BULGARIAN ACADEMY OF SCIENCES
JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
Sofia, 1991, Year XXII, No 1

Z. Petkov

Error Estimation of Direct Integration Process in Terms of Frequency Content of Input Signal

Introduction

Up to now the basic tool in the study of nonlinear behaviour of structures due to different types of dynamic loadings remains direct integration methods (DIM). Most of the problems require tracing of structural behaviour because nonlinearity is path dependent process during the time. Since time history responses are numerically obtained the problems of stability and accuracy are a subject of the number of research works during the past two decades. Time step appears as a result of time discretization and it is the most important factor in error analysis. In order to control stability properties and accuracy a set of numerical coefficients (SNC) are involved. The influence of these coefficients on the errors can also be significant and should be studied. For most DIM, SNC must previously be valued. Knowing the errors expected is an important side of the computational process.

Since a lot of comprehensive works [1], [2] have been devoted to unconditional stability the attention is focussed on free vibration response of a linear system. Second order of the error with respect to the time step seems to be the best possible accuracy for one step methods [2], [1]. The same order of accuracy in free vibration case, but without any "overshooting" phenomena, is achieved in [5] and [6].

However, for most engineering applications such as earthquake and wind response of structures, related to nonhomogeneous difference operators numerical errors should be regarded as dependent on the input signal. Unfortunately, such treatments are not enough at present. Preumont [4] has reported frequency domain analysis of some temporal operators and the measure of the error is the deviation between exact and discretized transfer functions.

The purpose of this work is to provide an explicit expression suitable for nonhomogeneous linear operators. Such result will allow to give a realistic assessment of the validity of final results. Next come practical applications on some methods for example [5] and [6] and recommendations for the choice of SNC.

Basic truncation error expression

The following assumptions hold:

1. Input signal is presented by a finite number of samples. A linear law between two points is assumed. Time step is considered to be a constant during the analysis.

2. A linear nonhomogeneous system of difference equations is considered. That means all stiffness coefficients remain constant and one can draw attention to a single equation, obtained as a product of mode superposition transforms [1].

It is well known that one step integration process can be expressed in operating form in such a manner:

(1) $$\mathbf{X}_{n+1}=\mathbf{A}\,.\,\mathbf{X}_n+\mathbf{G}\,.\,\mathbf{p}_{n+1},$$

where $\mathbf{A}$ and $\mathbf{G}$ are approximate amplification and load operators, respectively. Matrix $\mathbf{A}$ is of order (3×3), the order of $\mathbf{G}$ is (3×2). More details on their structure will be discussed later on. Equation (1) includes the following notations in a vector form:

(2) $$\mathbf{X}^T_{n+1}=[d_{n+1},\quad hv_{n+1},\quad h^2a_{n+1}],$$
$$\mathbf{p}^T_{n+1}=[p(t_n),\quad p(t_{n+1})]=[p_n,\ p_{n+1}]$$

in which d_{n+1}, v_{n+1} and a_{n+1} are approximate values of displacements, velocities and accelerations at time t_{n+1}, p_n and p_{n+1} are the discrete values of load function, acting at time t_n and t_{n+1}, respectively. The time step is denoted by h.

Let's consider the exact solution for the displacements $d(t)$, velocities $v(t)$ and accelerations $a(t)$. All of these functions are continuous with respect to time t, thus the following relations are valid:

(3) $$v(t)=\dot{d}(t),\quad a(t)=\dot{v}(t)=\ddot{d}(t).$$

Exact solution, similarly to Eq. (1) can be presented in one step form

(4) $$\mathbf{X}(t_{n+1})=\mathbf{E}\,.\,\mathbf{X}(t_n)+\mathbf{R}\,.\,\mathbf{p}_{n+1}.$$

where

(5) $$\mathbf{X}^T(t_{n+1})=[d(t_{n+1}),\quad hv(t_{n+1}),\quad h^2a(t_{n+1})].$$

The operators $\mathbf{E}$ and $\mathbf{R}$ have the same meaning and order as $\mathbf{A}$ and $\mathbf{G}$ from Eq. (1). The difference is that these operators encorporate relationships from the exact solutions. Further details can be found in [1], [2] and [4]. A particular form for $\mathbf{E}$ and $\mathbf{R}$ will ce given below. The vector of local truncation error τ at time t_n, due to distinct operators, is defined [1] as:

(6) $$\boldsymbol{\tau}(t_n)=(\mathbf{E}-\mathbf{A})\,.\,\mathbf{X}(t_n)+(\mathbf{R}-\mathbf{G})\mathbf{p}_{n+1}$$

and

(7) $$\boldsymbol{\tau}^T(t_n)=[\tau_1(t_n),\quad \tau_2(t_n),\quad \tau_3(t_n)].$$

