

GENERAL MECHANICS

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Kinematics of a Three-Degrees-of-Freedom Manipulator with Parallel Topology. Part I. Direct Kinematics

1. Introduction

This paper (in two parts) presents the complete kinematic analysis of a 3-dimensional mechanism with a parallel topology having three degrees of freedom. The possibility of utilizing this mechanism as a robotic manipulation system is examined.

Devices with open kinematic chains and motors in series are prevailing in the practice of robot engineering. Certain limitations are characteristic of systems with a closed kinematic type, such as reduced zone of reachability (at equal number of degrees of freedom), reduced manoeuvrability, etc. Simultaneously, robots of parallel topology have also a number of advantages: greater robustness, improved accuracy, better dynamic characteristics, ease of placing the motors in the base, etc. Furthermore, an accumulation of errors from the individual drives is produced in robots having drives in series, which is avoided in the parallel topology robots.

In this connection recently there has been greater interest in investigations in robots of parallel topology. In the prevailing part of the works dedicated to this subject, the investigations and developments have been directed towards six-degrees-of-freedom in-parallel actuated robots, of the type of the Stewart Platform [1], [2]. A systematic review on in-parallel actuated mechanism are presented in [3]. Kinematic analysis of a three-degrees-of-freedom (3-DOF) in-parallel actuated manipulator is presented in [4], [5], [6] and [7]. The solutions of direct and inverse kinematic problems are expressed in terms of Euler angles [4], [5], [6]. Lee and Shah [4] also examine the influence of physical constraints on the range of motion of such a manipulator and discuss some applications of a 3-DOF in-parallel actuated manipulator as part of a manipulation system. Waldron, Raghavan and Roth [8] present a rate and force decomposition of a six-degrees-of-freedom manipulator system, which is a combination of three series joints and three in-parallel actuated joints. Dynamic analysis of a 3-DOF in-parallel actuated manipulator is presented in [5] and [6] using the Lagrangian approach. In [9] the authors discuss an application of 3-DOF in-parallel actuated mechanism.

The geometric model of a parallel robot having three degrees of freedom is considered in [13] and [14]. Kinematic analysis of a 3-DOF in-parallel actuated mechanism is also presented in [10], [11] and [12]. In [11] the authors examine a possible application of this mechanism as part of a manipulation system and give a kinematic analysis of the whole system.

This article presents an analytical method of analysis of a 3-DOF mechanism of parallel topology. This method eliminates the use of Euler angles. The solutions of the direct and inverse kinematic problems are given explicitly. In this part are given the solutions of the direct kinematic problem of the 3-DOF mechanism of parallel topology.

Consider a closed three-dimensional mechanism of parallel topology (Fig. 1) of the 3RRS type (three symmetrical chains, each containing two rotational (R) joints and one spherical (S) joint).

Driving is effected by means of DC or stepping electric motors, angles φ_i being defined as control parameters. The S-joints with B_i centers in the upper platform and the R-joints with A_i centers in the lower platform are situated through 120° on circles having radii R_1 and R_2 , the lower platform taken as fixed.

Any change of the φ_i ($i=1, 2, 3$) angles results in changes of the distances $L_i=A_iB_i$ (Fig. 2) according to the formula

$$(1) \quad L_i = (M_i^2 + N_i^2 - 2M_iN_i \cos \varphi_i)^{1/2},$$

where M_i and N_i are the lengths RR and RS respectively (Fig. 1). Segments A_iB_i are projected onto the A_iO_i straight lines.

In the direct kinematic problem, for fixed angles φ_i (or L_i lengths) the position of the upper mobile platform is required, i. e. the coordinates of points B_i or point O_1 in the $O_2X_2Y_2Z_2$ coordinate system.

In the inverse kinematic problem, it is necessary to determine angles φ_i (or L_i lengths) as functions of the position of the mobile platform.

Further on we give explicit close form solutions of the direct kinematic problem for this class of mechanisms.

