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Solution Ways of a Classical FSI Problem

Introduction

The problem considered below arises as a first methodical step helping the understanding of more complicated cases of fluid-structure interaction (**FSI**), e. g. the cases of a mutual interaction of some structural components through the liquid in the process of their all together interaction with the liquid [1, 2, 7, 10, 12]. Consider the hydroelastic system (**HES**), shown in Fig. 1. It consists of a vibrating plate (a channel wall) and the inviscid, compressible and heavy liquid flowing along the longitudinal coordinate z. The flow velocity is considerably lower than the sound velocity c in the liquid and that is why the liquid is accepted as stationary one.

Let us suppose that the small plate vibration can be described by function $w(x, z, t)$ of the middle plate surface displacements. Assuming that this function does not depend on z-coordinate, we shall consider cylindrical plate vibrations [11]. So, the problem becomes planar, and along the longitudinal coordinate only unit dimension is taken.

Function $w(x, t)$ has to be an integral of the partial differential equation (**PDE**):

$$(1)\qquad (EI)\, w^{IV} + \bar{m}\ddot{w} = \text{loading},$$

where $w^{IV} = \partial^4 w/\partial x^4$; $\ddot{w} = \partial^2 w/\partial t^2$; $\bar{m}$ — plate mass per 1 m^2; E — young modulus and I — inertia momentum. Additionally w must satisfy the boundary conditions at $x=0$ and $x=h$

$$(2)\qquad x=0,\ h:\ w=(\partial^2 w/\partial x^2)=0.$$

Evidently, the consideration of arbitrary conditions having technological sense is not difficult to be accomplished.

Suppose that the small plate vibrations initiate into the liquid acoustical waves with small amplitudes which can be described by velocity potential function $\varphi(x, y, z; t)$. Again we shall suppose that the problem does not depend on coordinate z, so $\varphi = \varphi(x, y; t)$. This function has to be a solution of the PDE:

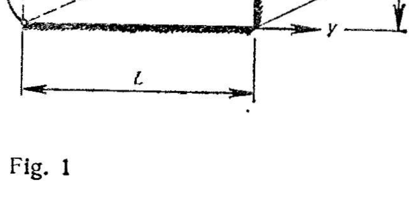

Fig. 1

$$(\partial^2\varphi/\partial x^2)+(\partial^2\varphi/\partial y^2)-(1/c^2)(\partial^2\varphi/\partial t^2)=0. \tag{3}$$

If function $\varphi(x, y; t)$ is known, actual velocity $\vec{v}$ and the acoustical pressure p_a can be determined by following the relations

$$\vec{v}=\operatorname{grad}\varphi,\ p_a=-\rho(\partial\varphi/\partial t), \tag{4}$$

(ρ — liquid density).

Velocity potential function φ has additonaly to satisfy the boundary conditions:

$$(\partial\varphi/\partial x)_{x=0}=0,\quad (\partial\varphi/\partial x)_{x=h}=0\,; \tag{5}$$

$$(\partial\varphi/\partial y)_{y=0}=\dot{w},\quad (\partial\varphi/\partial y)_{x=L}=0. \tag{6}$$

The subordinated problem is to determine the **HES** dynamical characteristics — the natural frequencies and the corresponding vibration modes.

Loadings

In more detailed form Eq. (1) must be:

$$(EI)\,w^{IV}+\bar{m}\ddot{w}=q(x, t)-p_0-p_g+p_a, \tag{7}$$

where:

$q(x, t)$ — external dynamical loading;
$p_0=\text{const}$ — internal initial pressure;
$p_g=-\rho\,g(h-x)$ — gravitational pressure;
$p_a=-\rho(\partial\varphi/\partial t)$ at $y=0$ — acoustical (**FSI**) pressure.

All loadings accounted are shown in Fig. 2. Obviously, if $q(x, t)$ is taken with sign "+", the loadings p_0, p_g and p_a must be taken as negative ones. Mentioning that the acoustical loading has the following character, we shall state that it is directed in all times in direction opposite to the dynamical deflection/displacement $w(x, t)$. The statical components p_0, p_g will not be taken into account.

