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Some Remarks on the Fokker-Planck Equation in Velocity and Phase Space

1. Introduction

In some theoretical approaches to various physical problems theory of stochastic processes seems to give very useful and enlightening suggestions. A stochastic approach may also be successfully applied as a starting point in order to explain the development of some metabolic diseases [1, 2], so that implications of this theory appear of grear interest in some fields of pure and applied science.

In many aspects of stochastic theory we must use fluctuating forces [3, 4] as a necessary parameter: the purpose of the present paper is to demonstrate that the problem of finding the distribution function can be reduced to the solution of the Fokker-Planck equation: this method will require less restrictive assumptions about the statistical properties of the fluctuating force; of particular interest may be the fact that when further restrictions are imposed, they can be expressed as boundary conditions for the solution of the Fokker-Planck equation, which seems to appear the most suitable model for comparison with the results of the microscopic theory.

2. Discussion

A very intuitive feeling of the Brownian motion is at the basis of the assumptions which must be done in order to describe the framework of the phenomenological theory about the statistical properties of the fluctuating force $A(t)$.

It is known that the characteristic time for the variation of the macroscopic quantities is much longer than the time interval between two successive collisions of Brownian particle with the particles of the fluid, so that it is necessary to assume a time interval Δt such that during Δt macroscopic quantities change by a negligible amount

$$\frac{\langle v(t+\Delta t)-\langle v(t)\rangle}{\langle v(t)\rangle} \ll 1\,; \tag{1}$$

$A(t+\Delta t)$ and $A(t)$ are uncorrelated, containing $A(t)$ a large number of fluctuations and being the Brownian particle much heavier than the fluid particle. During a time interval Δt the acceleration of the Brownian particle due to the action of the fluctuating force will be

(2) $$\dot{v}(\Delta t)=\int_{t}^{t+\Delta t} d\xi A(\xi);$$

we can assume that this acceleration depends only on the time interval Δt, so that we can neglect memory effects; moreover, this acceleration is due to the superposition of a great number of random accelerations.

Before dealing with the problem of finding the complete probability distribution in phase space $P(x, v, t \mid x_0, v_0)$ to find the particle in x with velocity v at time t, being x_0 the initial displacement and v_0 the initial velocity, we can try to find a more simple probability $P(v, t \mid v_0)$.

The formal solution of the Langevin equation takes the form

(3) $$v-v_0e^{-\beta t}=e^{-\beta t}\int_0^t e^{\beta\xi}A(\xi)\,d\xi$$

in equation (3) both sides have the same probability distribution, and having

(4) $$\alpha=e^{-\beta t}\int_0^t e^{\beta\xi}A(\xi)\,d\xi,$$

it is possible to write:

(5) $$\alpha=\sum_{j=1}^{n}\exp\left[-\beta(t-j\Delta t)\right]\dot{v}(\Delta t)=\sum_{j=1}^{n}\alpha_j$$

dividing the time interval from 0 to t into n intervals Δt; the distribution function of α takes also the form [5, 6]

(6) $$P(\alpha)=\left[4\pi q\int_0^t d\xi e^{2(\xi-t)}\right]^{-3/2}.\exp\left[-\mid\alpha\mid^2/4q\int_0^t d\xi e^{-2\beta(t-\xi)}\right],$$

from which we obtain the velocity distribution function as:

(7) $$P(v, t\mid v_0)=\left[2\pi q\frac{1}{\beta}(1-e^{-2\beta t})\right]^{-3/2}.\exp\left[-\frac{\beta}{2q}\frac{\mid v-v_0e^{-\beta t}\mid^2}{1-e^{-2\beta t}}\right].$$

If we consider long times (i. e., if $\beta t\gg 1$) we also have

(8) $$P(v, t\rightarrow\infty\mid v_0)=(2\pi q/\beta)^{-3/2}\exp\left[-\beta v^2/2q\right],$$

and therefore, we obtain an irreverisble evolution towards a Gaussian distribution, and this distribution will be independent from v_0. It is to note that in a discussion 'a priori' nothing implies a Maxwell-Boltzmann equilibrium; moreover, if we add this condition as a further requirement, we must choose the diffusion coefficient in velocity space to be

(9) $$q=\frac{KT\beta}{M}.$$

The distribution function in velocity space may also completely be determined by equations (7) and (9).

3. Results and conclusions

In the previous section we have obtained the probability distribution function corresponding to the condition that at $t=0$ the velocity is v_0; let us now assume that time intervals Δt are characterized by the fact that macroscopic quantities do not vary very much, whereas, the fluctuating force may change several times.

In the case of a Markov process, the probability distribution function $P(v, t+\Delta t)$ must satisfy the equation

$$(10) \qquad P(v,\ t+\Delta t)=\int d(\Delta v)\, P(v-\Delta v,\ t)\,\Psi(v-\Delta v;\ \Delta v),$$

in which the term $\Psi(v;\ \Delta v)$ indicates a transition probability if velocity increases of Δv in a time Δt.