Eq. (6) after taking into account (4) can be written in the form:

(8) $$\mathbf{X}(t_{n+1})=(\boldsymbol{\tau}t_n)+\mathbf{A}\,.\,\mathbf{X}(t_n)+\mathbf{G}\,.\,\mathbf{p}_{n+1}.$$

Similarly, for the time t_n and t_{n-1} one can write:

(9) $$\mathbf{X}(t_n)=\boldsymbol{\tau}(t_{n-1})+\mathbf{A}\,.\,\mathbf{X}(t_{n-1})+\mathbf{G}\,.\,\mathbf{p}_n,$$
$$\mathbf{X}(t_{n-1})=\boldsymbol{\tau}(t_{n-2})+\mathbf{A}\,.\,\mathbf{X}(t_{n-2})+\mathbf{G}\,.\,\mathbf{p}_{n-1}.$$

The next step is to derive an expression for the error $\tau^d(t_n)$ which appears in the displacement response. To achieve that, all velocities and accelerations should be eliminated from Eqs (8) and (9). The general purpose of these subsequent manipulations is to operate only on displacement dependent terms. It can be shown [3] that such elimination process leads to one value multistep integration. The final result after performing elimination phase is:

$$(10)\qquad \tau^d(t_n) = \frac{1}{h^2}\tau_1(t_n)-(a_{22}+a_{33})\frac{1}{h^2}\tau_1(t_{n-1}) + (a_{22}a_{33}-a_{23}a_{32})\frac{1}{h^2}\tau_1(t_{n-2})$$

$$+a_{13}\frac{1}{h^2}\tau_3(t_{n-1})+(a_{23}a_{12}-a_{22}a_{13})\frac{1}{h^2}\tau_3(t_{n-2})+a_{12}\frac{1}{h^2}\tau_2(t_{n-1})+(a_{32}a_{13}-a_{33}a_{12})\frac{1}{h^2}\tau_2(t_{n-2}),$$

where a_{ij} are the elements of $\mathbf{A}$.

When two pairs of operators are known, the components (7) are simply obtained by Eq. (6) and then replaced in Eq. (10). The essential advantage of Eq. (10) is that it includes both types of the errors, caused by homogeneous and nonhomogeneous operators. The corresponding expression in [2] does not contain the second type errors. The present expression allows to consider both types of the error simultaneously or separately.

Error estimation for two integrators

Two implicit one step DIM [5] and [6] will be discussed in order to illustrate application of Eq. (10). Both methods are unconditionally stable and do not produce overshooting effects when h grows infinitely. If no dumping is included, approximate and exact equilibrium equations are presented in the form:

$$(11)\qquad a_{n+1}+(1-\alpha)\omega^2 d_{n+1}+\alpha\omega^2 d_n=p_{n+1};$$

$$(12)\qquad a(t)+\omega^2 . d(t)=p(t),$$

where ω is natural frequency, α is numerical coefficient, introduced by Hilber [2] to improve accuracy and dissipative ability.

Integration operators have the structure:

$$\mathbf{A}=[a_{ij}]=\frac{1}{D}\begin{bmatrix} 1-\alpha b\Omega^2 & 1 & a \\ -d\Omega^2 & 1+(1-\alpha)(b-d)\Omega^2 & c \\ -\Omega^2 & -(1-\alpha)\Omega^2 & -(1-\alpha)a\Omega^2 \end{bmatrix};\quad \mathbf{G}=[g_{ij}]=\frac{1}{D}\begin{bmatrix} 0 & bh^2 \\ 0 & dh^2 \\ 0 & h^2 \end{bmatrix},$$

where (13) $D=1+(1-\alpha)b\Omega^2$, $\Omega=\omega h$ for the first [5] method and

$$\mathbf{A}=[a_{ij}]=\frac{1}{D}\begin{bmatrix} 1-\alpha bd\Omega^2 & a+b & b.c \\ -d\Omega^2 & 1+(1-\alpha)ad\Omega^2 & c \\ -\Omega^2 & -(1-\alpha)(a+b)\Omega^2 & -(1-\alpha)bc\Omega^2 \end{bmatrix};\quad \mathbf{G}=[g_{ij}]=\frac{1}{D}\begin{bmatrix} 0 & bdh^2 \\ 0 & dh^2 \\ 0 & h^2 \end{bmatrix}$$

$$(14)\qquad D=1+(1-\alpha)bd\Omega^2,\quad \Omega=\omega h$$

for the second [6] method.