We shall suppose that at moment $t=t_0$; where $t_0=-\infty$; the external loading has stopped to act, and being conservative, the **HES** has continued to vibrate until the actual moment t. Then

$$(EI)\,w^{IV}+\bar{m}\ddot{w}=-\rho\,(\partial\varphi/\partial t)_{y=0}. \tag{8}$$

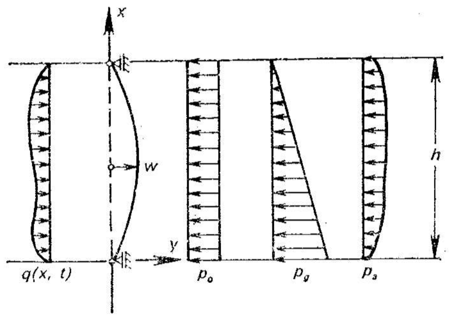

Fig. 2

Free vibrations of the HES components

Let us consider homogeneous boundary problem

(9) $$(EI)\,w_0^{\mathrm{IV}}+\bar{m}\ddot{w}_0=0 \quad x=0,\ h;\quad w_0=\ddot{w}_0=0.$$

The integral can be supposed in the form

(10) $$w_0=W_0(x)\exp(i\sigma t),\quad \text{where}$$

(11) $$W_0(x)=C_1\cos(\Omega x)+C_2\sin(\Omega x)+C_3ch(\Omega x)+C_4sh(\Omega x),$$

Ω being determined by the relation $\Omega^4=\bar{m}\omega^2/(EI)$

Satisfying the boundary conditions, one will obtain the dispersion equation

(12) $$C_2\sin(\Omega h)=0,\quad C_2\neq 0.$$

It provides the chain of transforms

(13) $$\sin(\Omega h)=0,\quad \Omega h=m\pi,\quad m=0,\ 1,\ 2,\ \dots,\ \infty;\quad \Omega_m=m\pi/h$$

which at the end determines the dynamical characteristics, playing role of a wave number in the integral

(14) $$w_0=\sum_m C_m\sin(\Omega_m x)\exp(i\,\omega_m t).$$

The circular frequencies ω_m are namely, the natural frequencies sought. As Ω are already determined, one can determine ω_m as:

(15) $$\omega_m=(m\pi/h)^2\sqrt{(EI/\bar{m})}.$$

Consider the homogeneous boundary problem

(16) $$\Delta\varphi_0-(1/c^2)\,\ddot{\varphi}_0=0;$$

(17) $$x=0,\ h:\ (\partial\varphi_0/\partial x)=0;\quad y=0,\ L:\ (\partial\varphi_0/\partial y)=0.$$

Its solution will be

(18) $$\varphi_0=\sum_r\sum_s i\,\omega_{rs}\varepsilon_{rs}A_{rs}\cos(k_r x)\cos(k_s y)\exp(i\,\omega_{rs}t),$$

where A_{rs} are unknown dimensioning constants and ε_{rs} takes into account the cases when one of the subscripts (r or s) is equal to zero. The wave numbers

(19) $$k_r=(r\pi/h),\quad k_s=(s\pi/L),\quad (r,\ s=0,\ 1,\ 2,\ \dots,\ \infty),$$

together with the circular frequencies ω_{rs} have to satisfy the dispersion condition

(20) $$-(r\pi/h)^2-(s\pi/L)^2+(\omega_{rs}/c)^2=0.$$

Free vibrations of the HES (I)

The problem we must solve now can be described mathematically by the SPDE:

(21) $$(EI)\,w^{\mathrm{IV}}+\bar{m}\ddot{w}=-\rho\,(\partial\varphi/\partial t)_{y=0};$$

(22) $$\Delta\varphi-(1/c^2)\,\ddot{\varphi}=0;$$

(23) $$(\partial\varphi/\partial y)_{y=0}=\dot{w}$$

and the boundary conditions:

(24) $$x=0,\ h:\ w=w''=0 \quad (\partial\varphi/\partial x)=0;$$

(25) $$y=L:\ (\partial\varphi/\partial y)=0.$$

One can presuppose that the functions sought w & φ are representable in the forms:

(26) $$w(x;\ t)=W(x).\sin(\sigma t), \quad \varphi(x,\ y;\ t)=\Phi(x,\ y)\,\sigma\cos(\sigma t)\cdot$$

Putting these expressions into Eqs (21)—(25) we shall obtain:

(27) $$W^{\mathrm{IV}}(x)-(\bar{m}\,\sigma^2/EI)\,W(x)=(\rho\sigma^2/EI).\Phi(x,\ 0);$$

