BULGARIAN ACADEMY OF SCIENCES
JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
Sofia, 1991, Year XXII, No 2

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# On Planar Oscillations of Gyrostat on Elliptic Orbit

## 1. Introduction

We consider the motion of gyrostat around its centre of mass $O$ in Newtonian force field. We assume that rotors are aligned along satellite principal axes and mass centre $O$ follows the elliptic orbit aroung attracting centre $O_1$ (Fig. 1) and

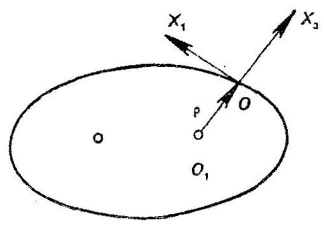


Fig. 1

(1)
$$\rho=\frac{P}{1+e\cos\nu},\quad \frac{d\nu}{dt}=\omega_0=\sqrt{\frac{\mu}{P}}(1+e\cos\nu)^2=\frac{\sqrt{\mu P}}{\rho^2},$$
$$\dot{\omega}_0=-2e\sin\nu\,.\,\mu/\rho^3,\quad (\dot{}=d/dt),$$

where $\rho=O_1O$ is the polar radius vector, $\nu$ — the true anomaly, $P$ — the parameter of orbit, $e$ — the eccentricity, $\mu=Mf$, $f$ gravity constant and $M$ is the mass of the attracting centre.

Except the inertial reference frame $O_1\vec{\xi}\vec{\eta}\vec{\zeta}$, we consider also two reference frames (with origin in mass centre $O$) — a moving reference frame $Ox_1x_2x_3$ with axes along satellite principal central inertia axes and an orbital reference frame $OX_1X_2X_3$, which is introduced as follows: $OX_3$ is along radius vector of the orbit, axes $OX_1$, $OX_2$ are along transfersal and normal to the orbit, respectively.

In the restricted case of the problem, equations of motion around mass centre are [1]:

(2)
$$\mathbf{A}\dot{\vec{\omega}}+\dot{\vec{\mathbf{k}}}+\vec{\omega}\times\mathbf{A}\,.\,\vec{\omega}+\vec{\omega}\times\vec{\mathbf{k}}=\frac{3\mu}{\rho^3}\mathbf{X}_3\times\mathbf{A}\,.\,\mathbf{X}_3;$$

(3) $$\dot{\mathbf{k}}+\mathbf{I}\dot{\vec{\omega}}=\mathbf{m};$$

(4) $$\dot{\mathbf{X}}_1=\mathbf{X}_1\times\vec{\omega}-\omega_0\mathbf{X}_3,\quad \dot{\mathbf{X}}_2=\mathbf{X}_2\times\vec{\omega},\quad \dot{\mathbf{X}}_3=\mathbf{X}_3\times\vec{\omega}+\omega_0\mathbf{X}_1,$$

where $\vec{\omega}$ is the angular velocity of gyrostat, $\mathbf{k}$ — gyrostatic torque, $\mathbf{m}$ — controlling torque applied to the rotors. The inertia tensor $\mathbf{A}$ about satellite mass centre in the moving reference frame takes from $\mathbf{A}=\mathrm{diag}(A_1, A_2, A_3)$ and the inertia tensor of rotors $\mathbf{I}=\mathrm{diag}\,(I_1, I_2, I_3)$, respectively.

Eliminating $\dot{\mathbf{k}}$ from (2) and (3) we have

(5) $$\mathbf{B}\dot{\vec{\omega}}+\mathbf{m}+\vec{\omega}\times\mathbf{A}.\vec{\omega}+\vec{\omega}\times\mathbf{k}=\frac{3\mu}{\rho^3}\mathbf{X}_3\times\mathbf{A}.\mathbf{X}_3,$$

where $\mathbf{B}=\mathbf{A}-\mathbf{I}$.

It is shown in [1], that availability of rotors in satellite implies existence of relative equilibria positions i. e. positions in which satellite is fixed in orbital frame, and $\mathbf{k}$ and $\mathbf{m}$ are calculated for each equilibria (control via rotors).

