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On the Optimal Control of the Oriented Plastic Forming Process of Perfectly Plastic Plates

Statement of the problem

Oriented forming of thinwalled elements with controlled indentors is a contemporary technology that creates meaningful problems with respect to theory of plasticity. The processes are unstationary and the final form of the elements could be quite complicated.

In the work presented a comparatively simple case that has its own importance for applications, and is also a test for the methods worked out for optimal control of such processes, is considered.

The action of pressing with an absolutely rigid sphere (as an instrument) on a thinwalled plate made of perfectly plastic material (the element being formed) is considered. The sphere is translated (downwards) with the aim of forming the plate to a spherical shell. The instrument is applied with not great acceleration, and the material of the plate is strongly dissipative. These assumptions allow us not to take into account the wave processes occurring when applying the action, and to model the material as rigid perfectly plastic.

The problem we put is connected with the control of the process of forming the thin plate, i. e. with determining the sphere centre velocity, aiming to obtain the given form. The strain energy lost during the process is accepted as criterion for the process quality. We are looking for such a velocity of the sphere centre $v^0(t)$ in a given finite time interval $t \in [0, t_f]$, that would provide us the assessment of the thinwalled element final form with minimum strain energy expenses.

Solution of the basic problem

The basic problem, i. e. the direct problem consists of determining the displacements the corresponding stress-strain state and the final form of the element as well as the strain energy spent for its forming, given the kinematics of the spherical instrument. Similar kind of a direct problem without controlling the process is solved in [1]. When solving the problem as an optimization one, usually it is necessary to solve the direct problem many times. That is the reason to consider its solution briefly.

Let us introduce a spherical coordinate system with centre O in the sphere centre, radius r and angle α (Fig. 1). We assume that all the variables that characterize the stress-strain state of the plate, as well as the particle velocities do not depend on the angle α, but only on the radius r and the time t. Let us introduce the next notation $r'=r+R$, where $R=\text{const}$ is the sphere radius.

We make the following basic assumptions when solving the direct problem.

1) All the points of the sphere surface, as well as of the plate, lying on the corresponding normals have normal velocities $v_n=v_n(r, t)$ only.

2) If a point of the upper surface of the plate does not belong to the contact surface, it has a velocity $v_n(0, t)=0$. The same holds for the remaining points of the same normal.

3) The material of the plate is rigid perfectly plastic and satisfies the Mizess condition.

4) The problem has complete spherical symmetry, and it follows then that only normal stresses and strains will occur in the plate, i. e. $\sigma_n(r, t)\neq 0$, $\sigma_\Phi(r, t)=\sigma_\vartheta(r, t)\neq 0$, $\varepsilon_n(r, t)\neq 0$, $\varepsilon_\Phi(r, t)=\varepsilon_\vartheta(r, t)\neq 0$.

5) We suppose the velocity of the sphere centre $v^0(t)$ changing slowly, so that one could neglect the partial time derivative of the normal particle velosity, i. e. $\frac{\partial v_n}{\partial t}=0$, leaving the convective term only.

From assumptions 1) and 2) it follows that plastic strains in the plate will occur only on the contact surface and below it.

The motion equation written for an arbitrary point of the plate will be as follows:

$$\rho v_n \frac{\partial v_n}{\partial r}=\frac{\partial \sigma_n}{\partial r}+2\frac{\sigma_n-\sigma_\Phi}{r'}, \tag{1}$$

where ρ is the material density of the plate. Following the geometry equations, the components of the strain rate tensor could be expressed as follows:

$$\dot{\varepsilon}_n=\frac{\partial v_n}{\partial r}; \quad \dot{\varepsilon}_\Phi=\dot{\varepsilon}_\vartheta=\frac{v_n}{r'}. \tag{2}$$

We obtain from the plastic incompressibility condition:

$$\frac{\partial v_n}{\partial r}+2\frac{v_n}{r'}=0. \tag{3}$$

From assumption 3) it follows:

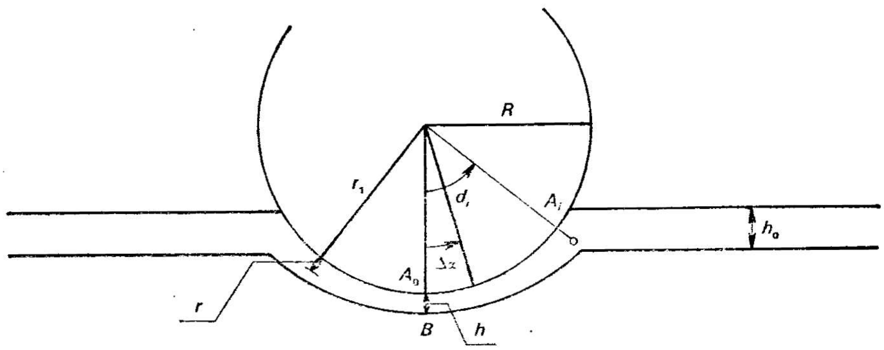

Fig. 1

(4) $S - S_\Phi = \sigma_n - \sigma_\Phi = \sqrt{3}\tau_p, \quad S_n = \sigma_n - \sigma_M, \quad S_\Phi = S_\vartheta = \sigma_\Phi - \sigma_M, \quad \sigma_M = 1/3(\sigma_n + 2\sigma_\Phi),$

where τ_p is the shear yield limit for the plate material. The plastic flow rule is written it the form:

$$\frac{v_n}{r'} = \lambda S_\Phi = \lambda S_\vartheta; \quad \frac{\partial v_n}{\partial r} = \lambda S_n, \quad \lambda > 0, \tag{5}$$

where λ is the plastic flow parameter.

After integrating the plastic incompressibility equation (3) with respect to r, we obtain:

$$v_n(r, t) = \frac{C(t)}{r'^2}. \tag{6}$$

Let the normal velocity of the corresponding point of the sphere surface be $v_0(t)$. Then the boundary condition on the contact surface sphere-plate will look like this:

$$v_n(0, t) = v_0(t) = v^0(t)\cos\alpha \quad \text{(see Fig. 1)} \tag{6'}$$

After determining $C(t)$ from (6') and substituting in (6) we obtain for the velocity;

$$v_n(r, t) = v_0(t)\left[R + \int_0^t v_0(\tau)\,d\tau\right]^2 / r'^2. \tag{7}$$

Formula (7) gives the velocity of each point of the plate in each moment t, given the velocity of the corresponding point from the sphere surface. Having obtained the normal velocities, we turn to (1) to determine the stresses.

$$\sigma_n(r, t) = \frac{2\rho A^2(t) v_0^2(t)}{4} \cdot \frac{r'^4 - (R+h)^4}{r'^4 (R+h)^4} + 2\sqrt{3}\tau_p \ln\left[\frac{(R+h)}{r'}\right], \tag{8}$$

where $h = h(t)$ is the current thickness and $A = \left[R + \int_0^t v_0(\tau)\,d\tau\right]^2$. And for the stresses on the contact surface one obtains:

$$\sigma_n(0, t) = p_n(t) = \frac{2\rho A^2(t) v_0^2(t)}{4h'} 2\sqrt{3}\tau_p \ln h. \tag{8'}$$

When one knows the velocity distribution one can determine the displacements in each point of the plate at any moment t

$$u_n(r, t) = \int_0^t v_n(r, \tau)\,d\tau. \tag{9}$$

The strain energy spent up to the moment t can be expressed as follows:

$$E(t) = \int_s p_n(t) \cdot u_0(t)\,ds, \tag{10}$$

where S is the contact surface and $u_0(t) = u_n(0, t)$.

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Numerical procedure for simulating the process of plastic forming with a spherical instrument

Using the solution obtained for determining the plate stress-strain state and its geometry, in case of small plastic deformations, the following procedure for simulating the process in case of large deformations has been worked out.

