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Sensitivity Analysis for Shape Optimization by the BEM

1. Introduction

The problem of the shape optimization with design sensitivity analysis have been studied in the recent decade by different authors. Some of them investigate the problem on the grounds of differentiation of the discretized (FEM) system (Bennett and Botkin, Botkin [1, 2]), others have used the differentiating the continuum equations (Choi and Hang [3], Chun and Hang [4], Dems and Mroz [5]). The first method is the so-called direct method in which the calculations of the sensitivity derivatives have some disadvantages: the change in shape leads as usual to the distortion of the FE-mesh and the derivatives have a spurious components which are uncontrollable with respect to accuracy of the solution. The second method, used in the present paper, is based on the conception of material derivatives and only then discretization of the problem is possible. Discretization is made by BEM in the present work. Using BEM we overcome some disadvantages of the FEM, namely, finite element meshregeneration, difficulties in sensitivity derivatives calculation, stemming from inaccurate boundary representation. The main priorities of the BEM are: reduced set of equations; a smaller amount of data used; no interpolation error inside the domain; less number of joints on the surface of body in the comparison with FEM, discretization only on the boundary of the body and lack of the necessary for mesh regeneration in the domain.

Soares et al. [6, 7] have formulated the problem of optimization of the shape geometry in terms of the BEM.

Burczynski and Adamczyk [8] have studied the BEM application to a multiparameter optimal shaping using a maximum stiffness criterion design. Dems and Mroz [9] have formulated the optimal problem by means of FE approach.

Analyses and theories of shape optimization have been presented by Banichuk [10], Kikuchi et al. [11], and other authors as well.

The aim of the present paper is to show the advantages of the hybrid utilization of the BEM as a discretizing technique and design sensitivity analysis as an optimizing shape approach. We investigate the optimal shape of a hole in a metal plate as an example. Some authors have studied this problem (Dhir [12, 13], Durelli et al. 14]), but in addition we have examined the inelastic case to that aim applying the Gourevich's model (Gourevich [15]).

2. Stress-strain state of a plate in differential and BEM description

2.1. Stress-strain state of a plate in differential form

The entire set of equations describing the elastic (inelastic) behaviour of a plate has the form:

$$\sigma_{ij,j}=0,$$

(1) $$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \quad (i, j, k=1, 2),$$

$$\sigma_{ij}=\lambda(\varepsilon_{kk}-\varepsilon^0_{kk})\delta_{ij}+2G(\varepsilon_{ij}-\varepsilon^0_{ij}),$$

where σ_{ij}, ε_{ij} are the stress and strain tensors; ε^0_{ij} is the inelastic residual strain as predicted by the Maxwell-Gourevich model for metals; λ, G are the Lame constants; u_i are the components of the displacement vector.

The tractions τ_i in the points of the boundary Γ of the plate are:

(2) $$\tau_i=\tau_{ij}n_j=G\left[(u_{i,j}+u_{j,i})n_j+\frac{2\nu}{1-2\nu}u_{k,k}n_i-2\varepsilon^0_{ij}n_j\right],$$

where n_j are the components of the unit outward normal to at that points. The main problem is to find the optimal shape of the boundary of the hole in a rectangular metal plate, subjected to an unaxial tension. The geometrical characteristics and the boundary conditions of the considered plate are illustrated in Fig. 1.

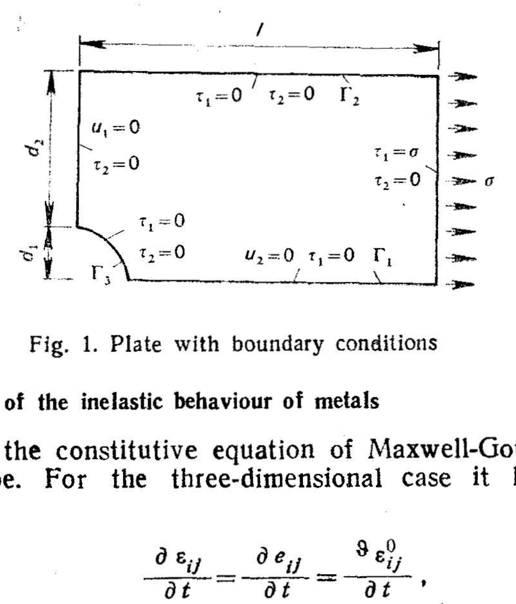

Fig. 1. Plate with boundary conditions

2.2. Physical model of the inelastic behaviour of metals

Consider briefly the constitutive equation of Maxwell-Gourevich. This equation is of a viscoplastic type. For the three-dimensional case it has the form (Gourevich [15]):

