BULGARIAN ACADEMY OF SCIENCES
JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
Sofia 1991, Year XXII, No 4

N. Bontcheva

Influence of Damage on Plastic Localization in Anisotropically Hardening Materials

1. Introduction

Many experiments showed that bands of localized plastic deformation initiate under certain conditions during metal forming processes [1]. Theoretically the problem was widely discussed. Most of the authors consider conditions for plastic localization bands initiation in homogeneous strain fields [2, 3, 4, 5]. The case of nonhomogeneous plane strain and coupled thermoplastic processes was considered in [7] for a rigid-plastic isotropically hardening material. Conditions for initiation of plastic localization bands in anisotropically hardening materials are obtained in [8].

Plastic strains inside the localization bands are much higher than in the rest of the material and they may cause void initiation and nucleation. The development of this damage process leads to softening of the material. A coupling between the thermoplastic and the damage processes exists. A mechanical model, describing the material damage during metal forming processes was proposed in [6]. In [9] this model was used to obtain the necessary conditions for plastic localization bands initiation in isotropically hardening rigid-plastic materials. The case of anisotropically hardening materials will be discussed here.

2. Mechanical model

The mechanical model of anisotropically hardening materials was proposed in [10]. The case of nonsymmetric anisotropic hardening was considered in [11]. According to that model a yield condition of the Mises type exists:

$$F=II_{\bar{s}}^2+\frac{1}{2}A\Sigma_{III}^2-[\varphi(\chi)+\tau_p^*]^2=0, \tag{2.1}$$

where

$$II_{\bar{s}}^2=\frac{1}{2}\bar{s}_{ij}\bar{s}_{ij},\quad \Sigma_{III}=\bar{s}_{ij}s_{ij}^{\mu},\quad \chi=\frac{\Sigma_{III}}{II_{\bar{s}}II_{\mu}},\quad II_{\mu}^2=\frac{1}{2}s_{ij}^{\mu}s_{ij}^{\mu}. \tag{2.2}$$

$\bar{s}_{ij}=s_{ij}-s_{ij}^{\mu}$ is the active stress deviator; $s_{ij}=\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}$ is the deviator of the Cauchy stress tensor σ_{ij}; s^{μ}_{ij} is the microstress deviator. The material functions A and τ^*_p depend on the strain rate

$$\beta=\frac{d\gamma}{dt}=\sqrt{\frac{1}{2}\lambda_{ij}\lambda_{ij}}\,, \tag{2.3}$$

on the plastic strain

$$\gamma=\int\limits_0^t \beta\, dt \tag{2.4}$$

and on the temperature T. The material is rigid-plastic and the strain rate coincides with the plastic strain rate:

$$\lambda_{ij}=\frac{1}{2}\left(v_{i,j}+v_{j,i}\right) \tag{2.5}$$

where v_i is the velocity vector. The function $\varphi(\chi)$ takes into account the nonsymmetric character of the subsequent yield surface and depends on the factor χ, expressing the difference between the directions of the preliminary and the subsequent loading [11].

According to [6] damage has a softening effect, expressed by the scalar damage variable D, appearing in the yield condition. In that case (2.1) takes the form:

$$F\equiv II^2_{\bar{s}}+\frac{1}{2}A\Sigma^2_{III}-(1-D)^2[\varphi(\chi)+\tau^*_p]^2=0. \tag{2.6}$$

If no damage is available $D=0$ and (2.6) coincides with (2.1).

The flow rule is associated with the yield condition (2.6) and it has the form

$$\lambda_{ij}=\frac{d\lambda}{dt}F_{ij}, \quad \lambda_{kk}=0, \tag{2.7}$$

where

$$F_{ij}=\frac{\partial F}{\partial \sigma_{ij}}=\bar{s}_{ij}+A\,\Sigma_{III}\,s^{\mu}_{ij}-\frac{2(\varphi+\tau^*_p)(1-D)^2}{II_{\bar{s}}II_{\mu}}\varphi'\left(s^{\mu}_{ij}-\frac{\Sigma_{III}}{2\,II^2_{\bar{s}}}\bar{s}_{ij}\right). \tag{2.8}$$

The equation of evolution for the microstresses is [11]

$$\frac{d\,s^{\mu}_{ij}}{dt}=Q^{\mu}\lambda_{ij} \tag{2.9}$$

where $Q^{\mu}=Q^{\mu}(\gamma, \beta, T)$ is a material function.