(28) $$\Delta\Phi(x,\ y)+(\sigma^2/c^2)\,\Phi(x,\ y)=0;$$

(29) $$W(x)=(\partial\Phi/\partial y)_{y=0};$$

(30) $$x=0,\ h:\ W(x)=W''(x)=0, \quad (\partial\varphi/\partial x)=0;$$

(31) $$y=L:\ (\partial\Phi/\partial y)=0.$$

Taking

(32) $$\Phi(x,\ y)=X(x).Y(y)$$

one can prove that the solution of Eq. (28) can be expressed in the form:

(33) $$\Phi(x,\ y)=\sum_r{}'\varepsilon_r\,[A_r/\cos(k_rL)]\cos(r\pi x/h)\cos[k_r(L-y)];$$

(34) $$k_r^2+(r\pi/h)^2=(\sigma/c)^2,$$

k_r & σ being unknown.

Eq. (27) now will turn into

(35) $$W^{\mathrm{IV}}-(\bar{m}\,\sigma^2/EI)\,W=(\rho\sigma^2/EI)\sum_r{}'\varepsilon_rA_r\cos(r\pi x/h)$$

with an integral

(36) $$W(x)=W_0(x)+(\rho\sigma^2/EI)\sum_r{}'\varepsilon_r\,(A_r/\beta_r)\cos(r\pi x/h),$$

$W_0(x)$ being taken from Eq. (11), and β_r being determined as

(37) $$\beta_r=(r\pi/h)^4-(\bar{m}\,\sigma^2/EI).$$

Eq. (29) will now provide the relation:

(38) $$W_0(x)+(\rho\sigma^2/EI)\sum_r{}'(A_r/\beta_r)\,\varepsilon_r\cos(r\pi x/h)$$
$$=\sum_r{}'\varepsilon_rA_rk_r\operatorname{tg}(k_rL)\cos(r\pi x/h)$$

which can be transformed into

(39) $$W_0(x)=\sum{}'\varepsilon_rA_r\,[k_r\operatorname{tg}k_rL)-(\rho\sigma^2)\,\beta_rEI)]\cos(r\pi x/h).$$

The constants A_r can be determined in the form

$$A_r=(2/hZ_r)\int_0^h W_0(x)\cos(r\pi x/h)\,dx, \tag{40}$$

where

$$Z_r=k_r\,\mathrm{tg}(k_rL)-(\rho\sigma^2/\beta_r EI). \tag{41}$$

Substituting (41) into (38) one can obtain A_r as a linear function on constants $C_1C_2C_3$ & C_4:

$$A_r=(2/hZ_r)(C_1a_{1r}+C_2a_{2r}+C_3a_{3r}+C_4a_{4r}), \tag{42}$$

where

$$\left\{\begin{aligned} a_{1r}&=\int_0^h\cos(\Omega x)\cos(r\pi x/h)\,dx, & a_{2r}&=\int_0^h\sin(\Omega x)\cos(r\pi x/h)\,dx,\\ a_{3r}&=\int_0^h ch(\Omega x)\cos(r\pi x/h)\,dx, & a_{4r}&=\int_0^h sh(\Omega x)\cos(r\pi x/h)\,dx,\end{aligned}\right. \tag{43}$$

Putting (41) into (36) we shall obtain the vibration mode

$$\begin{aligned} W(x)=C_1&\Big[\cos(\Omega x)+\sum_r(\rho\sigma^2a_{1r}/\beta_rEI)\cos(r\pi x/h)\Big]\\ +C_2&\Big[\sin(\Omega x)+\sum_r(\rho\sigma^2a_{2r}/\beta_rEI)\cos(r\pi x/h)\Big]\\ +C_3&\Big[ch(\Omega x)+\sum_r(\rho\sigma^2a_{3r}/\beta_rEI)\cos(r\pi x/h)\Big]\\ +C_4&\Big[sh(\Omega x)+\sum_r(\rho\sigma^2a_{4r}/\beta_rEI)\cos(r\pi x/h)\Big], \end{aligned} \tag{44}$$

i. e. it is depending on the same four unknown constants, the liquid influence, however, being expressed by the series in the square brackets.