(3) $$\frac{\partial\varepsilon_{ij}}{\partial t}=\frac{\partial e_{ij}}{\partial t}=\frac{\partial\varepsilon^0_{ij}}{\partial t},$$

where e_{ij} are the elastic strains:

(4) $$\varepsilon_{ik}=\frac{1}{2G}\left(\sigma_{ik}-\frac{3\nu}{1+\nu}p\,\delta_{ik}\right).$$

The residual strains rate is:

(5)
$$\frac{\partial \varepsilon_{ik}^{0}}{\partial t}=\frac{3}{2}(\sigma_{ik}-\delta_{ik}p)\frac{1}{\eta_0}.$$

The viscosity ratio η_0 in equation (5) for the residual strain is:

(6)
$$\frac{1}{\eta_0}=\frac{1}{\eta_0^0}exp\left\{\frac{1}{m_0}\left[\gamma^0 p+\frac{3}{2}(\sigma_{rr}-p)\right]\right\},$$

where t is the time, δ_{ik} is the Kronecker delta, $p=\frac{1}{3}\sigma_{ii}$, m_0 is the logaritmic modulus of the residual strain rate, γ^0 is the volumetric modulus, η_0^0 is the initial viscosity modulus for the residual strain. The method of determination of the physical parameters of the Maxwell-Gourevich model is based on the constancy of the deformation velocity.

The advantage of physical parameters of Gourevich consists in its ability to describe both the loading curve with existing horizontal asymptote and the unloading one. In this case it is not necessary to know in which part of the curve σ-ε the behaviour of the material should be described, as it is with the classical plasticity models (Fig. 2).

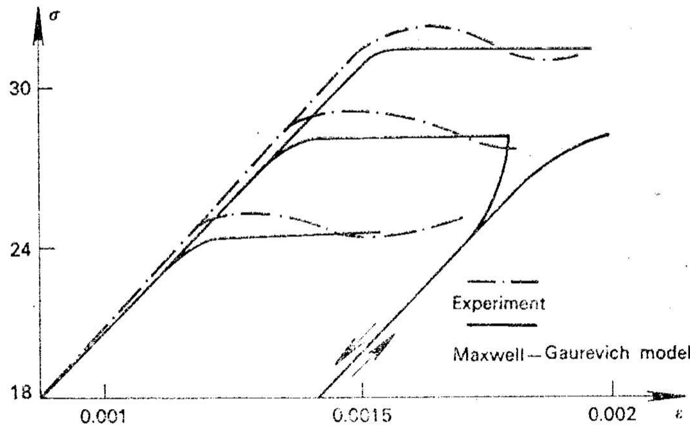

Fig. 2 τ-ε curve of Gourevich model

2.3. Stress-strain state of a plate in BEM description

The equivalent BIE approach reduces the set of equations (I) to the form:

(7)
$$[C_{ij}]\{u_i\}=\int_{\Gamma}([U_{ij}^{*}]\{\tau_i\}-[T_{ij}^{*}]\{u_i\})\,d\Gamma$$
$$+2G\int_{\Gamma}[U_{ij}^{*}]\{\varepsilon_{ik}^{0}n_k\}\,d\Gamma-2G\int_{\Omega}[U_{ij}^{*}]\{[\varepsilon_{ik,\,k}^{0}\}\,d\Omega,$$

where U_{ij}^{*} and T_{ij}^{*} are the components of the displacement and the traction vectors in the x_i-direction due to a unit point load in the x_j-direction. These functions are obtainable from the Kelvin's singular solution:

(8)
$$U^*_{ij}=-\frac{1}{8\pi(1-\nu)G}\{(3-4\nu)\delta_{ij}\ln r-r,_i r,_j\},$$

$$T^*_{ij}=-\frac{1}{4\pi(1-\nu)r}\{[(1-2\nu)\delta_{ij}+2r,_i r,_j]\frac{dr}{dn}+(1-2\nu)(r,_i n_j-r,_j n_i)\},$$

where $r_i=x_i-x_{oi}$ is the distance from a field point to the source point, $r=(r_i r_i)^{1/2}$.

The matrix $[C_{ij}]$ is determined by making use of the rigid displacements of the body:

(9)
$$[C_{ij}]=-\int_\Gamma [T^*_{ij}]\,d\Gamma.$$

Along the boundary and in the region Ω, N-linear boundary elements and M-triangular plane elements (internal cells) are respectively used. The discretized version of the boundary integral equations (7) reads:

(10)
$$[C_{ij}]\{u_i\}=\sum_N\int_{\Gamma_N}([U^*_{ij}]\{\tau_i\}-[T^*_{ij}]\{u_i\})\,d\Gamma_N$$

$$+2G\sum_N\int_{\Gamma_N}[U^*_{ij}]\{\varepsilon^0_{ik}\,n_k\}\,d\Gamma_N-2G\sum_M\int_{\Omega_M}[U^*_{ij}]\{\varepsilon^0_{ik,k}\}\,d\Omega_M.$$

The BEM mesh for the problem is shown in Fig. 3.

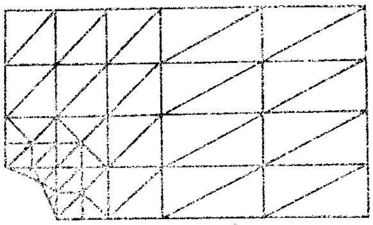

Fig. 3. The BEM mesh

A linear approximation for the displacements, tractions and strains is introduced. It is established by Brebbia et al. [16] that among the variety of elements used for discretization, the linear ones are of acceptable accuracy in comparison with elements of higher order and they are convenient for numerical realization.

Numerical discretization transforms equation (10) into an algebraic system of the type:

(11)
$$[A]\{u\}-[B]\{\tau\}=\{D\},$$

where coefficient matrices $[A]$ and $[B]$ contain integrals of the kernels and the shape functions; vector $\{D\}$ if formed from the known displacements and tractions on the boundary Γ and from the strains. Once all the displacements and tractions are determined over the entire boundary, a discretized version of the following equations is used to calculate displacements and stresses in the internal points of the body:

$$\{u_j\}=\int_\Gamma([U^*_{ij}]\{\tau_i\}-[T^*_{ij}]\{u_i\})\,d\Gamma$$

(12)
$$+2G\int_{\Gamma}[U^{*}_{ij}]\{\varepsilon^{0}_{ik}n_k\}\,d\Gamma-2G\int_{\Omega}[U^{*}_{ij}]\{\varepsilon^{*}_{ik}n_k\}\,d\Omega,$$

$$\sigma_{ij}=G\left\{(u_{i,\,j}+u_{j,\,i})+\frac{2\nu}{1-2\nu}U_{k,\,k}\,\delta_{ij}-2\varepsilon^{0}_{ij}\right\}.$$

3. Formulation of the optimal problem for free boundary

Due to the symmetry let us consider a quarter of plate with boundary $\Gamma=\Gamma_1 U \Gamma_2 U \Gamma_3$, where Γ_3 is the boundary of hole. On the parts Γ_1 and Γ_2 the fields of displacement and traction are respectively given (Fig. 1). We assume that optimal boundary shape is described by a shape vector function $\varphi(p_i)$, Fig. 4. Now the optimization problem is reduced to determination of this function, which is expressed by:

(13)
$$x'(p_i)=x(p_i)-\varphi(p_i),$$

where $x'(p_i)$ is the modified initial boundary configuration $x(p_i)$.

