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GENERAL MECHANICS

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Kinematics of a Three-Degrees of Freedom Manipulator with Parallel Topology Part II: Inverse Kinematics and Application

1.Introduction

This paper presents a kinematic analysis of a three-degrees-of-freedom (3-DOF) mechanism with parallel topology. In this part are given the solutions of the inverse kinematic problem of this mechanism. The inverse problem is solved in a compact form which is very useful for the real time control. A kinematic analysis of a manipulation system containing a 3-DOF mechanism is presented. The constraints imposed on the physical-geometric parameters of the real 3-DOF mechanism with parallel topology are also considered.

Since this paper is connected with the companion one [1] (part I), it is convenient to continue the numeration of the equations.

2. Inverse kinematics

In the inverse kinematic problem the orientation of the mobile platform and the vector $\mathbf{r}_0 = \mathbf{O}_2\mathbf{O}_1$ are given, while the generalized coordinates, i.e., the angles φ_i (i=1,2,3) (Fig. 1) are sought.

We assume that the platform has an axis of symmetry O_1Z_1, i.e. rotations along this axis are indistinguishable. Hence, the orientation of the mobile platform is determined only by the unit vector

(23) $$\mathbf{k} = (x,\ y,\ z)^{T},\quad x^2 + y^2 + z^2 = 1\ ,$$

along axis O_1Z_1 (Fig. 2). As in this type of mechanism the projection of $\mathbf{r}_o'$ onto the plane $O_2X_2Y_2$ depends on $\mathbf{k}$, only the z_0 projection of $\mathbf{r}_0$ along axis O_2Z_2 should be given.

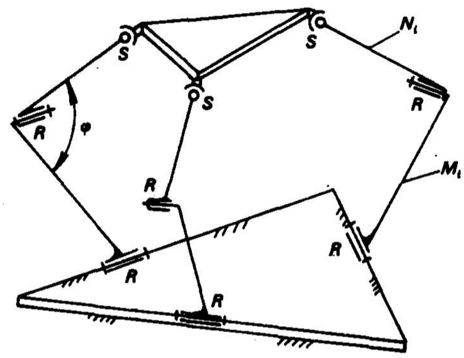

Fig. 1

The coordinates of $\mathbf{k}$ in the $O_1X_1Y_1Z_1$ coordinate system are 0,0,1. Whence, according to (23),

$$(x,\ y,\ z)^{T} = \mathbf{T}(0,0,1)^{T}\ .$$

Therefore the transformation matrix $\mathbf{T}$ has the form

(24) $$\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} & x \\ t_{12} & t_{22} & y \\ t_{31} & t_{32} & z \end{bmatrix}\ .$$

From the orthogonality of matrix $\mathbf{T}$ and equations (18)-(20) [1], two equations for v_1 are produced. A third equation is obtained from (22) [1], i.e.

$$\frac{1}{3}\left(v_1 + v_3 - 2v_2\right) = -x \,,$$

(25)
$$\frac{1}{\sqrt{3}}\left(v_1 - v_3\right) = -y \,,$$

$$\frac{R_1}{3}\left(v_1 + v_2 + v_3\right) = z_0 \,.$$

From system (25) we get

$$v_1 = -\frac{1}{2}x - \frac{\sqrt{3}}{2}y + \frac{z_0}{R_1} \,,$$

(26)
$$v_2 = x + \frac{z_0}{R_1} \,,$$

$$v_3 = -\frac{1}{2}x + \frac{\sqrt{3}}{2}y + \frac{z_0}{R_1} \,.$$

The unknown quantities u_1, u_2, u_3 are found from the three equations of system (4) [1],

(27)
$$u_1^2 + u_2^2 + u_1u_2 = 3 - (v_1 - v_2)^2 \,,$$

(28)
$$u_2^2 + u_3^2 + u_2u_3 = 3 - (v_2 - v_3)^2 \,,$$

(29)
$$u_3^2 + u_1^2 + u_3u_1 = 3 - (v_3 - v_1)^2 \,.$$

To solve system (27)-(29) we introduce the notation

(30)
$$P = u_1 + u_2 + u_3 \,,$$

(31)
$$Q = u_1u_2 + u_2u_3 + u_3u_1 \,,$$

(32)
$$W^2 = (v_1 - v_2)^2 + (v_2 - v_3)^2 + (v_3 - v_1)^2 \,.$$

Adding eguations (27)-(29), while keeping in mind (30)-(32), we obtain

(33) $$2P^2 - 3Q = 9 - W^2 .$$

If we set $C_{ij} = 3 - u_i^2 - u_j^2 - u_i u_j$, then it follows from (27)-(29)