The damage variable changes according to the law [6]

$$\frac{dD}{dt}=\begin{cases}\dfrac{D^*_c}{\varepsilon_R-\varepsilon_D}R_v\beta & \text{if} \quad \gamma\geq\dfrac{\sqrt{3}}{2}\dfrac{\varepsilon_D}{R_v}\\ 0 & \text{if} \quad \gamma<\dfrac{\sqrt{3}}{2}\dfrac{\varepsilon_D}{R_v}\end{cases} \tag{2.10}$$

where $D_c^*=\frac{2}{\sqrt{3}}D_c$ and D_c, ε_R and ε_D are material constants, described in [6]. R_v is a factor, taking into account the triaxiality of the stress state [6].

The temperature equation yields:

$$\frac{dT}{dt}=\frac{k_T}{\rho c_T}T_{,ii}+Q, \tag{2.11}$$

where K_T is the conductivity coefficient, ρ is the material density, c_T is the specific heat per unit volume and

$$Q=\frac{k}{\rho c_T}\bar{\sigma}_{ij}\lambda_{ij} \tag{2.12}$$

is the strain rate energy, turning into heat. The Taylor-Quinney coefficient k takes the value 0.9-0.95.

These are the basic equations, describing the problem. It is worth here mentioning the following: The anisotropic hardening of the materials is a result of preliminary and actual plastic deformation. The damage is isotropic, described by means of a scalar variable D. The influence of the plastic anisotropy on damage is taken into account only by means of the plastic strain rate invariant β, which depends on the strain rate field, determined by the associated flow rule. Anisotropic damage, which could take place in connection with the oriented plastic strain is not considered here.

Eqs (2.3) and (2.7) yield

$$\beta=\frac{d\lambda}{dt}II_F, \tag{2.13}$$

where

$$II_F=\sqrt{\frac{1}{2}F_{ij}F_{ij}}, \tag{2.14}$$

The consistency condition $\frac{dF}{dt}=0$ has the form

$$\frac{dF}{dt}=\frac{\partial F}{\partial \sigma_{ij}}\frac{d\sigma_{ij}}{dt}+\frac{\partial F}{\partial s_{ij}^{\mu}}\frac{ds_{ij}^{\mu}}{dt}+\frac{\partial F}{\partial \gamma}\frac{d\gamma}{dt} +\frac{\partial F}{\partial \beta}\frac{d\beta}{dt}+\frac{\partial F}{\partial T}\frac{dT}{dt}+\frac{\partial F}{\partial D}\frac{dD}{dt}=0, \tag{2.15}$$

or

$$\frac{dF}{dt}=F_{ij}\frac{d\sigma_{ij}}{dt}+S_{ij}\frac{ds_{ij}^{\mu}}{dt}+F_{\gamma}\frac{d\gamma}{dt}+F_{\beta}\frac{d\beta}{dt}+F_T\frac{dT}{dt}+F_D\frac{dD}{dt}=0, \tag{2.16}$$

where

$$S_{ij}=\frac{\partial F}{\partial s_{ij}^{\mu}}=-\bar{s}_{ij}+A\,\Sigma_{III}(\bar{s}_{ij}-s_{ij}^{\mu})-\frac{2(\varphi+\tau_p^*)(1-D)^2}{II_{\bar{s}}\,II_{\mu}}\varphi'\left[\left(1+\frac{\Sigma_{III}}{2II_{\bar{s}}^2}\right)\bar{s}_{ij}\right. \tag{2.17}$$

$$-\left(1+\frac{\Sigma_{III}}{2II_{\mu}^{2}}\right)s_{ij}^{\mu}\Bigg],$$

$$F_{\gamma}=\frac{\partial F}{\partial\gamma}=\frac{1}{2}\Sigma_{III}^{2}A_{\gamma}-2(\varphi+\tau_{p}^{*})(1-D)^{2}(\varphi_{\gamma}+\tau_{\gamma}),$$

