BULGARIAN ACADEMY OF SCIENCES
JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
Sofia, 1991, Year XXII, No 2

FLUID MECHANICS

A. Yarin

The Collective Effect in Disperse Systems — an Approach Based on the Renormalization Group Technique*

In the present paper consideration is given to the problem of the value of hydrodynamic drag on an individual spherical particle in a system of similar particles in an unbounded fluid flow. The drag value has been calculated using the renormalization group approach for the cases of location of particles in sites of one-, two- and three-dimensional lattices. The Stokesian and inertial (at moderate values of Reynolds number) regimes of flow have been investigated. It is shown that the drag value abruptly decreases in the presence of a large number of particles. The approach outlined makes it possible to tackle the problem of large groups of solid particles and drops motion in a gaseous medium without resorting to ideas of two interpenetrating continua (gas and "gas of particles").

The problem of the value of hydrodynamic drag of a particle surrounded by a large number of other particles in an unbounded (at infinity) flow arises in connection with the motion of clouds of particles, in determining the shape of macromolecular structures from the viscometric data and the like (Happel and Brenner [3]). In similar situations the drag value decreases as a result of the fact that in a collective system particles screen each other, reducing the effective difference of velocities of each of them and of the ambient fluid. For many-particle systems in an unbounded Stokesian flow the drag value was first calculated by Smoluchowski (Happel, Brenner [3]). His result corresponds to the limit of weak interactions.

Consider a system of spheres of equal radius a. In the case of one-dimensional location ($d=1$) the centres of the spheres lie on a straight line Γ and are spaced at l apart. In the cases of two- and three-dimensional ($d=2$ and 3) locations of the spheres their centres fill the sites of square ($L\times L$; L is the length of the side, $L\gg l$) and simple cubic ($L\times L\times L$) lattices. Let us assume for a certainty that the system of spheres is enveloped by a flow with a velocity U in the direction normal to: a straight line Γ ($d=1$); a plane in which the centres of the spheres are located ($d=2$); and one of the faces of the cubic lattice ($d=3$). We also investigate the cases where the flow velocity is directed parallel to: a straight line Γ ($d=1$); and a plane in which the centres of the spheres are located ($d=2$). First consider the case of creeping Stokesian flow at a Reynolds number calculated using the sphere diameter, $\mathrm{Re}\to 0$. Let us take a group of m spheres. We note that in the cases of $d=2$ and 3, as such a group

---

* The present work was presented at the National Congress of Theoretical and Applied Mechanics, held in Varna, 1989.

we take spheres the centres of which are filled by sites of square $(m^{1/2}l \times m^{1/2}l)$ and cubic $(m^{1/3}l \times m^{1/3}l \times m^{1/3}l)$ lattices. The total drag $f_m = F(m, y, f_0)$ acts on the group of m spheres, where F is an unknown function, $y = l/a$ and $f_0 = 6\pi\mu a U$ is the drag on a singly moving sphere (in the present case, the Stokes drag; and μ is the fluid viscosity). For reasons of dimensionality in the present situation we obtain

$$F(m, y, f_0) = f_0 \varphi(m, y), \tag{1}$$

where φ is another unknown function.

With the group mentioned above we associate a sphere of radius a_m such that the Stokes drag $6\pi\mu a_m U$, acting on it during the motion in the absence of other spheres, is f_m. From this and from equation (1) we obtain

$$a_m = a\varphi(m, y). \tag{2}$$

Take N/m groups of spheres of radius a, similar to the first one. Form from them a new group A which is geometrically similar to the primary group and has the same relative distance between the spheres y. In this combined group A the drag $F(N, y, f_0)/N$ acts on one sphere on the average, and on each of its constituent groups of m spheres, the drag

$$\psi_A = \frac{mF(N, y, f_0)}{N} = \frac{mf_0\varphi(N, y)}{N}. \tag{3}$$

Now combine N/m spheres of radius a_m into a group B which is geometrically similar to the primary group of m spheres of radius a. The centres of the spheres in the group B will be separated by a distance l'_m. Then on one sphere of radius a_m in the group B on the average acts the drag

$$\psi_B = \frac{F(N/m, y_m, f_m)}{N/m} = \frac{mf_m\varphi(N/m, y_m)}{N}, \quad y_m = \frac{l'_m}{a_m}. \tag{4}$$

On the other hand, the total drag, acting on the group A, is determined by the interaction between the N/m primary groups of m spheres and, consequently, it is