$$v_1 - v_2 = \pm \sqrt{C_{12}} ,$$

$$v_2 - v_3 = \pm \sqrt{C_{23}} ;$$

$$v_3 - v_1 = \pm \sqrt{C_{31}} ,$$

(34) $$\pm \sqrt{C_{12}} \pm \sqrt{C_{23}} \pm \sqrt{C_{31}} = 0 .$$

From (34) we get

(35) $$C_{12}^2 + C_{23}^2 + C_{31}^2 - 2\left(C_{12}C_{23} + C_{23}C_{31} + C_{31}C_{12}\right) = 0 .$$

It may be shown that equation (35) is equivalent to

(36) $$3\left(2P - Q - 3\right)\left(2P + Q + 3\right) = 0 .$$

For physically feasible solutions ($u_i \geq 0$) equation (36) yields

(37) $$2P - Q - 3 = 0 .$$

Now relations (37) and (33) give

$$P = \frac{1}{2}(Q + 3) , \quad Q^2 = 9 - 2W^2 ,$$

or

(38) $$Q = \pm\sqrt{9 - 2W^2} ,$$

(39) $$P = \frac{1}{2}\left(3 \pm \sqrt{9 - 2W^2} \right) .$$

For solutions $u_i \geq 0$ we take the + sign in (38) and (39)

(40) $$Q = \sqrt{9 - 2W^2} ,$$

(41) $$P = \frac{1}{2}\left(3 + \sqrt{9 - 2W^2} \right) .$$

The condition for the existence of a solution according to (40), (41) is

$$W^2 \leq \frac{9}{2} .$$

It follows from equations (27) and (28) that

$$(42) \qquad \left(u_3 - u_1\right)P = \left(v_3 - v_1\right)\left(2v_2 - v_1 - v_3\right) ,$$

from (27) and (29)-

$$(43) \qquad \left(u_2 - u_3\right)P = \left(v_2 - v_3\right)\left(2v_1 - v_2 - v_3\right) ,$$

and from (29) and (28)-

$$(44) \qquad \left(u_1 - u_2\right)P = \left(v_1 - v_2\right)\left(2v_3 - v_1 - v_2\right) .$$

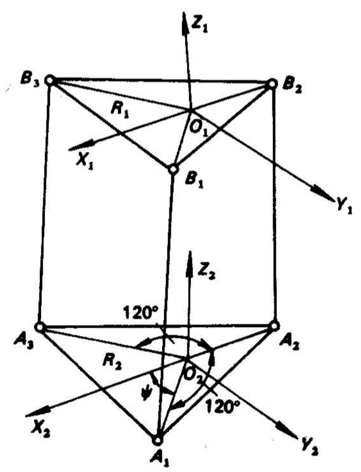

Fig.2

The solution of system (42) - (44) is

(45) $$u_1 = 1 + \left(v_1 - v_2\right)\left(v_3 - v_1\right)/P\ ,$$

(46) $$u_2 = 1 + \left(v_1 - v_2\right)\left(v_2 - v_3\right)/P\ ,$$

(47) $$u_3 = 1 + \left(v_2 - v_3\right)\left(v_3 - v_1\right)/P\ ,$$

After substituting v_i from (26), we get for P from (41)

$$P = \frac{3}{2}\left(1 + \sqrt{1-x^2-y^2}\right),$$

and it follows from (45)-(47) that

$$u_1 = 1 - \frac{\sqrt{3}\,x\,y + y^2}{1 + \sqrt{1-x^2-y^2}}\ ,$$

(48) $$u_2 = 1 - \frac{3x^2 - y^2}{2\left(1 + \sqrt{1-x^2-y^2}\right)}\ ,$$

$$u_3 = 1 + \frac{\sqrt{3}\,x\,y - y^2}{1 + \sqrt{1-x^2-y^2}}\ .$$

The elements of the transformation matrix **T** are now determinated from (18) - (20) [1], (24), (26) and (48):