Let us consider the solution of equation (4), (5) given in [1] in which axes of orbital and moving frames coincide

(6) $$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad \begin{aligned}\omega_1&=0\\ \omega_2&=\omega_0\\ \omega_3&=0;\end{aligned}$$

(7) $$\begin{aligned}m_1&=0 & k_1&=0\\ m_2&=-\dot{\omega}_0(A_2-I_2) & k_2&=\Lambda_0-\omega_0A_2\\ m_3&=0 & k_3&=0\quad(\Lambda_0=\mathrm{const.}).\end{aligned}$$

We fix this regime of rotors (7), that supports equilibria (6). Suppose that there is a perturbation in the plane of orbit $(O\mathbf{X}_1\mathbf{X}_3)$ which corresponds to a certain turning around $Ox_2$ $(O\mathbf{X}_2)$ in angle $\alpha$ (Fig. 2). The components of angular velocity and vectors of orbital frame in moving reference frame are

(8) $$\begin{aligned}\omega_1&=0\\ \omega_2&=\omega_0+\dot{\alpha}\\ \omega_3&=0\end{aligned}\qquad \begin{bmatrix}\cos\alpha&0&-\sin\alpha\\0&1&0\\ \sin\alpha&0&\cos\alpha\end{bmatrix}$$

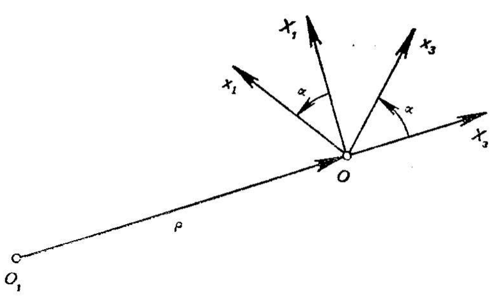


Fig. 2

After substituting (7) and (8) in (5) a single scalar equation is obtained

$$(A_2-I_2)\ddot{\alpha}=-\frac{3\mu}{\rho^3}(A_1-A_3)\sin\alpha\cos\alpha.$$

Denoting $n=3(A_1-A_3)/(A_2-I_2)$, $2\alpha=\theta$ and having in mind that $d/dt=\omega_0 d/d\nu$, the equation for the planar oscillations is reached

(9) $$\theta''-\frac{2e\sin\nu}{1+e\cos\nu}\theta'+n\frac{\sin\theta}{1+e\cos\nu}=0. \quad \left('=\frac{d}{d\nu}\right).$$

It may be also presented in Hamiltonian form

(10) $$\frac{dp}{d\nu}=-\frac{\partial H}{\partial\theta},\quad \frac{d\theta}{d\nu}=\frac{\partial H}{\partial p},$$

where $H=\frac{1}{2}\cdot\frac{p^2}{(1+e\cos\nu)^2}-(1+e\cos\nu)\,n\cos\theta$.

For comparison, the equation of planar oscillations of rigid satellite [2] is also given:

(11) $$\theta''-\frac{2e\sin\nu}{1+e\cos\nu}\theta'+\frac{3(A_1-A_3)}{A_2}\cdot\frac{\sin\theta}{1+e\cos\nu}=\frac{4e\sin\nu}{1+e\cos\nu}.$$

Therefore, the presence of rotors in the satellite (only one of them is active in this case) suppress the member on the right side of (11) and makes it homogeneous. It can be also seen that the inertia moment $I_2$ of the rotor is included in $n$, so this is an additional opportunity to change it.