The optimization problem consists in the following: maximize (minimize) a certain functional of synthesis of shape F in such a way that the area of the hole S should not exceed a certain given value S_0:

(14)
$$\max F \text{ for } S\leq S_0.$$

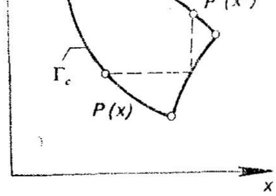

Fig. 4. Displacement of the boundary point P

The equivalent definition of this problem is to find the stationarity conditions for the functional:

(15)
$$\Phi=F-\lambda^{*}(S-S_0),$$

where λ^{*} is the Lagrange multiplier. The potentional energy $W=\int_{\Gamma_3}\sigma_{ij}(\varepsilon_{ij}-\varepsilon^{0}_{ij})\,d\Gamma_3$ is taken as a measure of total stiffness and the problem is transformed in a shape optimal structural design problem for maximum stiffness criterion.

Applying the material derivative formulas (Choi and Hang [3]) for variation of functional over a variable boundary Γ_3, one obtains the necessary optimality conditions in the following form:

(16)
$$\delta\varphi=\int_{\Gamma_3}\Phi\vec{n}\,.\,\delta\vec{\varphi}\,d\Gamma_3=0,$$

where

(17)
$$\Phi=W-\frac{d}{dn}\underline{(u\,.\tau)}+\underline{2Hu\,.\tau}-\lambda^{*}.$$

In equation (17) H is the curvature of the boundary Γ_3 in R^2. Since the tractions vanish in the whole boundary Γ_3 the underlined terms in the r. h. side of (17) vanish as well.

4. Discretization of optimality conditions

In the two-dimensional case under consideration we make the optimal boundary Γ_3 to be among the class of polygons, whose sides are the boundary elements $\Gamma^p (p=1,2,\dots M,$

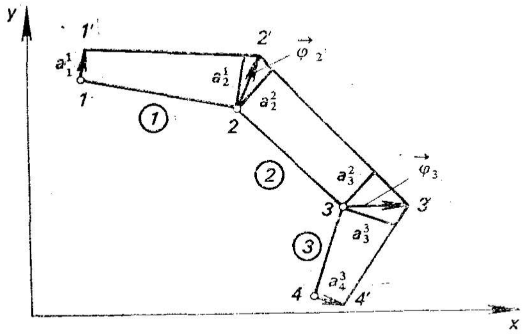

Fig. 5. Discretization and design parameters

M-number of boundary elements) of length L_p (Fig. 5). Following Burc-zynski and Adamczyk [8] a local coordinate system (A_p, ξ) is located at the origin A_p of every boundary element.

The function $\varphi_i^p(\xi)$ $(i=1, 2, \dots N$, N-number of nodes of $\Gamma_3)$ which generates the boundary Γ_3 — as follows:

(18)
$$\varphi_i^p(\xi)=\left(1-\frac{1}{L_p}\right) a_i^p+\frac{1}{L_p} a_{i+1}^p.$$

For the components a_i^p of the vector shape function $\varphi_i^p(\xi)$ we have (Fig. 5):

(19)
$$\varphi_i^p=\varphi_i^p(\xi, a_i^p); \quad \delta\varphi_i^p=\frac{\partial \varphi_i^p}{\partial a_i^p}\delta a_i^p ;$$

(20)
$$\begin{cases} a_i^p=n_k^p \varphi_k^p, & k=x, y \\ a_i^{p+1}=n_k^{p+1}\varphi_k^p, & i=1, 2, \dots, N; \quad p=1, 2, \dots, M. \end{cases}$$

We insert the optimal boundary generating function $\varphi(\xi)$ into equation (16) and not that the variation δa_i^p is arbitrary. The stationary conditions take the form of a set of $2N$-optimal, nonlinear algebraic equations, namely:

(21)
$$\frac{1}{L_p}\int_0^{L_p} W(\varepsilon)\xi\, d\xi=\frac{1}{2}\lambda^* L_p ,$$
$$\frac{1}{L_p}\int_0^{L_p} W(\varepsilon)(L_p-\xi)\, d\xi=\frac{1}{2}\lambda^* L_p, \quad p=1, 2, \dots M$$

from which $2N$-parameters a_i^p defining the shape of optimal boundary can be determined.

