics, corresponding to the equivalence principle established by Nitsiotas for unilateral elastostatics [3].

5. Numerical example

The structural system of Fig. 1 has been recently studied in [8] by an hybrid Laplace time-domain trial-and-error approach. The same problem is also studied here by the presented method for comparison reasons.

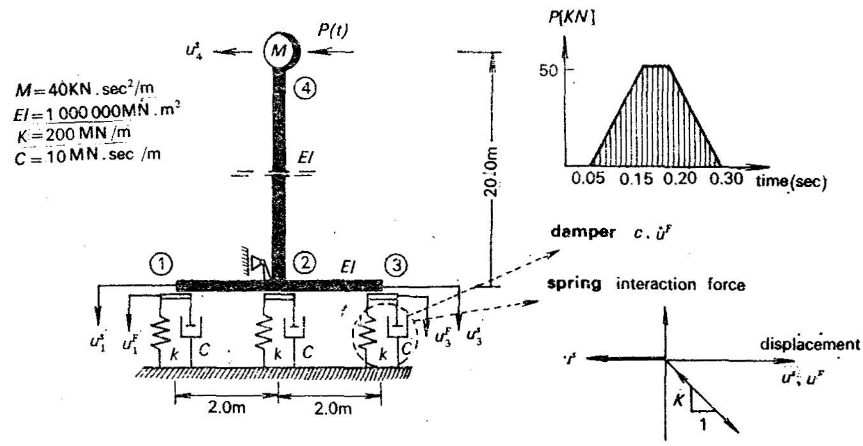

Fig. 1. Numerical example: system and loads 8

The problem concerns a simple lumped mass system positioned on three isolated spring-damper soil-elements. The mass is subjected to an impulsive load as depicted in Fig. 1*b*. Between the foundation and the soil-element *i*, ($i=1$, 2, 3) only negative forces (pressure) can be transmitted. So, if these forces tend to become positive, then uplifting υ_i of the foundation (that means relative retirement displacements between foundation and soil) will occur. The relevant Fig. 1*c* describes the unilateral contact behaviour of the soil-elements, for which the corresponding constitutive relations (4)-(7) are $\psi_i=\sigma_i$, $\psi_i\leq 0$, $\upsilon_i\geq 0$, $\psi_i\upsilon_i=0$ (no summing). With the data: $M=40$ KN.sec²/m, $k=200$ MN/m, $C=10$ MN.sec/m and $EI=10^6$ MN. m², the static displacement of the foundation is 0.667×10^{-3} m.

Starting from this static value, the displacement function of node 3 is presented in Fig. 2 for both cases, with and without uplifting of the foundation taken into account. The obtained results by the herein procedure are in a very good agreement with those of [8]. So, the displacement of the foundation will reach much larger values after uplift has taken place than before in the purely elastic case, where uplift is not taken into account. These results also show that the dynamic response of a structure containing unilateral constraints may be entirely different from that when the unilateral behaviour is considered as a bilateral one.

6. Concluding remarks

The herein presented numerical approach extends the use of Laplace transform to unilateral contact problems of structural dynamics. The Laplace domain is used to obtain an equivalent problem formulation, called Convolutional Linear Complementarity Problem (CLCP), with a considerably reduced number of unknowns. Moreover, the Laplace domain can be used to obtain, on the one hand, the linear response due to external actions and, on the other hand, an influence matrix. In both these cases, the

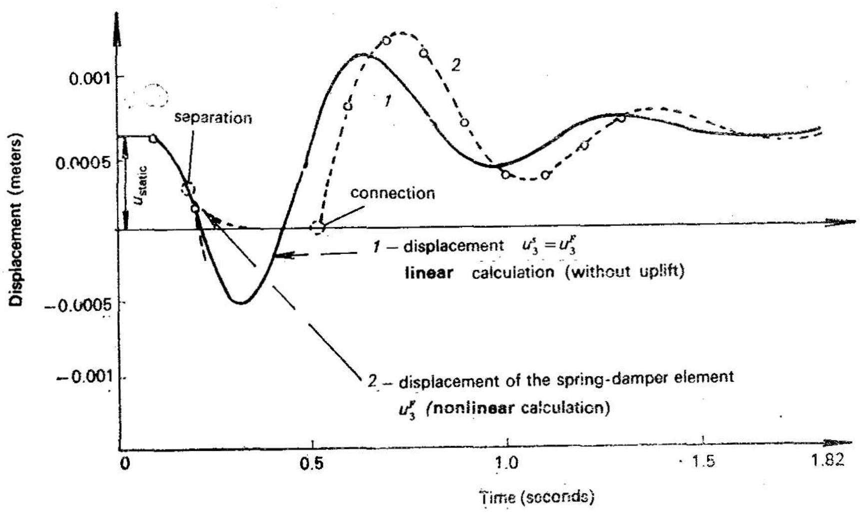

Fig. 2. Displacement of structure node 3

unilateral constraints are considered as bilateral ones and the finite or the boundary element method can be applied in the usual way for linear problems.

Further, for the solution of the Convolutional Linear Complementarity Problem, a time discretization and a recursive evaluation of convolution integrals are used. Thus, the numerical realization of the method is obtained by using available computer codes of the finite or/and the boundary element method, the Laplace transform and mathematical programming. Finally, this numerical approach can be extended to other inequality evolution problems of structural mechanics and mathematical physics.

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