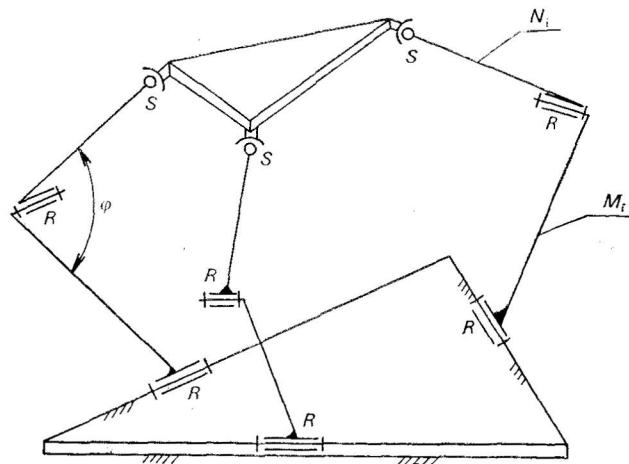


Fig. 1

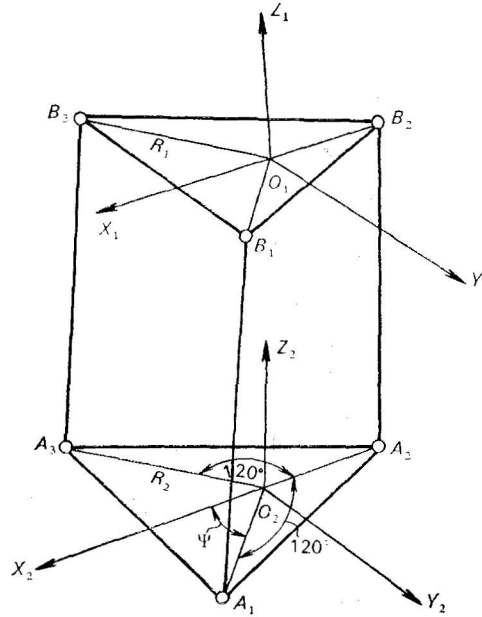


Fig. 2

2. Direct kinematic problem

Let us assume that the upper and lower platforms are connected to coordinate systems $O_1X_1Y_1Z_1$ and $O_2X_2Y_2Z_2$ (Fig. 2), where O_1 and O_2 are the centers of the circles on which the S - and R -joints are located. Axes O_1X_1 and O_2X_2 pass through points B_2 and A_2 , respectively. The remaining axes are selected so that the coordinate frames to be right-handed.

In order to find the transformation relating the coordinate systems, it is necessary to determine the coordinates of points B_1, B_2, B_3 in the $O_2X_2Y_2Z_2$ coordinate system. For this purpose we define the vectors

$$(2) \quad \begin{aligned} \mathbf{a}_i &= \mathbf{O}_2\mathbf{A}_i = (R_2 \cos \psi_i, R_2 \sin \psi_i, 0)^T, \\ \mathbf{b}_i &= \mathbf{O}_2\mathbf{B}_i = (U_i \cos \psi_i, U_i \sin \psi_i, V_i)^T; \\ & \quad i=1, 2, 3, \end{aligned}$$

where $\psi_i = (2i-1)\frac{\pi}{3}$ and U_i, V_i ($i=1, 2, 3$) are unknown quantities which are to be determined.

From the conditions

$$\begin{aligned} |\mathbf{a}_i - \mathbf{b}_i| &= L_i, \quad |\mathbf{b}_i - \mathbf{b}_j| = R_1 \sqrt{3}; \\ & \quad i, j=1, 2, 3; \quad j \neq i, \end{aligned}$$

we obtain a system of six quadratic algebraic equations for the unknown quantities U_i, V_i :

$$(3) \quad \begin{aligned} (U_i - R_2)^2 + V_i^2 &= L_i^2, \\ U_i^2 + U_j^2 + U_i U_j + (V_i - V_j)^2 &= 3R_1, \\ i, j &= 1, 2, 3; \quad j \neq i. \end{aligned}$$

Introducing the non-dimensional quantities $u_i = \frac{U_i}{R_1}$, $v_i = \frac{V_i}{R_1}$ and parameters $\lambda_i = \frac{L_i}{R_1}$, $\rho = \frac{R_2}{R_1}$, the system (3) takes the form

$$(4) \quad \begin{aligned} (u_i - \rho)^2 + v_i^2 &= \lambda_i^2, \\ u_i^2 + u_j^2 + u_i u_j + (v_i - v_j)^2 &= 3, \\ i, j &= 1, 2, 3; \quad j \neq i. \end{aligned}$$

The problem of existence of solutions of the quadratic system (4) can not be solved explicitly in the general case. That is why we shall derive necessary and sufficient conditions for existence of physically realizable solutions in some particular cases.