For the exact solution these operators are:

$$\mathbf{R}=[r_{ij}]=\begin{bmatrix} \frac{1}{\omega^2}\left(\frac{\sin\Omega}{\Omega}-\cos\Omega\right) & \frac{1}{\omega^2}\left(1-\frac{\sin\Omega}{\Omega}\right) \\ \frac{h}{\omega}\left(\frac{\cos\Omega-1}{\Omega}+\sin\Omega\right) & \frac{h}{\omega}\left(\frac{1-\cos\Omega}{\Omega}\right) \\ h^2\left(\cos\Omega-\frac{\sin\Omega}{\Omega}\right) & h^2\left(\frac{1}{\Omega}\sin\Omega\right) \end{bmatrix},$$

$$\mathbf{E}=[e_{ij}]=\begin{bmatrix} \cos\Omega & \frac{1}{\Omega}\sin\Omega & 0 \\ -\Omega\sin\Omega & \cos\Omega & 0 \\ -\Omega^2\cos\Omega & -\Omega\sin\Omega & 0 \end{bmatrix}.$$

Finally, for both methods SNC contains two free coefficients which need previously to be fixed:

$$(16)\qquad \left|\begin{array}{ll} a=\frac{1}{4}(1+\alpha)(1-\alpha-\zeta) & b=\frac{1}{4}(1+\alpha)(1+\alpha+\zeta), \\ c=\frac{1}{2}(1-\alpha-\zeta) & d=\frac{1}{2}(1+\alpha+\zeta) \end{array}\right.$$

for the first method and

$$(17)\qquad \left|\begin{array}{ll} a=\frac{1}{2}(1-\alpha-\zeta) & b=\frac{1}{2}(1+\alpha+\zeta), \\ c=\frac{1}{2}(1-\alpha-\zeta) & d=\frac{1}{2}(1+\alpha+\zeta) \end{array}\right.$$

for the second method.

It is established [5], [6] that greater values for α and ζ will cause stronger numerical dissipation and filtering effect on high and low frequencies, respectively. However, the behaviour of the error τ^d need to be cleared in view of input signal.

Taking into account (12), the 'i'-th component of (10) is transformed to give

$$(18)\qquad \tau_i(t_n)=[(e_{i1}-\Omega^2 e_{i3})-(a_{i1}-\Omega^2 a_{i3})]d(t_n)+(e_{i2}-a_{i2})\cdot hv(t_n) + \left[(r_{i1}-g_{i1})+\frac{\Omega^2}{\omega^2}(e_{i3}-a_{i3})\right]p_n+(r_{i2}-g_{i2})p_{n+1},\quad (i=1,\ 2,\ 3).$$

When h tends to zero, from (10) one can write:

$$(19)\qquad \lim_{h\to 0}\tau^d(t_n)=\frac{1}{2}\zeta\,.\,\omega\,.\,\Omega\,.\,v(t_n)+c_{-1}\cdot p_{n-1}+c_0\cdot p_n+c_1\cdot p_{n+1}+o(\Omega^2)$$

for the first method and

$$(20)\qquad \lim_{h\to 0}\tau^d(t_n)=\zeta\,.\,\omega\,.\,\Omega\,.\,v(t_n)+c_{-1}\cdot p_{n-1}+c_0\cdot p_n+c_1\cdot p_{n+1}+o(\Omega^2)$$

for the second method.

For the sake of simplicity the following auxiliary expressions are introduced:

(21) $$c_{-1}=a-c+\frac{1}{6}; \quad c_0=b-a-d+\frac{2}{3}; \quad c_1=\frac{1}{6}-b$$

for the first method and

(22) $$c_{-1}=bc-c+\frac{1}{6}; \quad c_0=bd-bc-d+\frac{2}{3}; \quad c_1=\frac{1}{6}-bd$$

for the second method.

Let's consider the second part of the error. Using the notation θ^d for this error the following expression is available:

(23) $$\vartheta^d(t)=c_{-1}\,.\,p(t-h)+c_0\,.\,p(t)+c_1\,.\,p(t+h)$$

After applying Laplace transform in both sides of Eq. (23) the result is:

$$\bar{\vartheta}^d(s)=(c_{-1}\,.\,e^{-sh}+c_0+c_1\,.\,e^{sh})\,.\,\bar{p}(s)=H(s)\cdot\bar{p}(s),$$

where

(24) $$\bar{p}(s)=\int_0^\infty p(t)\,.\,e^{-st}\,.\,dt, \quad \bar{\vartheta}^d(s)=\int_0^\infty \vartheta^d(t)\,.\,e^{-st}\,.\,dt,$$

$$s=j\,.\,\varphi, \quad j=\sqrt{-1}.$$

Note that s is a complex variable but φ is the frequency of input signal. Let's define dimensionless frequency putting;