The vibration mode obtained must satisfy the four boundary conditions (30). As they are homogeneous, a linear algebraic system of homogeneous equations will be obtained. The role of unknowns will be performed by the constants C_j, $(j=1, 2, 3, 4)$. Using the matrix mathematics this system can be written down in the form:

$$[b_{ij}]\{C_j\}=0, \quad (i,\ j=1,\ 2,\ 3,\ 4), \tag{45}$$

where $[b_{ij}]$ is a square matrix of the coefficients in front of the unknown constants forming the vector matrix $\{C_j\}$. The equation

$$\mathrm{Det}\,[b_{ij}(k_i,\ \sigma,\ \text{——})]=0 \tag{46}$$

together with the dispersion equation (34) will provide the natural frequencies sought. By the line in Eq. (46) the massive of all geometric and physical parameters is denoted, e. g. $[L,\ h,\ \bar{m},$ etc.].

Obviously the method of Leibenson [9], which was performed in the present paragraph suggests the important concept for the coupling of both dynamical problems (for the plate & for the liquid). In the same time it is cumbersome and difficult for practical use, (this was proved in the time of preparation of Ref. [6]).

Free vibrations of the HES (II)

Following the well known statements, [3, 4, 5, 8] as trial functions we have to use the mode-functions obtained in paragraph 3. However, the solution of the plate free vibration problem is:

$$w_0 = \sum_m C_m \sin(m\pi x/h) \exp(i\omega_m t), \tag{47}$$

but for our needs we shall use trial functions:

$$w_0 = \sum_m C_m \sin(m\pi x/h) \sin(\sigma t). \tag{48}$$

Analogously, the trial function φ is accepted in the form:

$$\varphi = \sum_r \varepsilon_r A_r \frac{\sigma}{\cos(k_r L)} \cos(r\pi x/h) \cos[k_r(L-y)] \cos(\sigma t). \tag{49}$$

Let us first satisfy the kinematical coupling condition:

$$\sum_m C_m \sin(m\pi x/h) = \sum_r \varepsilon_r A_r k_r \operatorname{tg}(k_r L) \cos(r\pi x/h) \tag{50}$$

in which both sides are expanded in series on different functions of x-coordinate, e. g. $\sin(m\pi x/h)$ & $\cos(r\pi x/h)$.

The dynamical connection between the HES components is:

$$(EI)\, w^{\mathrm{IV}} + \bar{m}\ddot{w} = \sum_r \varepsilon_r \rho\sigma^2 A_r \cos(r\pi x/h). \tag{51}$$

Substituting (here) w by (47), Eq. (50) will transform into

$$\sum_m [(EI)(m\pi/h)^4 - \bar{m}\sigma^2]\, C_m \sin(m\pi x/h) = \sum_r \varepsilon_r \rho\sigma^2 A_r \cos(r\pi x/h)\,; \tag{52}$$

$$\sum_m \bar{m}(\omega_m^2 - \sigma^2)\, C_m \sin(m\pi x/h) = \sum_r \varepsilon_r \rho\sigma^2 A_r \cos(r\pi x/h). \tag{53}$$

If one tries to develop L.H.S. of Eqs (50) & (53) in series on sine-functions will obtain the homogeneous system of equations

$$\sum_m C_m [\rho\sigma^2 - \bar{m}(\omega_m^2 - \sigma^2)\, k_r \operatorname{tg}(k_r L)] \,.\, I_{mr} = 0, \quad \text{where} \tag{54}$$

$$I_{mr} = \int_0^h \sin(m\pi x/h) \cos(r\pi x/h)\, dx = \frac{2m[1-(-1)^{m+r}]}{\pi(m^2 - r^2)}, \quad m \neq r. \tag{55}$$

Let is introduce the denotation

$$d_{rm} = [\rho\sigma^2 - \bar{m}(\omega_m^2 - \sigma^2)\, k_r \operatorname{tg}(k_r L)] \,.\, I_{mr} \tag{56}$$

to have a possibility to use the short matrix form

(57) $$[d_{rm}] . \{C_m\} = 0.$$

Again the condition

(58) $$\mathrm{Det}\,[d_{rm}(\sigma,\ k,\ \text{——})] = 0$$

will provide the natural frequencies, but they shall form already the spectrum of the HES. Obviously, the multitude of σ must be arranged by subscripts m, r and the new subscript n.