Let the surface of the sphere be divided into m elementary plates with centres in the points A_i with coordinates (R, α_i), $\alpha_i=(i-1)\Delta\alpha$, $i=0, 1, \ldots, m$ (Fig. 1). And let the time interval of the process $[0, t_f]$ be divided into k intervals, so that $t_j=j.\Delta t$, $j=0, 1, \ldots, k$, $t_f=t_k$. In these intervals the velocity v^0 is considered constant and consequently all the variables do not change with respect to time. In these intervals the strains are small and so the equations from the previous paragraph hold. Let us now have the following values for the points A_i, for the moment t_{j-1}, $i=0, 1, \ldots, m$:

1) distances to the plate $d_i(t_{j-1})=d_i^{j-1}$;

2) thicknesses $h_i(t_{j-1})=h_i^{j-1}$.

These quantities define the contact surface and the plate geometry in the moment $t_{j-1}=(j-1).\Delta t$. Let us now consider the next time moment $t_j=t_{j-1}+\Delta t$. Let us have the velocities on the contact surface

$$v_{0i}^j=v_{0i}(0, t_j)=\varkappa^{0j}\cos\alpha_i=v^0(t_j)\cos\alpha_i.$$

We shall determine the new geometry of the plate, the new contact surface, the stresses as well as the energy, for the moment t_j.

1) First we shall determine the velocities following (7)

$$v_{ni}^j(r)=(v^{0j}\cos\alpha_i A_i^j)/r'^2, \tag{11}$$

$$A_i^j=\left(R+\sum_{l=0}^{j} v_{0i}^l\right)^2.$$

2) Then we can obtain the displacements:

$$u_{ni}^j(r)=u_{ni}^{j-1}(r)+\Delta u_{ni}^j, \quad \Delta u_{ni}^j=v_{ni}^j\Delta t. \tag{12}$$

3) The new distances will be given by

$$d_i^j=d_i^{j-1}-u_{0i}^j, \quad v_{0i}^j=u_{0i}^{j-1}+v_{0i}^j\Delta t, \quad u_{0i}^j=u_{ni}^j(0). \tag{13}$$

4) The new thicknesses are given by:

$$h_i^j=h_i^{j-1}-u_{0i}^j+u_{ni}^j(h_i^{j-1}). \tag{14}$$

Thus we obtain the new geometry of the plate and the new contact surface.

5) The stresses on the contact surface are obtained following (8′):

$$p_{ni}^j=\frac{2\rho(A_i^j)^2 v^{0j}\cos\alpha_i}{4(h_i^j)^4}-2\sqrt{3}\tau_p\ln(h_i^j). \tag{15}$$

6) The energy spent up to the moment t_j will be given by the next formula:

$$E^j=E^{j-1}+\sum_{i=1}^{m} p_{ni}^j\Delta u_{0i}^j. \tag{16}$$

7) With the new geometry, contact surface and the new stresses and displacements we go back to the beginning of the procedure 1).

Thus we obtain the plate geometry, the contact surface, the velocities, the displacements, the stresses as well as the energy spent for all the moments $t_j=t_{j-1}+\Delta t$. The procedure is repeated till $t_j=t_k=t_f$, where t_f is the time needed for the process. So, finally for the complete energy of the process one obtains:

(17)
$$E^T=\sum_{i=0}^{m}\sum_{j=0}^{k} p_{ni}^j \Delta u_{0i}^j.$$

Thus, given the velocities of the sphere centre v^{0j} in the moments t_j, $j=0, 1, \ldots, k$, all the variables are obtained in all these moments, using relations (11)-(16), their values depending on v^{0j}. For instance $p_{ni}^j=p_{ni}^j(v^{0l}\,|\,l=0, 1, \ldots, j)$ and $u_{ni}^j=u_{ni}^j(v^{0l}\,|\,l=0, 1, \ldots, j)$, $E^j=E^j(v^{0l}\,|\,l=0, 1, \ldots, j)$. Thus for the complete energy one will have