$$F_{\beta}=\frac{\partial F}{\partial\beta}=\frac{1}{2}\Sigma_{III}^{2}A_{\beta}-2(\varphi+\tau_{p}^{*})(1-D)^{2}(\varphi_{\beta}+\tau_{\beta}),$$

$$F_{T}=\frac{\partial F}{\partial T}=\frac{1}{2}\Sigma_{III}^{2}A_{T}-2(\varphi+\tau_{p}^{*})(1-D)^{2}(\varphi_{T}+\tau_{T}),$$

$$F_{D}=\frac{\partial F}{\partial D}=2(\varphi+\tau_{p}^{*})^{2}(1-D),$$

$$A_{\xi}=\frac{\partial A}{\partial\xi},\quad \varphi_{\xi}=\frac{\partial\varphi}{\partial\xi},\quad \tau_{\xi}=\frac{\partial\tau_{p}}{\partial\xi},\quad \xi=\gamma,\ \beta,\ T.$$

Taking into account the flow rule (2.7), the expression (2.12) becomes

$$Q=\frac{k}{\rho c_{T}}\bar{s}_{ij}\lambda_{ij}=\frac{k}{\rho c_{T}}\frac{d\lambda}{dt}\bar{s}_{ij}F_{ij}=2\frac{k}{\rho c_{T}}\frac{d\lambda}{dt}(\varphi+\tau_{p}^{*})^{2}(1-D)^{2}$$
$$=2\frac{k}{\rho c_{T}}\frac{\beta}{II_{F}}(\varphi+\tau_{p}^{*})^{2}(1-D)^{2}. \tag{2.18}$$

Substituting (2.9), (2.11), (2.18), (2.13) and (2.10) into (2.16), we obtain

$$\frac{dF}{dt}=F_{ij}\frac{d\sigma_{ij}}{dt}+L_{\gamma}\frac{d\gamma}{dt}+L_{\beta}\frac{d\beta}{dt}+L_{0}=0, \tag{2.19}$$

where

$$L_{\gamma}=Q^{\mu}\frac{\Sigma_{II}}{II_{F}}+\frac{1}{2}\Sigma_{III}^{2}A_{\gamma}+\frac{k}{\rho c_{T}}\Sigma_{III}^{2}A_{T}(\varphi+\tau_{p}^{*})^{2}(1-D)^{2}\frac{1}{II_{F}}$$
$$-2(\varphi+\tau_{p}^{*})(1-D)^{2}(\varphi_{\gamma}+\tau_{\gamma})-4\frac{k}{\rho c_{T}}(\varphi+\tau_{p}^{*})^{3}(1-D)^{4}(\varphi_{T}+\tau_{T})\frac{1}{II_{F}}$$
$$+2\frac{D_{c}^{*}}{\varepsilon_{R}-\varepsilon_{D}}R_{v}(\varphi+\tau_{p}^{*})^{2}(1-D). \tag{2.20}$$

$$L_{\beta}=\frac{1}{2}\Sigma_{III}^{2}A_{\beta}-2(\varphi+\tau_{p}^{*})(1-D)^{2}(\varphi_{\beta}+\tau_{\beta}).$$

$$L_{0}=\left[\frac{1}{2}\Sigma_{III}^{2}A_{T}-2(\varphi+\tau_{p}^{*})(1-D)^{2}(\varphi_{T}+\tau_{T})\right]\frac{k_{T}}{\rho c_{T}}T_{ii},$$

$$\Sigma_{II}=S_{ij}F_{ij}.$$

$\frac{d\lambda}{d\tau}$, obtained from 2.19) has the form

$$\frac{d\lambda}{dt}=\frac{1}{II_{F}}\frac{d\gamma}{dt}=\left|\frac{F_{ij}\frac{d\sigma_{ij}}{dt}+L_{\beta}\frac{d\beta}{dt}+L_{0}}{II_{F}L_{\gamma}}\right|. \tag{2.21}$$

3. Plastic localization

The necessary conditions for plastic localizaion bands initiation under plain non-homogeneous plastic flow will be obtaned as conditions for bifurcation of the rates of the process parameters. If at time t_0 such a bifurcation takes place, inside a band of the width $2d$, plastic localization will be available inside this band at $t>t_0$.