## 2. Quasi-harmonic oscillations

In this paragraph we assume that $e\ll 1$ and $|\sin\theta-\theta|$ is a small value i. e. the oscillations do not differ much from linear ones. These oscillations are called quasi-harmonic [2] and there they are studied via averaging methods for equation (11). Here, we apply the same methods and algorithms as in [2] for studying quasy-harmonic oscillations of equation (9). Denoting $n=m^2$ (9) is transformed into the following form

(12) $$\theta''+m^2\theta=e[2\sin\nu\,.\,\theta'-\cos\nu\,.\,\theta'']+m^2(\theta-\sin\theta)=f(\nu,\theta,\theta',\theta'').$$

According to our assumptions the value

(13) $$f=e[2\sin\nu\,.\,\theta'-\cos\nu\,.\,\theta'']+m^2(\theta-\sin\theta)$$

is small and nevertheless the small parameter is implicit, average methods may be used. The following formulas introduce new variables:

(14) $$\theta=a\cos\psi,\quad \theta'=-am\sin\psi,\quad \psi=\varphi+m\nu.$$

After certain calculations, the result is the system

(15) $$\frac{da}{d\nu}=-\frac{1}{m}f\sin\psi,\qquad \frac{d\varphi}{d\nu}=-\frac{1}{ma}f\cos\psi,$$

which is not in normal form, because of the existence of the term $\theta''$ in $f$. But since we restrict the calculations to the first approximation, $\theta''=-am^2\cos\psi$ is taken. Denoting $f_0=f(\nu,\ a\cos\psi,\ -am\sin\psi,\ -am^2\cos\psi)$ the system (15) is presented in a normal form.

$$\frac{da}{d\nu} = -\frac{1}{m} f_0 \sin\psi,$$

(16)
$$\frac{d\varphi}{d\nu} = -\frac{1}{ma} f_0 \cos\psi,$$

$$d\nu/d\nu = 1.$$

A system with two fast variables ($\varphi$, $\nu$) and one slow $a$ is obtained. In the absence of resonance between the fast variables, an average of them is taken and in first approximation we have

$$\frac{da}{d\nu} = -\frac{1}{4\pi^2 m}\int_0^{2\pi}\int_0^{2\pi} f_0 \sin\psi \, d\psi \, d\nu = 0$$

or $a = a_0 = \text{const}$. The second equation gives $\psi = \psi(a)$ which means nonisochronity. If resonance is considered

(17)
$$\frac{d\psi}{d\nu} - \frac{d\nu}{d\nu} = 0$$

a new variable $\varkappa = \psi - \nu$ ($\psi = \varkappa + \nu$) is introduced in the neighbourhood of the resonance (17). Then the system (16) takes the form

$$\frac{da}{d\nu} = -\frac{1}{m} f_0 \sin(\varkappa + \nu),$$

(18)
$$\frac{d\varkappa}{d\nu} = m - 1 - \frac{1}{ma} f_0 \cos(\varkappa + \nu),$$

$$\frac{d\nu}{d\nu} = 1.$$

The system (18) already has two slow variables $a$ and $\varkappa$ (according to our assumption $m-1$ has the same order as $f_0$). After averaging by $\nu$, we have

$$\frac{da}{d\nu} = -\frac{1}{2\pi m}\int_0^{2\pi} f_0 \sin(\varkappa + \nu)\, d\nu = 0$$

i. e. again $a = a_0 = \text{const}$. Thus, no matter if we have resonance or not, quasi-harmonic oscillations in first approximation of (9) do not differ from linear. This result is natural because availability of one active rotor suppresses member $4e \sin\nu/(1 + e\cos\nu)$ of (11) which is responsible for all effects in resonance case.

### 3. Stability

After linearising the equation (9), change of variables $\theta = \gamma/(1 + e\cos\nu)$ is performed. Then the equation (9) obtains form

(19)
$$\gamma'' + g(\nu)\gamma = 0,$$

where $g(\nu) = (n + e\cos\nu)/(1 + e\cos\nu)$, $g(\nu + 2\pi) = g(\nu)$. Therefore, (19) is Hill equation. It is equivalent to the system

(19′)
$$z_1' = z_2, \quad z_2' = -g(\nu) z_1.$$

Sufficient condition of unstability is $g(\mathrm{v}) \leq 0$ $(g(\mathrm{v}) \neq 0)$, which gives $n+e\cos \mathrm{v} \leq 0$. Expressing condition for the maximum of left side function to be $\leq 0$, we obtain

$$n+e \leq 0 \quad \text{or} \quad \frac{3(A_1-A_3)}{(A_2-I_2)}+e \leq 0. \tag{20}$$