(18)
$$E^T=E^T(v^{0j}\,|\,j=0, 1, \ldots, k)$$

Optimal control of the forming process

The problem we state is the following. Find the vector of velocities of the sphere centre, defined as $\mathbf{v}^0=\{v^{0j}\,|\,j=0, 1, \ldots, k\}$, such that the strain energy E^T is minimal under the next constraints:

1) The minimal thickness during the whole process being greater than a given value $\hat{h}$

(18)
$$\mathbf{h}_0\geq\hat{\mathbf{h}},\quad \mathbf{h}_0^T=\{h_0^j\,|\,j=0, 1, \ldots, k\}\quad \hat{\mathbf{h}}^T=\{\hat{h}, \hat{h}, \ldots, \hat{h}\}.$$

2) The displacement $u_0^k(h)$ of the point B of the lower surface with radius $r|_{t=t_k}=h_0^k$ (see Fig. 1) in the moment $t=t_k$, being equal to a given value $\hat{u}$, defined by the forming matrix,

(19)
$$u^d=\hat{u},\quad u^d=u_0^k(h).$$

The first constraint is connected with the plate thickness. It is required for it to be greater than $\hat{h}$. As far as the thickness in point A_0 (Fig. 1) is the least one during the whole process, the constraint is for this thickness only. The second constraint concerns the form of the plate required. Giving the maximal displacement, i. e. the one in point A_0 in the moment $t_f=t_k$ we define the final form of the plate.
Hence, we have the following problem to solve:

(20)
$$\left|\begin{array}{l}\mathbf{v}^0: E^T(\mathbf{v}^0)\rightarrow\min;\quad \mathbf{v}^0=?\\ \mathbf{h}_0\geq\hat{\mathbf{h}}\\ u^d=\hat{u}.\end{array}\right.$$

Usually, in practice, the exact realization ot some velocity values in some time moments is not possible, because of imperfections of the forming techniques, errors in measuring, etc. That is why it is reasonable to adopt that we work with noised values of $\mathbf{v}$, $\mathbf{v}=\mathbf{v}^0+\boldsymbol{\eta}$. It follows then that the energy values will be also noised, due to the noising of $\mathbf{v}^0$ as well as the numerical method of obtaining E, i. e. $\tilde{E}=E+\delta E$. Then the problem (20) is transformed to the following one:

(21)
$$\left| \begin{array}{l} \mathbf{v}: M(\tilde{E}) \rightarrow \min; \quad \mathbf{v} = ? \\ \mathbf{h}_0 \geq \hat{\mathbf{h}} \\ u^d = \hat{u}. \end{array} \right.$$

Let us now assume that

(22)
$$M(\tilde{E}) = E \quad \text{and} \quad \Sigma^2(\tilde{E}) < L_E, \quad L_E = \text{const.},$$

where $M(.)$ denotes the mathematical expectation and Σ^2—the variance. In order to solve (21) we shall define the Lagrange functional:

(23)
$$\Phi(\mathbf{v}, \mathbf{h}_0, u^d) = E(\mathbf{v}) + \boldsymbol{\lambda}^T(\mathbf{h}_0 - \hat{\mathbf{h}}) + \mu(u^d - \hat{u}).$$

We are looking for the minimum of the mathematical expectation of Φ with respect to $\mathbf{v}$, $\boldsymbol{\lambda}$ and μ:

(24)
$$M\{\Phi(\mathbf{v}, \mathbf{h}_0, u^d) \rightarrow \min, \quad \mathbf{v} = ?, \quad \boldsymbol{\lambda} = ?, \quad \mu = ?$$

To find this minimum we shall apply the Robbins-Monro algorithm for stochastic approximation [2]. When using it $\mathbf{v}$, $\boldsymbol{\lambda}$ and μ are determined iteratively following the next equations:

Table 1

$t=0.1$	$t=0.3$	$t=0.5$	$t=0.7$	$t=0.$	$E.10^{15}$	α_q	q
0.0143	0.038	0.056	0.0690	0.0760	4.05	1	1
0.0160	0.045	0.059	0.067	0.065	2.875	1/2	2
0.0191	0.047	0.60	0.066	0.063	2.50	1/2	3
0.0215	0.050	0.061	0.065	0.062	2.005	1/3	4
0.0230	0.0520	0.0610	0.0645	0.0613	1.700	1/3	5
0.0238	0.0530	0.0615	0.0630	0.0619	1.400	1/5	10
0.0241	0.055	0.0612	0.0618	0.0615	1.378	1/14	20
0.0244	0.059	0.0612	0.0614	0.0615	1.360	1/25	30
0 0244	0.0611	0.0613	0.0613	0.0613	1.359	1/30	40
0.0245	0.0613	0.0613	0.0613	0.0613	1.351	1/40	50

$\tau_p = 4.2 \times 19^7$ N/m^2, $[t] = [s]$

Table 2

$t=\Delta t$	$t=3\Delta t$	$t=5\Delta t$	$t=7\Delta t$	$t=10\Delta t$	α_q	$E.10^{16}$	q
0.012	0.050	0.061	0.065	0.085	1	25.17	1
0.035	0.090	0.066	0.661	0.080	1	13.91	2
0.055	0.0615	0.668	0.069	0.079	1/2	7.85	3
0.060	0.035	0.0691	0.070	0.76	1/2	6.90	4
0.067	0.070	0.0696	0.0704	0.073	1/3	6.11	5
0.068	0.071	0.070	0.071	0.072	1/7	5.90	10
0.0685	0.0711	0.0712	0.0713	0.0714	1/14	5.30	20
0.069	0.0711	0.0715	0.0716	0.0715	1/20	5.15	30
0.0689	0.0713	0.0716	0.0715	0.0716	1/26	5.11	40
0.069	0.0715	0.0715	0.0715	0.0715	1/34	5.12	50

$\tau_p = 4.5 \times 10^8$ N/m^2, $\Delta t = 0.1$ s. $[E] = [N.m]$

$$\underline{\mathbf{v}}(q+1)=\underline{\mathbf{v}}(q)-\alpha_q\left\{\frac{\partial\Phi}{\partial\mathbf{v}}\right\}\Big|_q,$$

(25)
$$\mu(q+1)=\mu(q)-\alpha_q\left(\frac{\partial\Phi}{\partial\mu}\right)\Big|_q,$$

$$\boldsymbol{\lambda}(q+1)=\min\left(\mathbf{0},\ \boldsymbol{\lambda}(q)-\alpha_q\left\{\frac{\partial\Phi}{\partial\boldsymbol{\lambda}}\right\}\Big|_q\right),$$

where q is the number of iterations, $q=0,\ 1,\ \ldots\ \infty$.
The following relations are required for the sequence α_q,

$$\lim_{q\to\infty}\alpha_q=0,$$

(26)
$$\sum_{q=0}^{\infty}\alpha_q=0,$$

$$\sum_{q=0}^{\infty}\alpha_q<\infty.$$

The corresponding values of α_q are given in Tables 1 and 2. Robbins—Monro algorithm is convergent for any initial approximations $\mathbf{v}(0)$, $\boldsymbol{\lambda}(0)$ and $\mu(0)$ of the variables searched, provided that conditions (22) and (26) are fulfilled.

Some results and comments

Here we shall comment some results obtained U solving the problem considered above. The following values for the parameters have been used: initial thickness of