Introducing a convective orthogonal curvilinear coordinate system (η, ξ) so, that $\xi=\text{const}$ are lines parallel to the middle line L of the band and $\eta=\text{const}$ are straight lines, normal to L and assuming the localization band to be thin and material, we apply the straight normals hypothesis [7] and the basic equations describing the process take the form

$$\begin{gathered}
\lambda_{\alpha\beta}=\frac{1}{2}(v_{\alpha|\beta}+v_{\beta|\alpha}),\\
\sigma_{\alpha\beta|\beta}=0,\\
\lambda_{\alpha\beta}=\dot{\lambda}F_{\alpha\beta},\quad F_{\alpha\beta}=\frac{\partial F}{\partial\sigma_{\alpha\beta}},\quad \dot{\lambda}=\frac{\beta}{II_F},\\
\dot{T}=\frac{k_T}{\rho c_T}q_{\alpha|\alpha}+2\frac{k}{\rho c_T}(\varphi+\tau_p^*)(1-D)^2\frac{\beta}{II_F},\quad q_\alpha=T_{,\alpha}.\\
(\alpha,\ \beta=\eta,\ \xi)
\end{gathered}\tag{3.1}$$

Outside the localization band all process variables are unique. Inside the band the following bifurcation takes place

$$\Delta\lambda_{\alpha\beta}\neq 0,\quad \Delta\dot{\sigma}_{\alpha\beta}\neq 0,\quad \Delta\dot{s}^{\mu}_{\alpha\beta}\neq 0,\quad \Delta\dot{T}\neq 0,\quad \Delta v_\alpha\neq 0.\tag{3.2}$$

Due to the plastic incompressibility, plastic localization bands are shear bands [4]

$$\Delta\lambda_{\xi\xi}=0,\quad \Delta\lambda_{\eta\eta}=0,\quad \Delta\lambda_{\xi\eta}\neq 0.\tag{3.3}$$

The compatibility condition (3.1) leads to

$$\Delta\lambda_{\alpha\beta}=\frac{1}{2}(\Delta v_{\alpha|\beta}+\Delta v_{\beta|\alpha})\tag{3.4}$$

and the condition, that the solution is unique outside the bifurcation band yields

$$\Delta v_\alpha(-d)=\Delta v_\alpha(d)=0.\tag{3.5}$$

It follows from (3.3), (3.4) and (3.5) that

$$\Delta v_\xi=0,\quad \Delta v_\eta=\Delta v_\eta(\xi)\tag{3.6}$$

inside the localization band. Introducing the bifurcation vector g_α, $(\alpha=\eta, \xi)$, the relation (3.4) leads to [4]

$$\Delta\lambda_{\alpha\beta}=\frac{1}{2}(g_\alpha n_\beta+g_\beta n_\alpha)\tag{3.7}$$

and hence

$$\Delta\lambda_{\gamma\gamma}=g_\gamma n_\gamma=0,\quad g_\alpha(-d)=g_\alpha(d)=0,\tag{3.8}$$

or

(3.9) $$\Delta \lambda_{\xi\eta}=\frac{1}{2} g_{\eta}(\xi), \quad \Delta \lambda_{\xi\xi}=0, \quad \Delta \lambda_{\eta\eta}=0$$

and

(3.10) $$g_{\eta}(\xi)\neq 0, \quad g_{\xi}=0.$$

The flow rule $(3.1)_3$ leads to the relation

(3.11) $$\Delta \lambda_{\alpha\beta}=\Delta \dot{\lambda} F_{\alpha\beta}$$

and due to (3.7)