Table

O	d_{12}	O	d_{14} ...
d_{21}	O	d_{32}	O ...
O	d_{23}	O	d_{34} ...
d_{41}	O	d_{42}	O ...
:	:	:	:

The real calculation of d_{rm} shows that, if $r+m=2j, j=1, 2, \ldots, \infty$

(59) $$d_{rr} = d_{mm} = 0; \quad d_{rm} = 0.$$

if $r+m=2j+1$, then $d_{rm} \neq 0$. So, the matrix $[d_{rm}]$ will have the form of Table 1. The approximations used are underlined. Obviously, the first approximation do not provide any σ. But the use of two approximations provides the equation.

(60) $$d_{12} . d_{21} = [\rho\sigma^2 - \bar{m}(\omega_1^2 - \sigma^2) k_2 \operatorname{tg}(k_2 L)][\rho\sigma^2 - \bar{m}(\omega_2^2 - \sigma^2) k_2 \operatorname{tg}(k_2 L)] = 0$$

and the lowest spectrum of HES natural frequencies is fully determinable.

Numerical investigation

7.1. The purpose of the numerical calculations is to investigate the first natural frequency evolution depending on the geometry variation of the HES, considering two materials for the vibrating wall (steel, aluminium). From the theory it is seen that the HES behaviour can be described by two groups of parameters. The first of them is connected with the geometric dimensions, the second reflects the physical properties of the system. The physical data accepted are:

Water	sound velocity	— $c = 1450$ m/s at 20°C,
	mass density	— $\rho_2 = 1000$ kg/m^3;
Steel plate:	Young modulus	— $E = 2.1.10^{11}$ Pa,
	mass density	— $\rho_1 = 7850$ kg/m^3;
Aluminium plate:	Young modulus	— $E = 7.1.10^{10}$ Pa,
	mass density	— $\rho_1 = 2700$ kg/m^3.

The geometric parameters H and L are taken as variables correspondingly in the intervals $0.2 \leq H \leq 2.0$ m with a step 0,2 m, and $0.5 \leq L \leq 4$ m with a step 0.4 m. The thickness is accepted $\delta = 0.02$ m.

7.2. The numerical investigations (for steel and aluminium materials) are accomplished for the consequent approximations of 2×2, 4×4, 6×6 and 8×8 determinant order and the corresponding dispersion equations. As a matter of fact, the results of the last two approximations are very close. The tendency expected is confirmed — the increasing of H and L decreases the frequency. Additionally, it is seen that channels with large width have practically the same frequencies.

The comparison of the results for steel and aluminium shows that the material physical constants do not significantly influence the frequency — the corresponding curves are very close. On the other hand the frequencies of the HES are nearly 10 times higher than the frequencies of the "dry" beams=plates.

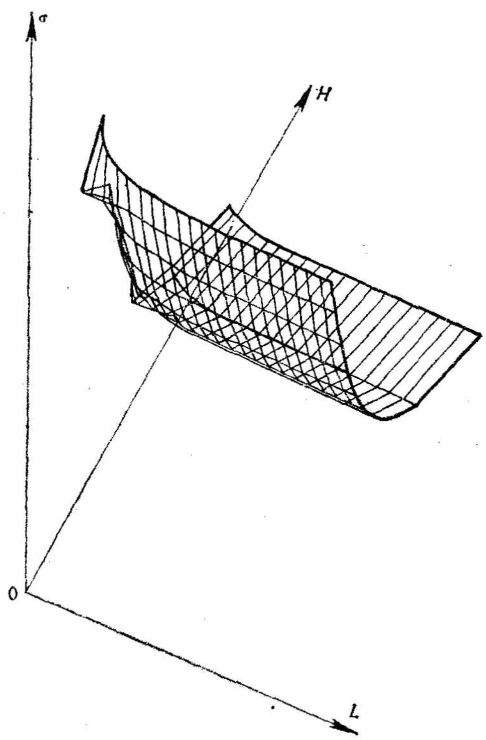

Fig 3.

Fig. 3 shows the dependance of the frequency on the geometrical parameters H and L. The relief performs this dependance with very good clearness.

Conclusions

8.1. The formation of loading members in R.H.S. of Eq. (1) is discussed. The following character of the acoustical loading is underlined.

8.2. Two solution schemes for the problem of HES dynamical characteristics determination are proposed. The analysis pointed out that Bubnov—Galerkin's method is more fruitful.

8.3. The numerical analysis shows that the influence of the mass of the liquid suppresses considerably the influence of the plate mass.

Acknowledgements

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discussions in which J. Horacek (Czechoslovakia), A. Popov (USSR) & J. Antunes (Portugal) have taken part.

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