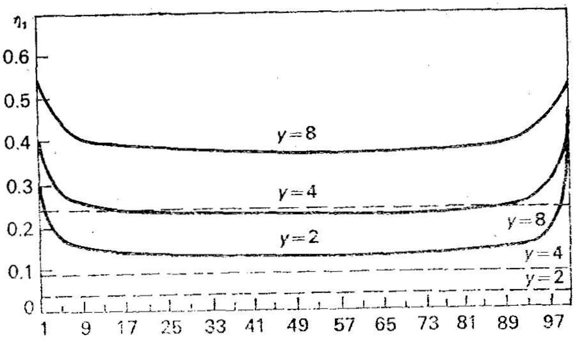

Fig. The correction factor of array of 101 spheres

$$F_1\left(\frac{N}{m}, \frac{l_m}{ma}, f_m, \varepsilon_i\right) = f_m \varphi_1\left(\frac{N}{m}, \frac{l_m}{ma}, \varepsilon_i\right) \tag{5}$$

$$l_m = m^{1/d} l, \quad i = 1, 2, \dots$$

Here F_1 and φ_1 are some unknown functions; and the degree of forcing-back the flow of each of the primary groups is characterized by a quantity ma and by a set of dimensionless parameters of its shape ε_i, which take into account, in particular, the existence of intervals between the spheres in the group.

Hence,

$$\psi_A = \frac{f_m \varphi_1(N/m, y/m^{1-1/d}, \varepsilon_i)}{N/m}. \tag{6}$$

Similarly for the group B we have

$$\psi_B = \frac{f_m \varphi_1(N/m, y_m, \varepsilon_i')}{N/m} \tag{7}$$

whereby ε_i' correspond to the interaction between spheres so that $\varphi_1(N/m, y_m$ $\varepsilon_i') = \varphi(N/m, y_m)$.

Let us choose l_m' so that

$$y_m = \frac{y}{m^{1-1/d}} \tag{8}$$

that is, taking into account (2) and the expression for y_m in (4), we put $l_m' = l\varphi(m, y)/m^{1-1/d}$.

The main assumption used hereafter is that the correction factor to the drag on a singly moving body (in the present case, f_m), which is determined by the collective hydrodynamic interaction, is practically independent of the shape of the body and it is determined only by the distance between bodies and by the effective dimensions of the bodies. In other words, the values of the function φ_1 are assumed to be independent of the value of the arguments ε_i.

The fact that the dependence of the correction factor η on the parameters of the shape ε_i should actually be very weak is confirmed by the following examples. When $y \gg 1$ and $N \gg 1$, using the familiar result of Smoluchowski [3] we can obtain the correction factor to the Stokes drag in the group A in the form ($d=1$; the flow normal to Γ)

$$\eta_{s1} = \frac{\varphi_{s1}(N, y)}{N} = 1 - \frac{3}{4y}\frac{1}{N}\sum_{k=1}^{N}\sum_{\substack{i=1 \\ i \neq k}}^{N}\frac{1}{|i-k|} = 1 - \frac{3}{2}\frac{\log N}{y}, \quad N \gg 1. \tag{9}$$

or in the form ($d=1$; the flow parallel to Γ)

$$\eta_{s2} = \frac{\varphi_{s2}(N, y)}{N} = 1 - \frac{3}{2y}\frac{1}{N}\sum_{k=1}^{N}\sum_{\substack{i=1 \\ i \neq k}}^{N}\frac{1}{|i-k|} = 1 - \frac{3\log N}{y}, \quad N \gg 1. \tag{10}$$

With increasing number of spheres, the η_{s1} and η_{s2} more and more differ from unity, which points to the dominant influence of the long-range interaction and, consequently, in the general case to the loss of information on the shape of widely spaced bodies which force back the flow near the body under consideration.

Another example is associated with the flow around two paralled elongated spheroids against which the Stokes flow runs normally to the semi-major axis. Calculation (Happel, Brenner [3]) indicates that the dependence η on the ratio of the semi-axes of spheroids is substantially weaker than that on the relative distance between them.