(3.12) $$F_{\alpha\beta}=\frac{1}{2}(\mu_{\alpha} n_{\beta}+\mu_{\beta} n_{\alpha}), \quad \mu_{\alpha}=\frac{1}{\Delta \dot{\lambda}} g_{\alpha}.$$

The consistency condition (2.19) leads to

(3.13) $$\Delta \dot{F}=F_{\alpha\beta}\Delta \dot{\sigma}_{\alpha\beta}+L_{\gamma}\Delta \dot{\gamma}+L_{\beta}\Delta \dot{\beta}=0.$$

Taking into account (3.12), the first term in (3.13) is

(3.14) $$F_{\alpha\beta}\Delta \dot{\sigma}_{\alpha\beta}=\frac{1}{2}(\mu_{\alpha} n_{\beta}+\mu_{\alpha} n_{\beta})\Delta \dot{\sigma}_{\alpha\beta}=0,$$

as due to equilibrium $\Delta \dot{\sigma}_{\alpha\beta} n_{\beta}=0$. The condition (3.13) takes then the form

(3.15) $$L_{\gamma}\Delta \dot{\gamma}+L_{\beta}\Delta \dot{\beta}=0.$$

This equation will be fulfilled in the following cases:

i. $\Delta \dot{\gamma}=0$, $\Delta \dot{\beta}=0$. No bifurcation takes place then.

ii. $\Delta \dot{\gamma}\neq 0$, $\Delta \dot{\beta}=0$. Bifurcation of strain rates is available, but no bifuration of strain accelerations exists. In that case the condition

(3.16) $$L_{\gamma}=0$$

should be fulfilled.

iii. $\Delta \dot{\gamma}=0$, $\Delta \dot{\beta}\neq 0$. Only bifurcation of strain accelerations is available. This leads at the next moment to bifurcation of strain rates too. The condition

(3.17) $$L_{\beta}=0$$

should be fulfilled in that case.

iiii. $\Delta \dot{\gamma}\neq 0$, $\Delta \dot{\beta}\neq 0$. This means bifurcation of strain rates and strain accelerations. The necessary conditions are

(3.18) $$L_{\gamma}=0, \; L_{\beta}=0.$$

Conditions (3.18) are the necessary conditions for initiation of plastic localization bands in the general case. Substituting (2.20) into (3.18) we obtain

(3.19) $$Q^{\mu}\Sigma_{II}+F_{\gamma}+2\frac{k}{\rho c_{T}}(\varphi+\tau_{p}^{*})^{2}(1-D)^{2}\frac{1}{II_{F}}F_{T}+\frac{D_{c}^{*}}{\varepsilon_{R}-\varepsilon_{D}}R_{v}F_{D}=0,$$
$$F_{\beta}=0.$$

This could be fulfilled in the particular case when $F_\alpha = F_\beta = F_T = F_D = 0$. This requires that

$$
\begin{aligned}
&\frac{A_\gamma}{\varphi_\gamma+\tau_\gamma}=\frac{A_\beta}{\varphi_\beta+\tau_\beta}=\frac{A_T}{\varphi_T+\tau_T}=\frac{4(\varphi+\tau_p^*)(1-D)^2}{\Sigma_{III}^2},\\
(3.20)\qquad &Q^\mu=-2\frac{D_c^*}{\varepsilon_R-\varepsilon_D}R_v(\varphi+\tau_p^*)^2(1-D)\frac{1}{\Sigma_{II}}.
\end{aligned}
$$

In the regions of the deformed body, where relations (3.20) are fulfilled, plastic localization is possible to take place. Substituting (2.20) into the more general case (3.19) we obtain

$$
\begin{aligned}
(3.21)\qquad &Q^\mu\Sigma_{II}+II_F\Big\{\frac{1}{2}\Sigma_{III}^2A_\gamma-2(1-D)(\varphi+\tau_p^*)\Big[(1-D)(\varphi_\gamma+\tau_\gamma)\\
&-\frac{D_c^*}{\varepsilon_R-\varepsilon_D}R_v(\varphi+\tau_p^*)\Big]\Big\}+\frac{2k}{\rho c_T}(\varphi+\tau_p^*)^2(1-D)^2\Big[\frac{1}{2}\Sigma_{III}^2A_T\\
&-2(1-D)^2(\varphi+\tau_p^*)(\varphi_T+\tau_T)\Big]=0,\\
&\frac{1}{2}\Sigma_{III}^2A_\beta-2(\varphi+\tau_p^*)(1-D)^2(\varphi_\beta+\tau_\beta)=0.
\end{aligned}
$$