In solving (4), the following three cases are possible [3]:

1. Case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$

Here

$$(5) \quad u_1 = u_2 = u_3 = u, \quad v_1 = v_2 = v_3 = v$$

and system (4) is reduced to

$$(6) \quad (u - \rho)^2 + v^2 = \lambda^2, \quad 3u^2 = 3.$$

The solution of (6) is

$$(7) \quad u = 1, \quad v = \sqrt{\lambda^2 - (1 - \rho)^2}.$$

The condition for the existence of a solution is

$$(8) \quad |1 - \rho| \leq \lambda.$$

2. Case $\lambda_1 = \mu$, $\lambda_2 = \lambda_3 = \lambda$, ($\lambda \neq \mu$)

Now we have

$$(9) \quad u_2 = u_3 = u, \quad v_2 = v_3 = v.$$

(analogous cases $\lambda_1 = \lambda_2 \neq \lambda_3$ and $\lambda_1 = \lambda_3 \neq \lambda_2$ are produced by index permutation) and system (4) takes the form

$$(10) \quad \begin{aligned} (u_1 - \rho)^2 + v_1^2 &= \mu^2, \\ u_1^2 + u^2 + u_1 u + (v_1 - v)^2 &= 3, \\ (u - \rho)^2 + v^2 &= \lambda^2, \quad 3u^2 = 3. \end{aligned}$$

Hence

$$(11) \quad u = 1, \quad v = \sqrt{\lambda^2 - (1 - \rho)^2}$$

and for (10) we obtain

$$(12) \quad (u_1 - \rho)^2 + v_1^2 = \mu^2, \quad u_1^2 + u_1 + (v_1 - v)^2 = 2.$$

The solution of (12) is

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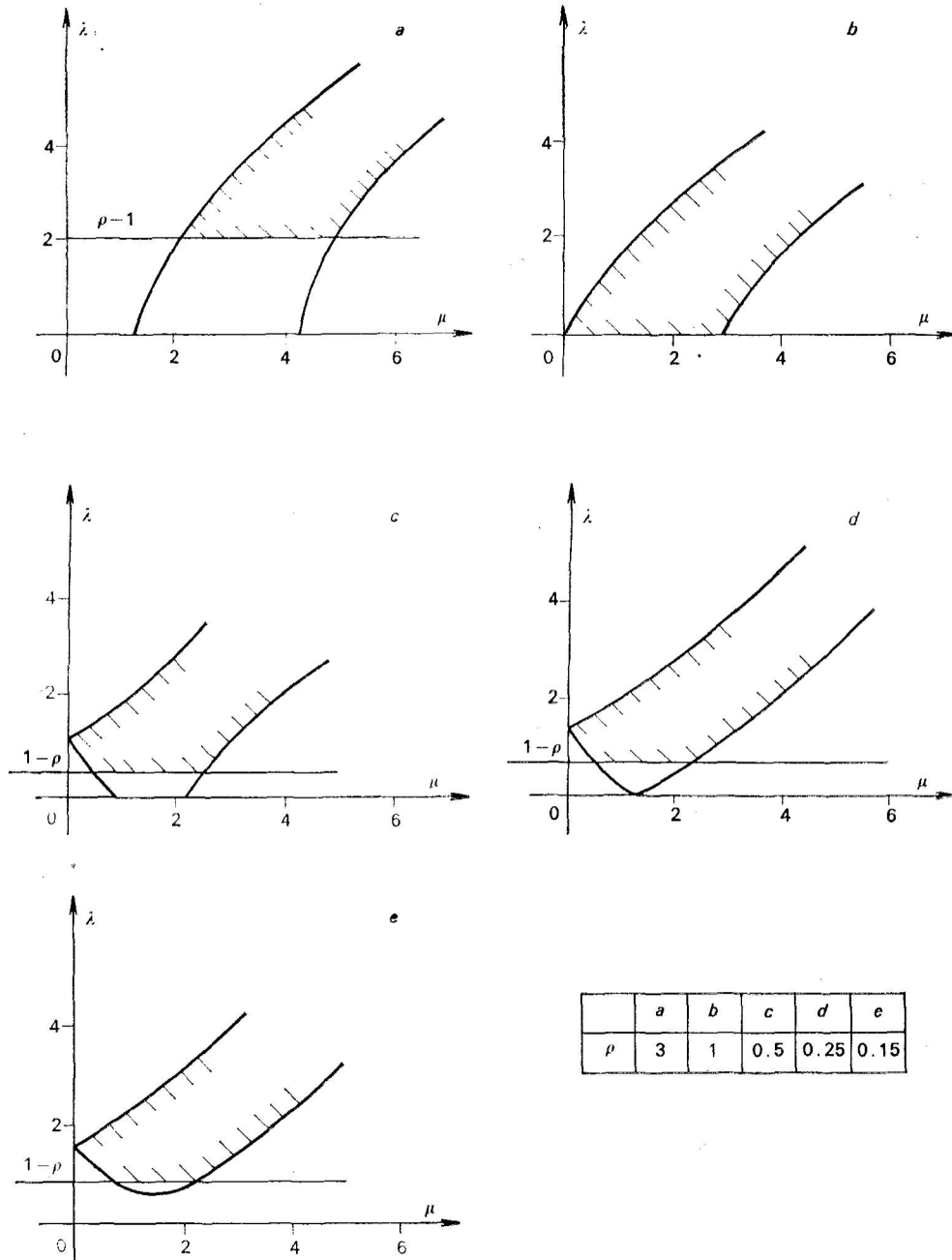