Expanding both sides of equation (10) in power series, we can obtain

$$(11) \qquad P(v,\ t)+\frac{\partial P}{\partial t}\Delta t+0\,(\Delta t)^2=\int d(\Delta v)\Big\{P(v,\ t)-\frac{\partial P}{\partial v_i}\Delta v_i + \frac{1}{2}\frac{\partial^2 P}{\partial v_i \partial v_j}\Delta v_i \Delta v_j + \cdots\Big\}\Big\{\Psi(v;\ \Delta v)-\frac{\partial \Psi}{\partial v_i}\Delta v_i+\frac{1}{2}\frac{\partial^2 \Psi}{\partial v_i \partial v_j}\Delta v_i \Delta v_j+\cdots\Big\}.$$

Writing also

$$(12) \qquad \langle\alpha\rangle=\int d(\Delta v)\,\alpha\Psi(v;\ \Delta v),$$

equation (11) may be rewritten as follows:

$$(13) \qquad \frac{\partial P}{\partial t}\Delta t+0\,(\Delta t)^2=-\frac{\partial}{\partial v_i}[P\langle\Delta v_i\rangle]+\frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}[P\langle\Delta v_i \Delta v_j\rangle]+0\,[\langle\Delta v_i \Delta v_j \Delta v_k\rangle].$$

Considering that in the Langevin equation all systematic effects are involved in the friction term and the fluctuating force is random, we can write

$$(14) \qquad \langle\int_0^{\Delta t} d\tau A(\tau)\rangle=0,$$

and

$$(15) \qquad \langle\int_0^{\Delta t} d\tau \int_0^{\Delta t} d\tau' A_i(\tau) A_j(\tau')\rangle$$

$$\sim \int_0^{\Delta t} d\tau \int_0^{\Delta t} d\tau' \delta(\tau-\tau') \sim \Delta t;$$

so that, taking the limit for $\Delta t \to 0$, it is possible to write the general Fokker-Planck equation for the velocity distribution function as

(16) $$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial v_i}\left[P\frac{\langle\Delta v_i\rangle}{\Delta t}\right]+\frac{1}{2}\frac{\partial^2}{\partial v_i\partial v_j}\left[P\frac{\langle\Delta v_i\Delta v_j\rangle}{\Delta t}\right].$$

Now, let us evolve from the Langevin equation

(17) $$\Delta v=-\beta v\Delta t+\dot{v}(\Delta t);$$

the transition probability, taking into account equation (6), may also be written as follows

(18) $$\Psi(v;\ \Delta v)=4\pi\beta KT\Delta t/M)^{-3/2}.\exp[-M\,|\Delta v+\beta v\Delta t\,|^2/4\beta KT\Delta t],$$

so that, having

(19) $$\langle\Delta v_i\rangle=-\beta v_i\Delta t,$$

and

(20) $$\langle\Delta v_i\Delta v_j\rangle=(2\beta KT/M)\,\delta_{i,j}+0(\Delta t)^2$$

we have the Fokker-Planck equation in the form

(21) $$\frac{\partial P}{\partial t}=\beta\left(\frac{\partial Pv_i}{\partial v_i}+\frac{KT}{M}\frac{\partial^2 P}{\partial v_i^2}\right);$$

the solution of this equation reduces at $t=0$ to a delta function

(22) $$P(v,\ 0\,|\,v_0)=\delta(v-v_0).$$

It is to note that this solution is given by equation (7), from which may be easily obtained the solutions corresponding to any arbitrary initial distribution.

A more general procedure may derive from the above considerations, in order to find an equation for the probability distribution $P(x, v, t)$ in phase space. Equation (10) must be changed in the form

(23) $$P(x,\ v,\ t+\Delta t)=\int\int P(x-\Delta x,\ v-\Delta v,\ t)\,\Psi(x-\Delta x,\ v-\Delta v;\ \Delta x,\ \Delta v)\,d(\Delta x)\,d(\Delta v),$$

and, considering the presence of an external force, we can evolve from Langevin equation:

(24) $$\Delta x=v\Delta t$$

and

(25) $$\Delta v=-(\beta v-F)\,\Delta t+\dot{v}(\Delta t),$$

to obtain

(26) $$\Psi(x,\ v;\ \Delta x,\ \Delta v)=\Psi(v;\ \Delta v)\,\delta(\Delta x-v\Delta t).$$

The effects of an external field may be simulated, considering also equation (18), adding the term — $F\Delta t$ in the exponential.

It is possible to generalize equation (21): a previous generalization of equation (13) enables us to write

(27) $$\left(\frac{\partial P}{\partial t}+v_i\frac{\partial P}{\partial x_i}\right)\Delta t+0(\Delta t)^2=-\frac{\partial P\langle\Delta v_i\rangle}{\partial v_i}+\frac{1}{2}\frac{\partial^2 P\langle\Delta v_i\Delta v_j\rangle}{\partial v_i\partial v_j}+0(\langle\Delta v_i\Delta v_j\Delta v_k\rangle),$$

from which, repeating all procedures described above, it is possible to write the Fokker-Planck equation in the from

$$\frac{\partial P}{\partial t}+v_i\frac{\partial P}{\partial x_i}+F_i\frac{\partial P}{\partial v_i}=\beta\left(\frac{\partial P v_i}{\partial v_i}+\frac{KT}{M}\frac{\partial^2 P}{\partial v_i^2}\right). \qquad (28)$$

It is to note that the arguments followed for the calculus of the Fokker-Planck equations in velocity space may be repeated if we do not consider a special choice for $\Psi(v;\ \Delta v)$.

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