So, in virtue of the assumption made according to (6)—(8) $\psi_A=\psi_B$ and, consequently, using (2)—(4) we arrive at the functional equation of the renormalization group (RG) (Bogolyubov and Shirkov [1])

$$\varphi(N,y)=\varphi(m,y)\,\varphi\left(\frac{N}{m},\ \frac{y}{m^{1-1/d}}\right),\quad \varphi(1,y)=1. \tag{11}$$

The functional equation (11) under the assumption of smoothness reduces to the differential equation

$$N\frac{\partial\varphi(N,y)}{\partial N}+\left(1-\frac{1}{d}\right)y\frac{\partial\varphi(N,y)}{\partial y}=\varphi(N,y)\left.\frac{\partial\varphi(m,y)}{\partial m}\right|_{m=1}. \tag{12}$$

In the case $d=1$ with the flow normal to Γ, using the expression for $\varphi(2,y)$ from Happell and Brenner [3], we obtain for the RG-function

$$\left.\frac{\partial\varphi(m,y)}{\partial m}\right|_{m=1}=\varphi(2,y)-\varphi(1,y)=1-\frac{3}{2y}+\frac{9}{8y_2}-\frac{59}{32y^3}+\frac{465}{128y^4}-\frac{15813}{3584y^5}=\nu_1(y). \tag{13}$$

At $d=1$ in the case of a flow parallel to Γ passing over the spheres, using the solution found by the method of reflection Happel, Brenner [3] we obtain

$$\begin{aligned}\left.\frac{\partial\varphi(m,y)}{\partial m}\right|_{m=1}&=\varphi(2,y)-\varphi(1,y)=1-\frac{3}{y}+\frac{9}{2y^2}-\frac{19}{4y^3}\\&+\frac{93}{8y^4}-\frac{387}{16y^5}+\frac{1197}{32y^6}-\frac{5331}{64y^7}+\frac{19821}{128y^8}-\frac{76115}{256y^9}=\nu_2(y).\end{aligned} \tag{14}$$

At $d=1$ in the case of a flow parallel to Γ passing over the spheres there is also the exact Stimson-Jeffery solution for two spheres [3], which gives

$$\begin{aligned}&\left.\frac{\partial\varphi(m,y)}{\partial m}\right|_{m=1}=\varphi(2,y)-\varphi(1,y)=\nu_2(y)=2\lambda(y)-1\\&\lambda(y)=\frac{4}{3}\sin h\alpha\sum_{n=1}^{\infty}\frac{n(n+1)}{(2n-1)(2n+3)}\left\{1-\frac{4\sin h^2\left(n+\frac{1}{2}\right)\alpha-(2n+1)^2\sin h^2\alpha}{2\sin h(2n+1)\alpha+(2n+1)\sin h\,2\alpha}\right\}\\&\alpha=\log\left[\frac{y}{2}+\left(\frac{y^2}{4}-1\right)^{1/2}\right]\\&\lambda(2)=0.645,\quad(\alpha=0).\end{aligned} \tag{15}$$

Calculations with the use of (14)—(15) give similar results over the whole range of values y.

Solving (12) at $d=1$ and taking into account (13)—(15), we find

$$\varphi(N,y)=N^{\nu(y)},\quad \eta_1=\frac{\varphi(N,y)}{N}=N^{\nu(y)-1}, \tag{16}$$

where $\nu(y)=\nu_1(y)$ with the flow normal to Γ and $\nu(y)=\nu_2(y)$ with the flow parallel to Γ.

In the weak interaction limit ($\log N/y \ll 1$) equations (9) and (10) are obtained from (16). On the other hand, formulas (16) can be obtained directly from (9)—(11) but the expression for $\nu(y)$ will have the same accuracy as the results (9) and (10) have, i. e. with neglect of the values of $O(y^{-2})$.

In the two-dimensional case (with the flow normal to a plane containing the centres of the spheres), as well as in the three-dimensional case it becomes possible to determine the RG-function only within $O(y^{-2})$. In the cases cited, using Smoluchowski's formula [3] we obtain

$$\varphi(2^d, y)=2^d\left(1-\frac{k}{y}\right),\quad k=\begin{cases}\frac{3}{4}\left(2+\frac{1}{\sqrt{2}}\right)=2.03, & d=2\\ \frac{3}{4}\left(4+\frac{4}{\sqrt{2}}+\frac{4}{3\sqrt{3}}\right)=5.7, & d=3.\end{cases}\tag{17}$$

In the same approximation with the aid of Smoluchowski's formula we find for the case of flow parallel to the plane containing the centres of the spheres ($d=2$) that

$$\varphi(2^d, y)=2^d\left(1-\frac{k}{y}\right),\quad k=\frac{9}{4}\left(1+\frac{1}{2\sqrt{2}}\right)=3.045,\quad d=2\tag{18}$$

Using (17) and (18) we get

$$\left.\frac{\partial\varphi(m, y)}{\partial m}\right|_{m=1}=1-\frac{k_1}{y},\quad k_1=\frac{2^d k}{2^d-1}\tag{19}$$

and the solution of (12) at $d=2$ and 3 will be

$$\begin{gathered}\varphi(N, y)=N\exp\left[\frac{1}{1-1/d}\,\frac{k_1}{y}\left(1-N^{1-1/d}\right)\right]\\ \eta_{2,3}=\exp\left[-\frac{1}{1-1/d}\,\frac{k_1}{y}\left(N^{1-1/d}-1\right)\right].\end{gathered}\tag{20}$$

With increasing the space dimensionality, the collective effect is enhanced and the value of η progressively decreases at fixed values of N and y.