(25) $$\lambda=\varphi\,.\,h$$

Then $H(\lambda)$ is a transfer function which transforms directly input signal into numerical error. The region of λ should be specified in accordance with the folding frequency φ_N thus:

(26) $$\max\lambda\leq h\varphi_N=\pi, \quad \varphi_N=\frac{\pi}{h}\cdot$$

Since $|H(\lambda)|$ is a real measure for the error, its behaviour will be studied as a function of λ, α and ζ. For both methods the following relations hold:

$$|H(\lambda)|=[(c_{-1}+c_1)^2(1-\cos\lambda)^2+(c_{-1}-c_1)^2\sin^2\lambda]^{1/2}$$

and also

(28) $$c_{-1}+c_1=-\frac{1}{2}\left(\alpha^2+\alpha\zeta+\frac{1}{3}\right), \quad c_{-1}+c_1=-\frac{1}{2}\left[(\alpha+\zeta)^2+\frac{1}{3}\right]$$

$$c_{-1}-c_1=\alpha+\frac{1}{2}\zeta, \quad c_{-1}-c_1=\alpha+\zeta$$

(first method); (second method)

Results and conclusions

In order to illustrate the correlation between $|H(\lambda)|$, and SNC a number of curves are studied using different SNC. The results are presented graphically in Fig. 1.

Shaded area for both methods shows the region occupied by the family of curves having different α and ζ, starting from $\alpha=\zeta=0$ (trapezoidal rule). It is assumed, that the largest value of ζ is 0.10, whereas α can reach 0.330. However, the maximum recommended value for α is 0.200 [5], [6]. The final conclusions can be summarized as follows:

1. The smallest error can be attained by the use of trapezoidal rule ($\alpha=\zeta=0$). In this case the error has a second order $o(h^2)$.

2. When $\zeta=0$ and $\alpha\neq0$ both methods produce identical results for the error. The difference appears when $\zeta\neq0$.

3. For equal ζ, employed in both methods, the error of the second method is stronger influenced. Thus one should take much more care for large ζ. Otherwise, numerical response will significantly be affected. It is obvious that the second method possesses stronger numerical dissipation.

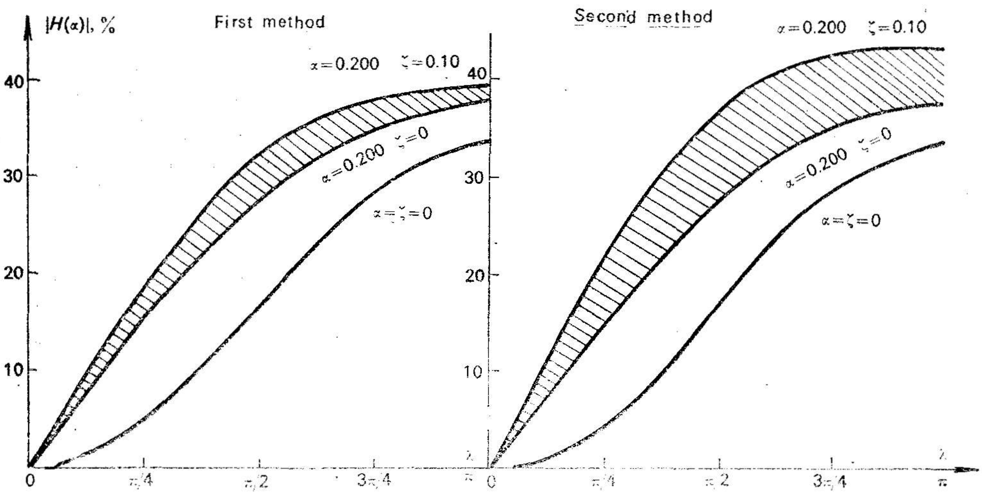

Fig. 1

References

1. Computational Methods for Transient Analysis. (Eds T. Belitschko, T. J. Hughes) North Holland, Elsevier Sci. Publ. B. V., 1983.
2. Hilbert, H. M. Analysis and design of numerical integration methods in structural dynamics (dissertation). Report No EERG 76-29, Berkeley, California, November, 1976.
3. Petkov, Z. B., S. G. Ganchev. Notes on accuracy analysis of forced vibrations in structural dynamics. — In: Trans. of the 9th Int. Conf. on Struct. Mech. in Reactor Techn. (Lausane 19-21 August, 1987).
4. Preumont, A. Frequency domain analysis of time integration operators. — Earth. Eng. Struct. Dyn., **10**, 1982, 691-697.
5. Симеонов, С. В. Върху един дискретен числен метод в строителната динамика. — Теор. прил. мех., **18**, 1987, № 2.
6. Id. Върху един числен метод в динамиката на деформируемите системи. — Ibid., **19**, 1988, № 1.

Received on December 12, 1988