These are the necessary conditions for initiation of plastic localization bands in anisotropially hardening materials, when damage is taken into account. The case of a material which is symmetrically hardening is easily obtained, substituting $\varphi=\varphi_\gamma=\varphi_\beta=\varphi_T=0$. If damage does not take place, then $D=0$ and (3.21) takes the form

$$
\begin{aligned}
(3.22)\qquad &Q^\mu\Sigma_{II}+II_F\Big[\frac{1}{2}A_\gamma\Sigma_{III}^2-2(\varphi+\tau_p^*)(\varphi_\gamma+\tau_\gamma)\Big]+2\frac{k}{\rho c_T}(\varphi+\tau_p^*)^2\Big[\frac{1}{2}\Sigma_{III}^2A_T\\
&-2(\varphi+\tau_p^*)(\varphi_T+\tau_T)\Big]=0,\\
&\frac{1}{2}\Sigma_{III}^2A_\beta-2(\varphi+\tau_p^*)(\varphi_\beta+\tau_\beta)=0.
\end{aligned}
$$

If the material is isotropically hardening, then $Q^\mu=A=\varphi=\Sigma_{III}=0$ and (3.21) yield

$$
\begin{aligned}
(3.23)\qquad &\tau_\gamma+\frac{2k}{\rho c_T}(1-D)\tau_p\tau_T-\frac{D_c^*}{\varepsilon_R-\varepsilon_D}\frac{1}{1-D}R_v\tau_p=0,\\
&\tau_\beta=0,
\end{aligned}
$$

which coincides with the result, obtained in [9].

References

1. Yanardhana, M. N., S. K. Biswas. Modes of deformation in aluminium rings subjected to static axial compression. — Int. J. Mech. Sci., 21, 1979, 699-712.
2. Lippmann, H. Velocity field equations and strain localization. — Int. J. Solids Struct., 18, 1986, 1399-1409.
3. Lippmann, H. Velocity field equations and strain localization in rigid-plastic materials. — Int. J. Solids Struct., 25, 1989, 459-464.
4. Rice, J. R. The localization of plastic deformation. — In: Koiter W. T. (Ed.), Theor. and Appl. Mech., North Holland Publ., 1976.
5. Rice, J. R., J. W. Rudnicki. A note of some features of the theory of localization of deformation. — Int. J. Solids Struct., 16, 1980, 597-605.
6. Lemaitre, J. Formulation of damage kinetic constitutive equations. — In: Krajcinovic D., J. Lemaitre (Eds), Continuum Damage Mechanics, Theory and Applications, CISM courses and Lectures No 295, 37-89.
7. Bontcheva, N., A. Baltov, St. Todorov, M. Pesheva. Plastic localization bands at nonhomogeneous plane strain and coupled thermoplastic process. — Theor. and Appl. Mech., 4, 1988, 85-94.
8. Bontcheva, N. Plastic localization in thermoplastic materials with deformational anisotropy. — ZAMM, 68, 1988, T159-T160.
9. Bontcheva, N., A. Baltov. Plastic localization in damaged materials. — Anniversary volume Prof. Lippmann, Springer-Verlag, Berlin, 1991, 242-251.
10. Балтов, А., Н. Бончева, Р. Казанджиев, Ив. Радованов, Ст. Воденичаров. Метод за изследване на микронапреженията в пластични материали, чувствителни към скоростта на деформация. — Теорет. и прил. мех., 2, 1982, 73-91.
11. Bontcheva, N., A. Baltov, St. Todorov. A rule of nonsymmetric anisotropic hardening. — Theor. and Appl. Mech., 1, 1985, 70-83.

Received on February 1, 1991