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P. Simeonov
D. Bainov

Theory of Lyapunov's Characteristic Exponents for Impulsive Differential Systems II. Inequalities of Wazewski and Lyapunov for Impulsive Systems. Reducible Systems

1. Introduction

In the present paper we develop our consideration from [1] of questions on the theory of the characteristic exponents for the linear impulsive system solutions.

$$
\begin{aligned}
&\frac{dx}{dt}=A(t)x, \quad t\neq\tau_k,\\
&\Delta x=B_k x, \quad t=\tau_k, \quad k=1, 2, \dots
\end{aligned}
\tag{1}
$$

As in [1] assume the following conditions (A) fulfilled:

A1. $B_k \in \mathbf{C}^{n\times n}$, $\det(E+B_k)\neq 0$, $(k=1, 2, \dots)$.

A2. $\tau_0=0<\tau_1<\tau_2<\dots$, $\lim\limits_{k\to\infty}\tau_k=+\infty$.

A3. The matrix-valued function A: $\mathbf{R}_+\to\mathbf{C}^{n\times n}$ is continuous for $t\in\mathbf{R}_+$, $t\neq\tau_k$, and for $t=\tau_k$ it has a discontinuity of the first kind and is continuous from the left.

We shall use the following notation: $\mathbf{R}_+=[0, +\infty)$; $\mathbf{C}^n$ is an n-dimensional complex Euclidean space with elements $x=\operatorname{col}(x_1, \dots, x_n)$, scalar product $(x\,|\,y)=\sum\limits_{k=1}^{n} x_k y_k$ of the elements x, y and norm $|x|=(x\,|\,x)^{1/2}$; $\mathbf{C}^{n\times n}$ is the space of $(n\times n)$-matrices $A=(a_{ij})$ with entries $a_{ij}\in\mathbf{C}$; $\|A\|=\sup\{|Ax|: x\in\mathbf{C}^n, |x|=1\}$ is the norm of the matrix $A\in\mathbf{C}^{n\times n}$; E is the unit $(n\times n)$-matrix; $A^*=(a^*_{ji})$ is the conjugate matrix of the matrix $A=(a_{ij})$; $\operatorname{Tr}A=\sum\limits_{k=1}^{n} a_{kk}$ is the trace of the matrix $A=(a_{ij})$.

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2. Wazevski's inequality for impulsive systems

Let $\Lambda(t)$ and $\lambda(t)$ be the greatest and the smallest eigenvalues of the matrix $A^H(t)=\frac{1}{2}(A(t)+A^*(t))$ and let Λ_k^2 and λ_k^2 be the greatest and the smallest eigenvalues of the matrix $(E+B_k)^*(E+B_k)$, $(k=1, 2, \dots)$.

We shall formulate an analogue of Wazewski's inequality for impulsive systems.

Theorem 1 [2]. Let conditions (A) hold. Then for any $t_0 \in R_+$, $x_0 \in C^n$ and $t>t_0$ the following inequality holds

$$|x_0| \prod_{t_0<\tau_k<t} \lambda_k \exp\left(\int_{t_0}^{t} \lambda(s)ds\right) \leq |x(t; t_0, x_0)| \leq |x_0| \prod_{t_0<\tau_k<t} \Lambda_k \exp\left(\int_{t_0}^{t} \Lambda(s)\, ds\right).$$

Corollary 1. The extended spectrum $\bar{\omega}$ of system (1) is contained in the interval $[g, G]$, where

$$g=\chi\left[\prod_{\tau_k<t} \lambda_k \exp\left(\int_0^t \lambda(s)ds\right)\right], \quad G=\chi\left[\prod_{\tau_k<t} \Lambda_k \exp\left(\int_0^t \Lambda(s)\, ds\right)\right].$$

Corollary 2. If $G<0$, then system (1) is asymptotically stable.

3. Lyapunov's inequality for impulsive systems

Let $D_k=\det(E+B_k)$, $(k=1, 2, \dots)$.

Theorem 2. Let conditions (A) hold and let the spectrum of system (1) be finite. Then Lyapunov's inequality is valid

$$\sigma_x \geq \limsup_{t\to+\infty} \frac{1}{t} \ln\left(\prod_{\tau_k<t} |D_k| \exp\left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s)ds\right)\right). \tag{2}$$

Proof. Let $X(t)=\{x_1(t), \dots, x_n(t)\}$ be an arbitrary fundamental matrix of system (1) and let σ_x be the sum of the characteristic exponents of the basis $x_1, \dots, x_n$ [1], i. e.

$$\sigma_x=\sum_{k=1}^{n} \chi[x_k].$$

By formula (3) of [1]

$$X(t)=U_{k+1}(t, \tau_k) \prod_{j=k}^{1} (E+B_j)U_j(\tau_j, \tau_{j-1})X(O) \tag{3}$$

for $\tau_k<t\leq\tau_{k+1}$, where $U_k(t, s)$ is the Cauchy matrix of the system

$$\frac{dx}{dt}=A(t)x, \quad (\tau_{k-1}<t\leq\tau_k).$$

In view of Liouville's equality

$$\det U_j(\tau_j, \tau_{j-1}) = \exp\left(\int\limits_{\tau_{j-1}}^{\tau_j} \operatorname{Tr} A(s)ds\right)$$

from (3) we obtain that

$$\det X(t) = \det X(O) \prod_{\tau_k<t} D_k \exp\left(\int\limits_0^t \operatorname{Tr} A(s)\, ds\right). \tag{4}$$

By arguments as in [3], II, § 7 we conclude that

$$\chi[\det X(t)] \leq \sum_{k=1}^{n} \chi\,[x_k] = \sigma_x. \tag{5}$$

From (4) and (5), (2) immediately follows.