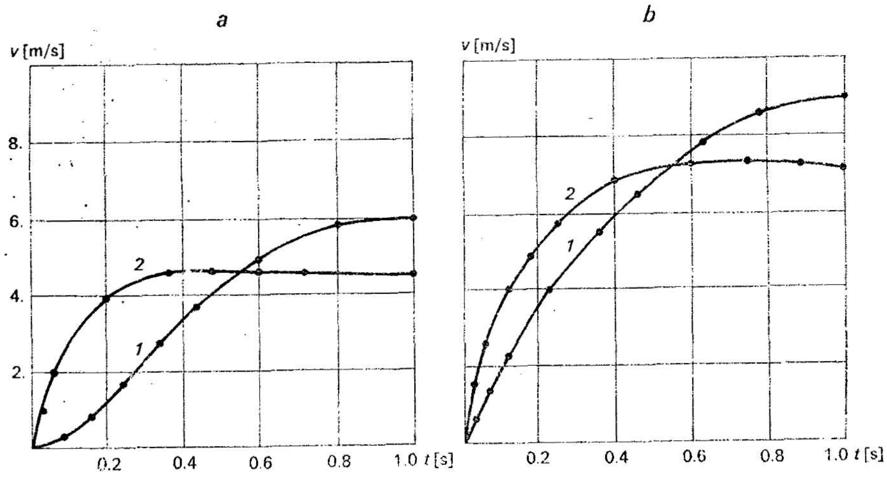

Fig. 2. *a* — *1* — initial approximation, *2* — $v^0(t)$, obtained after solving the optimization problem , $\tau_p=4.2\times10^7$, N/m²; *b* — *1* — initial approximation, *2* — $v^0(t)$, obtained after solving the optimization problem, $\tau_p=4.5\times10^8$ N/m²

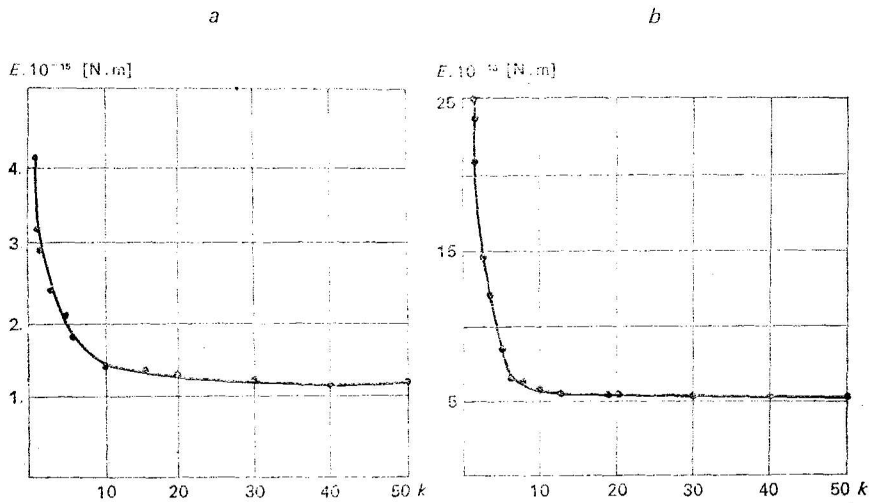

Fig. 3. a — τ_p=4.2×10^7 N/m^2; b — τ_p=4.5×10^8 N/m^2

the plate $h=0.04$ m, sphere radius $R=0.2$ m, time of the process $t_f=1$ s, displacement of point B $\hat{u}=R/4$. Two different kinds of materials have been investigated, an aluminium alloy and a steel, with different yield points [1]. In Tables 1 and 2 the results for the velocity $\mathbf{v}(q)$ in various time moments, obtained after different number of iterations q, are given. In the tables the corresponding values of the strain energy $E(q)$ are given, too. The results obtained for both materials considered are given on Figures 2a, b (the velocity diagrams) and 3a and b (the energy diagrams). It is easily observed (from Tables 1 and 2) that the convergence of the procedure in both cases is rapid for the first iterations especially for $k=0, 1, \ldots, 5$. Further, after $k=10$, the convergence becomes very slow. For obtaining an adequate accuracy one usually needs about 50 iterations. We have to mention here that the results obtained for the velocities of the sphere centre are the same despite the initial approximations for $\mathbf{v}$. This result holds for both materials investigated, and it is due to the unconditional convergence of the algorithm applied. It is easilly observed that for both materials the velocity of the sphere centre tends to a rectangular form, i. e. achieving some value and holding it constant during the process.

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