The condition of the polygon's area to be constant is:

$$\frac{1}{2}\sum_{i=1}^{M}\vec{r}_p\,.\,\vec{n}_p\,L_p=S_0, \tag{22}$$

where $\vec{r}_p$ is the radius vector of the node p.

The matrix form the equations (21) and (22) is:

$$F_m(a_i^j)=0 \quad \begin{array}{l}(m=1,\ 2,\dots 2N+1)\\ (\,i=1,\ 2,\dots\ N)\\ (\,j=1,\ 2,\dots\ M).\end{array} \tag{23}$$

The solution of these equations is rather difficult due to nonlinearity of the functions F_m. One can overcome it by applying the Newton-Raphson iterative procedure in the following form:

$$\left[\frac{\partial F_m}{\partial a_i^j}\right].\{\Delta a_i^j\}=-F_m, \tag{24}$$

where: $\{\Delta a_i^j\}$ are increments of design parameters a_i^j and $[\partial F_m/\partial a_i^j]$ has the form:

$$\begin{bmatrix} \frac{\partial F_1}{\partial a_1^1} & \frac{\partial F_1}{\partial a_2^1} & \frac{\partial F_1}{\partial a_2^2} & 0 & 0 & \dots & \frac{\partial F_1}{\partial \lambda^*} \\ \frac{\partial F_2}{\partial a_1^1} & \frac{\partial F_2}{\partial a_2^1} & \frac{\partial F_2}{\partial a_2^2} & 0 & 0 & \dots & \frac{\partial F_2}{\partial \lambda^*} \\ & \frac{\partial F_3}{\partial a_2^1} & \frac{\partial F_3}{\partial a_2^2} & \frac{\partial F_3}{\partial a_3^2} & \frac{\partial F_3}{\partial a_3^3} & 0 & \frac{\partial F_3}{\partial \lambda^*} \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ \frac{\partial F_{2N+1}}{\partial a_1^1} & \dots & \frac{\partial F_{2N+1}}{\partial a_2^2} & \dots & \dots & \frac{\partial F_{2N+1}}{\partial a_N^M} & 0 \end{bmatrix}$$

For $(i+1)$-iteration one has:

$$(a_i^j)^{l+1}=(a_i^j)^l+(\Delta a_i^j)^l\,. \tag{25}$$

The components of matrix $[\partial F_m/\partial a_i^j]$ can be obtained explicitly by analytical differentiation of the optimal conditions. The radius vector of the p-th node in the $(l+1)$-th iteration can be expressed as:

$$\vec{r}_p^{\,l+1}=\vec{r}_p^{\,l}-\vec{\varphi}_p^{\,l}\,. \tag{26}$$

The optimality process ends in two succeeding steps when the values of the design parameters are sufficiently closed. To realize the above mentioned iterative procedure we use the algorithm proposed by Burczynski and Adamczyk [8].

5. Computational algorithm

Schematically this algorithm is given as follows:

1. Set up an assumed shape of the boundary to be optimized. Perform a discretization of the boundary Γ_3 by BEM.

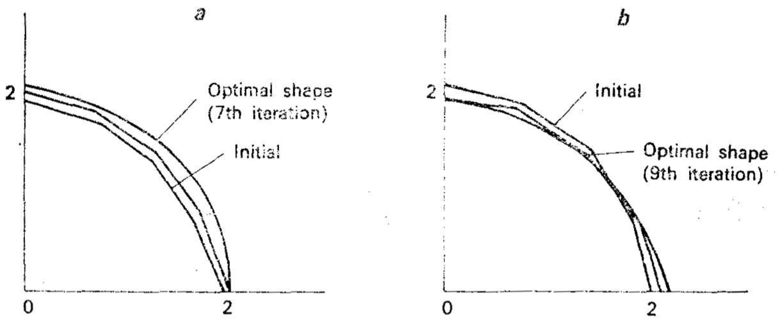

Fig. 6. Numerical results
a) elastic case
b) inelastic case

2. Compute the nodal displacements, stresses and strains for every boundary element of varying shape.
3. Compute the strain potential energy for the boundary elements and matrix F_m.
4. Compute the analytical expression for derivatives of the matrix $[\partial F_m / \partial a_i^j]$.
5. Compute the shape parameter increments Δa_i^j and check whether the obtained values of Δa_i^j are smaller than the given evaluation tolerance, i. e.