The drag value abruptly decreases in systems with a large number of particles ($N \gg 1$) owing to collective screening.

Of interest are estimates for the case of the inertial flow around the system of spheres. In the range $2\times10^4 \lesssim \mathrm{Re} \lesssim 2\times10^5$ the sphere drag coefficient $c_{\mathcal{D}}$ is practically independent of the value of Reynolds number (Landau and Lifshitz [4]). It is reasonable to suppose that in this range of values of Re the correction to the sphere drag coefficient is also independent of Reynolds number. Consequently,

$$f_0=\frac{1}{2}C_{\mathcal{D}}\rho U^2\pi a^2,\quad f_m=\frac{1}{2}C_0\rho U^2\pi a_m^2=f_0\varphi(m, y)\tag{21}$$

whence

$$a_m=a\,[\varphi(m, y)]^{1/2}\tag{22}$$

(ρ is the fluid density in the incoming flow).

A further train of our reasoning is similar to the analysis for the case of creeping Stokes flow so that

$$l'_m = l[\varphi(m, y)]^{1/2}/m^{1-1/d} \tag{23}$$

and we again arrive at equations (11) and (12).

In the cese of inertial flow the expression for the RG-function at $d=1-3$ seems to be of the form (19) at sufficiently large values of y but the value of k_1 must naturally differ from the quantities given in (13)—(15) and (17)—(19). In principle, in the inertial regime the coefficients k_1 at $d=1-3$ can be determined experimentally, whereupon the results are still given by the relations (16) and (20) with new values of k_1. In the case $d=3$ the value of k_1 can be estimated as follows. Consider a group of eight spheres at the vertices of a cube in the case $y=2$ (i. e. the spheres are in contact). Replace this group by an effective sphere of radius $a_*=2a$, which has the same volume. The value of the drag on this group will be

$$\frac{1}{2} C_{\mathscr{D}} \rho U^2 \pi a^2 \varphi(8,2) \simeq \frac{1}{2} C_{\mathscr{D}} \rho U^2 \pi a_*^2 \tag{24}$$

Approximating the function $\varphi(8, y)$ over the whole range of variation in $y \geq 2$ by the expression $\varphi(8, y)=8(1-k/y)$, from (24) we find $k=1$ and $k_1=8/7$. In the cases $d=1$ and 2 similar estimation should apparently give a large error.

Comparing the value of $k_1=8/7$ obtained here with the value of $k_1=6.51$ for the case of the Stokes flow, we ascertain that the screening of particles in the inertial regime is significantly less (while η_3 is greater) than that in the creeping flow.

Compare the results, obtained here for the case of the Stokes flow parallel to $\Gamma(d=1)$, with the results of numerical solution of the similar problem for an array of 101 sphere arranged one after another along a straight line (Gluckman et al. [2]). Using (15) and (16), we obtain at $N=101$

$$\eta_1 = \begin{cases} 0.0378, & y=2 \\ 0.0941, & y=4 \\ 0.235, & y=8 \end{cases} \tag{25}$$

The values (25) are represented in figure 1 by dashed lines and the results of calculations by Gluckman et al. [2], by solid curves. The discrepancy between the numerical (corresponding to internal spheres) and analytical results seems to come from the fact that the value of $N=101$ is insufficiently large and calls for further analysis.

The author is grateful to E. V. Teodorovich for the discussion of technical aspects of the RG technique and also to V. M. Entov and A. S. Ioselevich for helpful critical discussions.

References

1. Bogolyubov, N. N., D. V. Shirkov. Introduction to the Theory of Quantized Fields, Wiley-Interscience, 1980.
2. Gluckman, M. J., R. Pfeffer, S. Weinbaum. A new technique for treating multiparticle slow viscous flow: axisymmetric flow past spheres and spheroids. — J. Fluid Mech., 50, 1971, 705.
3. Happel, J., H. Brenner. Low Reynolds Number Hydrodynamics, 2nd edn, Noordhoff, 1973.
4. Landau, L. D., E. M. Lifshitz. Fluid Mechanics, Pergamon Press, 1959.

Received on October 22, 1990