Theorem 2 is proved.

Theorem 3. Let conditions (A) hold, let the spectrum of system (1) be finite and for the basis

$$x_1, \ldots, x_n \tag{6}$$

et Lyapunov's equality be valid:

$$\sigma_x = \limsup_{t\to+\infty} \frac{1}{t} \ln\left(\prod_{\tau_k<t} |D_k| \exp\left(\int\limits_0^t \operatorname{Re}\operatorname{Tr} A(s)ds\right)\right).$$

Then the basis (6) is normal.

Proof. If we suppose that the basis (6) is not normal, then by [1] (Theorem 4), there exists a normal basis $z_1, \ldots, z_n$ for which

$$\sigma_z < \sigma_x = \limsup_{t\to+\infty} \frac{1}{t} \ln\left(\prod_{\tau_k<t} |D_k| \exp\left(\int\limits_0^t \operatorname{Re}\operatorname{Tr} A(s)\, ds\right)\right),$$

which contradicts Lyapunov's inequality (2).

Theorem 3 is proved.

4. Reducible systems

Definition 1. The matrix-valued function $L\colon \mathbf{R}_+ \to \mathbf{C}^{n\times n}$ is said to be Lyapunov's matrix if:

1. $L(t)$ is continuously differentiable for $t \in \mathbf{R}_+$, $t \neq \tau_k$ and for $t = \tau_k$ the derivative $\frac{dL}{dt}$ has a discontinuity of the first kind and is continuous from the left.

2. $L(t)$ and $\frac{dL}{dt}$ are bounded in $\mathbf{R}_+$.

3. $\inf\limits_{t\in\mathbf{R}_+} |\det L(t)| > 0.$

Remark 1. As in [3] we can prove that if $L(t)$ is Lyapunov's matrix in the sense of Definition 1, then its inverse matrix $L^{-1}(t)$ is also Lyapunov's matrix.

Definition 2. The linear transformation

$$y=L(t)x \tag{7}$$

where $L(t)$ is Lyapunov's matrix, $x\in \mathbf{C}^n$ and $y\in \mathbf{C}^n$, is said to be Lyapunov's transformation.

Lemma 1. Under Lyapunov's transformation (7) carried out in the linear system (1) the characteristic exponents of its solutions are preserved.

Lemma 1 is proved as in [3].

Definition 3. System (1) is said to be reducible if by a Lyapunov's transformation it can be reduced to a system of the form

$$\frac{dy}{dt}=Ay, \tag{8}$$

where $A\in \mathbf{C}^{n\times n}$.

Theorem 4. Let conditions (A) hold.

Then system (1) is reducible if and only if some of its fundamental matrices can be represented in the form

$$X(t)=L(t)e^{At} \tag{9}$$

where $L(t)$ is Lyapunov's matrix and $A\in \mathbf{C}^{n\times n}$.

Proof. Necessity. By means of Lyapunov's transformation $x=L(t)y$ let system (1) be reduced to system (8). Then the normalized fundamental matrix $Y(t)$ of system (8) has the form

$$Y(t)=e^{At}$$

Hence,

$$X(t)=L(t)Y(t)=L(t)e^{At}.$$

Sufficiency. Let the fundamental matrix $X(t)$ of system (1) have the form (9). Then

$$L(t)=X(t)e^{-At},$$

$$L^{-1}(t)X(t)=e^{At}.$$

We carry out the transformation $x=L(t)y$. Let $x(t)$ be a solution of system (1). Then the function

$$y(t)=L^{-1}(t)x(t)=L^{-1}(t)X(t)X^{-1}(0)x(0)=e^{At}X^{-1}(0)x(0)$$

is a solution of system (8). Theorem 4 is proved.

Theorem 5. Let the following conditions be fulfilled:

1. Conditions (A) hold.
2. The matrix $A(t)$ and all solutions of system (1) are bounded in $\mathbf{R}_+$.
3. $\prod\limits_{\tau_k<t} |D_k| \exp\left(\int\limits_0^t \operatorname{Re}\operatorname{Tr} A(s)\,ds\right)\geq a>0.$

Then system (1) is reducible to the system

$$\frac{dy}{dt}=0. \tag{10}$$

Proof. Let $X(t)$ be a fundamental matrix of system (1). Then from condition 2 of Theorem 5 it follows that

$$\|X(t)\|\leq N_1,\quad \|\frac{dX}{dt}(t)\|\leq \|A(t)\|\ \|X(t)\|\leq N_2$$

where $N_1>0$ and $N_2>0$ are constants.

By condition 3 of Theorem 5 we have that

$$|\det X(t)|=|\det X(0)|\prod_{\tau_k<t}|D_k|\exp\left(\int_0^t \operatorname{Re}\operatorname{Tr}A(s)ds\right)\geq|\det X(0)|\,a>0.$$

Hence, $X(t)$ is Lyapunov's matrix and by the transformation $x=X(t)y$ system (1) is transformed into system (10).

Theorem 5 is proved.

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