When $e=0$, (20) converts in the sufficient condition for unstability of trivial solution of small oscillations in circular orbit i. e. $A_1 \leq A_3$. For sufficient condition for stability is used Liapunov's integral criterion

$$g(\mathrm{v}) \geq 0, \quad 0 \leq 2\pi \int_0^{2\pi} g(\mathrm{v})\, d\mathrm{v} \leq 4.$$

Both conditions give

$$e \leq n \leq 1-\frac{\pi^2-1}{\pi^2}\sqrt{1-e^2}, \quad \text{where}$$

$$e<e^* = \frac{\pi^4-(\pi^2-1)^2}{\pi^4+(\pi^2-1)^2} = 0.1064 \ldots \qquad \text{(Fig. 3)} \tag{21}$$

One may say that this picture is not satisfactory with respect to stability, because it is known that every point from $(n, 0)$ $n>0$ corresponds to a stable system. Here we use the assumptions from the previous paragraph: $n=m^2$ and $e \ll 1$. Our purpose further is to consider in the plane of parameters $(m, e)$ domains of dynamical unstability which approach $(m, 0)$. It may be done in one of the following ways: to express condition $|\mathrm{tr} A|=2$ which gives the boundaries of unstability domains (here $A$ is the matrix of the mapping for period for system (19')) [3] or direct computation of these boundaries in first and second approximation [4]. As the trace of $A$ is calculated explicitly in special cases and equation $|\mathrm{tr}\, A|=2$ is solved approximately, we use se-

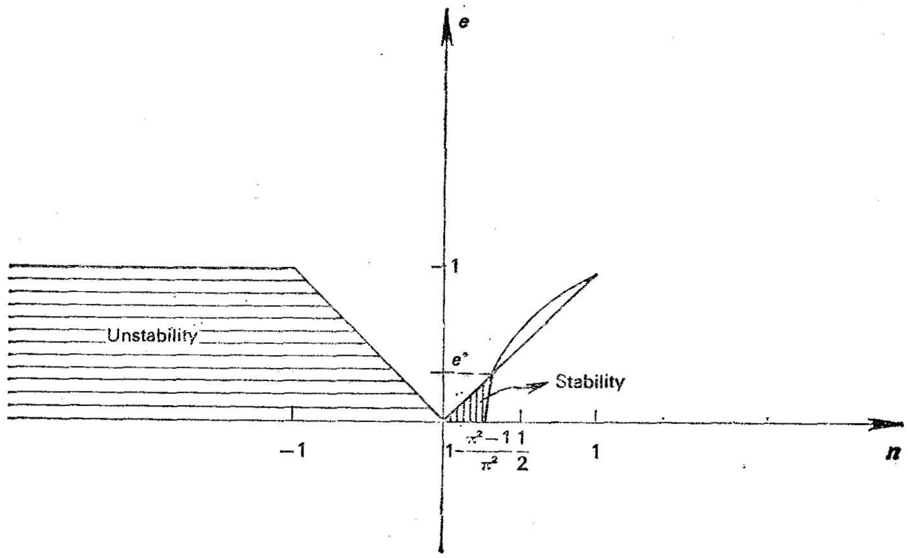


Fig. 3

cond approach and restrict calculations with first approximation of the boundaries of unstable regions.

The system (19') takes form

(22) $$\mathbf{z}' = [C + eB + \cdots]\,\mathbf{z},$$

where $\mathbf{z} = (z_1, z_2)^T$ and $C = \begin{bmatrix} 0 & 1 \\ -m^2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ (m^2 - 1)\cos\nu & 0 \end{bmatrix}$.