$$\sqrt{\sum_{i,j} (\Delta a_i^j)^2} \leq \varepsilon^* .$$

If so, print the nodal coordinates of the optimal boundary, if not — go to step. 2.

6. Numerical results

As a numerical example we consider a quarter of steel plate with a hole in it (Fig. 1). The geometric and mechanical characteristics are given in Table I. The tension tractions are $\sigma = 6.25$ kg/mm^2. The numerical results for stresses obtained by linear BE are compared with analytical solution of Timoshenko [17] and the difference is 2%. The optimal shape of the hole is obtained by the above described numerical algorithm for elastic and inelastic cases. In the first case the initial boundary Γ_3 (see Fig. 6 a) is assumed to be a polygon with the prescribed value of area of the hole $S_0 = 3.14159$ mm^2. The numerical algorithm ends at tolerance $\varepsilon^* = 0.01$ and the optimal shape obtained is a circle. For inelastic case the initial boundary Γ_3 is assumed to be a regular polygon with $S_0 = 3.14159$ mm^2 and $\varepsilon^* = 0.1$. The material is behaving linearly in some regions and non-linearly in others. The distribution of plastic zones around the hole is given by Hadjikov et al. [18]. The optimal shape of boundary is elliptical with approximate values for axes $a=1.8$ mm and $b=2.22$ mm (see Fig. 6 *b*).

The computational time for 7-iterations is t=37 sec for the elastic case and t=56 sec for the inelastic case.

7. Conclusions

The problem of optimal shape sensitivity analysis pose much more mathematical difficulties in comparison with the optimal problems with fixed boundaries. In fact, the sensitivity analysis for optimal shaping is the iterative procedure for reaching the global minimum of the sensitivity functional Φ, which uses the information from the set of solutions of this functional giving the stationary values of Φ at fixed boundaries.

The nonlinear system of the optimality conditions for the functional Φ has the following difficulties:

1. The loss of accuracy by numerical differentiation. The derivatives are eased for solving the nonlinear algebraic equations by Newton-Raphson method.

2. Unstability of Newton-Raphson iterative procedure for obtaining the new boundary in the case when the initial one is very far from the optimal boundary. In the present paper as an analytical function of design parameters a_i^j (the boundary nodes coordinates) and the optimal conditions are expressed by them, respectively.

The stability of the Newton-Raphson iterative procedure is reached by appropriate choice of the primary solution.

This is done on the grounds of the Pattern method (Oda and Yamazaki [19]), and some relations between the stress-strain state and the boundary's geometry, which guarantees the right direction to the global minimum of Φ.

In the present paper BE-mesh regeneration changes automatically because of the fact that the boundary nodes coordinates are the design parameters.

Some authors (Chaudouet and Afzali [20]) use the non-linear programming to solve the problem of optimal shaping. They introduce corrective coefficient for stress on the boundary, giving the right direction to the minimum of Φ, or they use the limited interval for design parameters in order to decrease the number of iterations.

We reckon that the combination of BEM, sensitivity analysis and Pattern method is a successful and convenient method for solving shape optimization problems not only in the elastic case, but in inelastic case as well.

Table 1

The geometric and mechanical characteristics of plate

d_1 [mm]	d_2 [mm]	l [mm]	ν	E [kg/mm^2]	γ^0	m_0 [kg/mm^2]	η_0 [kg.s/mm^2]
2	6	14	0.3	21000	0	1.65	$0.3844.10^{12}$

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