Fig. 3

$$(13) \quad \begin{aligned} u_1 &= \frac{v^2(2\rho-1)+(2\rho+1)(\rho^2-\mu^2+2)+2v\sqrt{D}}{(2\rho+1)^2+4v^2} = \frac{(2\rho-1)\lambda^2-(2\rho+1)\mu^2+6\rho^2+3+2v\sqrt{D}}{4\lambda^2+12\rho-3}, \\ v_1 &= \frac{2v(v^2+\mu^2+\rho^2+\rho-2)+(2\rho+1)\sqrt{D}}{(2\rho+1)^2+4v^2} = \frac{2v(\lambda^2+\mu^2+3\rho-3)+(2\rho+1)\sqrt{D}}{4\lambda^2+12\rho-3}, \end{aligned}$$

where

$$(14) \quad D = 9\mu^2 - (v^2 - \mu^2 + \rho^2 + \rho - 2)^2 = 9\mu^2 - (\lambda^2 - \mu^2 + 3\rho - 3)^2.$$

The conditions for existence of a solution in accordance with (11), (13) and (14) are

$$(15) \quad \mu^2 - 3\mu - 3(\rho - 1) \leq \lambda^2 \leq \mu^2 + 3\mu - 3(\rho - 1), \quad |1 - \rho| \leq \lambda.$$

The domains in the (μ, λ) -plane determined by (15) for different values of the parameter ρ are shown at Fig. 3a to 3e.

3. General Case

In this case the quadratic system (4) has no explicit solution. The solution should be found numerically. No necessary and sufficient conditions in terms of parameters λ, ρ (of the type of (8) or (15)) are known for the existence of physically attainable solutions.

After calculation of u_i, v_i from (4) the solution of the direct kinematic problem is given from

$$(16) \quad \begin{aligned} \mathbf{r}_2 &= \mathbf{T}\mathbf{r}_1 + \mathbf{r}_0; \quad \mathbf{r}_i = (x_i, y_i, z_i)^T \in \mathbb{R}^3, \\ \mathbf{T} &= [t_{ij}] \in \mathbb{R}^{3,3}, \end{aligned}$$

where

x_2, y_2, z_2 are the coordinates of a selected point from the mobile platform in the coordinate system $O_2X_2Y_2Z_2$; x_1, y_1, z_1 are the coordinates of the same point in the coordinate system $O_1X_1Y_1Z_1$; x_0, y_0, z_0 are the coordinates of point O_1 in the coordinate system $O_2X_2Y_2Z_2$; t_{ij} are the direction cosines of axes O_1X_1, O_1Y_1, O_1Z_1 in the $O_2X_2Y_2Z_2$ system.