The eigenvalues of $C$ are $\Lambda_{1,2} = \pm i\omega_1 = \pm im\,(\omega_1 = m)$ and eigenvectors are $\mathbf{c}_1 = (1, im)^T$, $\mathbf{c}_{-1} = \bar{\mathbf{c}}_1$. Resonance values of $m$ are obtained from the condition $2\omega_1 = k$, $k = 0, 1, 2\ldots$ Therefore, when

(23) $$m = k/2 \qquad k = 0,\ 1,\ 2\ \ldots$$

the trivial solution of (22) is unstable (parametric excitation). According to the basic theorem ([4], p. 193) boundaries of unstability regions in first approximation which approach the point $(k/2, 0)$ are given with

$$\frac{k}{2} + \chi_1^{(k)} e + 0(e^2) < m < \frac{k}{2} + \chi_2^{(k)} e + 0(e^2),\ e > 0,$$

where

$$\chi_{1,2}^{(k)} = \frac{-\chi_{-1,-1} - \chi_{1,1} \mp 2\,|\chi_{-1,1}^{(k)}|}{2}$$

and where

$$\chi_{-1,-1} = (B^{(0)}\mathbf{c}_{-1}, \mathbf{c}_{-1})|_{m=k/2},\ \chi_{1,1} = (B^{(0)}\mathbf{c}_1, \mathbf{c}_1)|_{m=k/2}$$
$$\chi_{-1,1}^{(k)} = (B^{(k)}\mathbf{c}_{-1}, \mathbf{c}_1)|_{m=k/2},\quad k = 0,\ 1,\ 2,\ \ldots$$

$B^{(k)}$ are Fourier coefficients for matrix $B(m, \nu)$. As $B^{(k)}$, $k \neq 1$ are nill matrices, $\chi_{-1,-1} = \chi_{1,1} = 0$, $|\chi_{-1,1}^{(1)}| = \frac{3}{8}$, $\chi_{-1,1}^{(k)} = 0$, $k \neq 1$. Therefore,

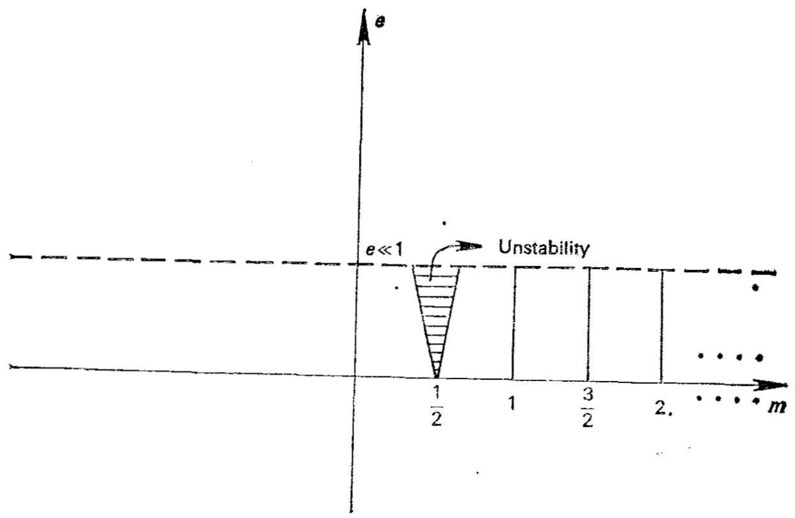


Fig. 4

$$\chi_{1,2}^{(1)} = \pm\frac{3}{8}, \quad \chi_{1,2}^{(k)} = 0 \quad k \neq 1.$$

Thus, in first approximation unstable regions are

$$m = k/2 \quad k = 0,\ 2,\ 3 \ldots$$

(24) $$\frac{1}{2} - \frac{3}{8} e + 0(e^2) < m < \frac{1}{2} + \frac{3}{8} e + 0(e^2). \quad \text{(Fig. 4)}$$

The analysis confirms remarks in [3], that although theoretically parametric excitation is observed in infinately many points $m = k/2$, $k = 0, 1, 2, \ldots$, practically it appears in small $k$ and strongest when $k = 1$. If $k$ is large, unstable regions are very narrow, even not existing at all [4].

## References

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*Received on April 10, 1990*