Denote by $\mathbf{b}_i^0 = (b_{i1}, b_{i2}, b_{i3})^T$ the vectors \mathbf{b}_i from (2) for $U_i = u_i, V_i = v_i$. Then, axes O_1X_1, O_1Y_1, O_1Z_1 are co-directional to vectors $\mathbf{X}^0 = (\mathbf{b}_1^0 - \mathbf{b}_2^0) + (\mathbf{b}_3^0 - \mathbf{b}_2^0)$, $\mathbf{Y}^0 = (\mathbf{b}_1^0 - \mathbf{b}_3^0)$ and $\mathbf{Z}^0 = \mathbf{X}^0 \times \mathbf{Y}^0$, respectively. To determine the direction cosines t_{ij} we use the formula

$$(17) \quad t_{ij} = \frac{T_{ij}}{d_j},$$

where

$$\begin{aligned} \mathbf{X}^0 &= (T_{11}, T_{21}, T_{31})^T, \quad \mathbf{Y}^0 = (T_{12}, T_{22}, T_{32})^T, \\ \mathbf{Z}^0 &= (T_{13}, T_{23}, T_{33})^T, \end{aligned}$$

and d_1, d_2, d_3 are the norms of the vectors $\mathbf{X}^0, \mathbf{Y}^0, \mathbf{Z}^0$. Hence, for t_{ij} we obtain

$$(18) \quad \begin{aligned} t_{11} &= \frac{1}{6}(u_1 + 4u_2 + u_3), \\ t_{21} &= \frac{\sqrt{3}}{6}(u_1 - u_3), \end{aligned}$$

$$\begin{aligned}
 t_{31} &= \frac{1}{3}(v_1 - 2v_2 + v_3), \\
 t_{12} &= t_{21}, \quad t_{22} = \frac{1}{2}(u_1 + u_3), \\
 t_{32} &= \frac{\sqrt{3}}{3}(v_1 - v_3); \\
 t_{13} &= \frac{1}{3}[u_1(v_2 - v_3) + u_3(v_2 - v_1)], \\
 t_{23} &= \frac{\sqrt{3}}{9}[u_1(v_3 - v_2) + 2u_2(v_3 - v_1) + u_3(v_2 - v_1)], \\
 t_{33} &= \frac{1}{3}(u_1u_2 + u_2u_3 + u_1u_3).
 \end{aligned}
 \tag{19}$$

For the coordinates of point O_1 in the coordinate system $O_2X_2Y_2Z_2$ we have

$$\begin{aligned}
 x_0 &= \frac{R_1}{3}(b_{11}^0 + b_{21}^0 + b_{31}^0), \\
 y_0 &= \frac{R_1}{3}(b_{12}^0 + b_{22}^0 + b_{32}^0), \\
 z_0 &= \frac{R_1}{3}(b_{13}^0 + b_{23}^0 + b_{33}^0),
 \end{aligned}
 \tag{21}$$

or, if expressed by u_i, v_i ,

$$\begin{aligned}
 x_0 &= \frac{R_1}{6}(u_1 - 2u_2 + u_3) \\
 y_0 &= \frac{\sqrt{3}}{6}R_1(u_1 - u_3), \\
 z_0 &= \frac{R_1}{3}(v_1 + v_2 + v_3).
 \end{aligned}
 \tag{22}$$

Relations (16)—(22), in view of (4), give the solution of the direct kinematic problem. In the particular cases, the solution is produced by (5), (7) or (9), (11), (13), (14). Case 1 is trivial and is given here for completeness. Note, however, that cases 1 and 2 are important for mechanisms with discrete positioning (e. g. with only two possible values of L_i or φ_i). For such mechanisms the relations (5), (7), (8) and (9) (11), (13)—(15) give the complete solution of the direct kinematic problem [10].

3. Summary

In this paper the direct kinematics of the three-degrees-of-freedom mechanism of parallel topology are presented. The solutions of the inverse kinematic problem, the restrictions on the mobile platform movement and the applications of the mechanism of parallel topology to manipulation systems are presented in a companion paper [15].

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