(49) $$t_{11} = 1 - \frac{x^2}{1+z}\ , \qquad t_{21} = t_{12} = -\frac{xy}{1+z}\ ,$$
$$t_{31} = -x\ , \quad t_{13} = x\ , \quad t_{22} = 1 - \frac{y^2}{1+z},$$
$$t_{32} = -y\ , \quad t_{23} = y\ , \quad t_{33} = z\ .$$

Having the quantities u_i, v_i we are in a position to give the solution of the inverse kinematic problem in an explicit form. First, using (4) [1] we determine

$$\lambda_i = \sqrt{(u_i-\rho)^2 + v_i^2}$$

and

(50) $$L_i = R_i \sqrt{(u_i-\rho)^2 + v_i^2} ,$$

The angles φ_i are now obtained by

(51) $$\varphi_i = \arccos\frac{M_i^2 + N_i^2 - L_i^2}{2\, M_i\, N_i} ; \qquad i = 1,2,3 .$$

Relations (50) or (51) in view of (48), (26) give the solution of the inverse kinematic problem for this type of mechanism.

4. Restrictions on the mobile platform movement

The geometrical restrictions on the mobile platform movements are a result of the inequalities $u_i \geq 0$ $(i=1,2,3)$ and $v_{min} \leq v_i \leq v_{max}$, where

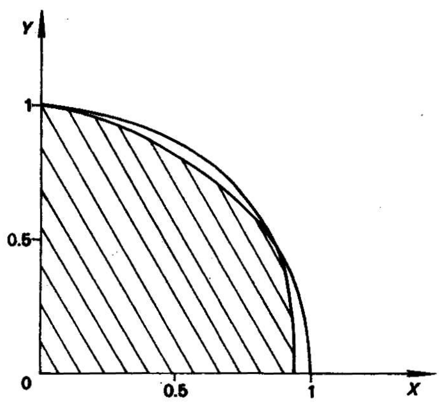

Fig.3

$$v_{min} = \frac{V_{min}}{R_1}, \qquad v_{max} = \frac{V_{max}}{R_1}$$

From the non-negativity of u_i and equations (48) we obtain a system of three inequalities. The solution of this system for x, y gives the domain of the possible orientation of the mobile platform (Fig. 3).

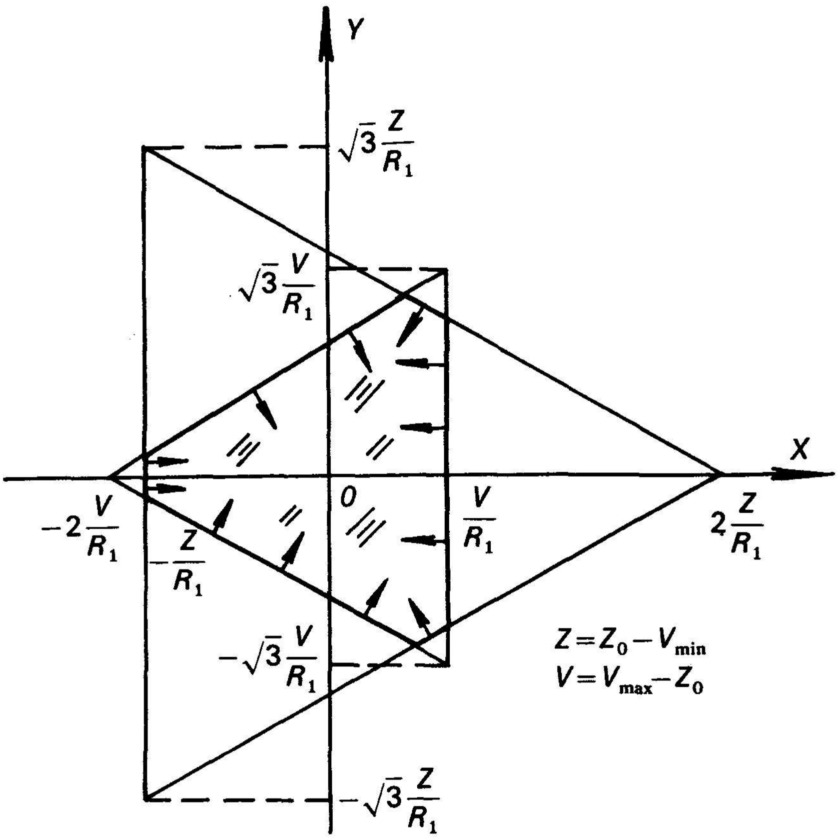

Fig. 4

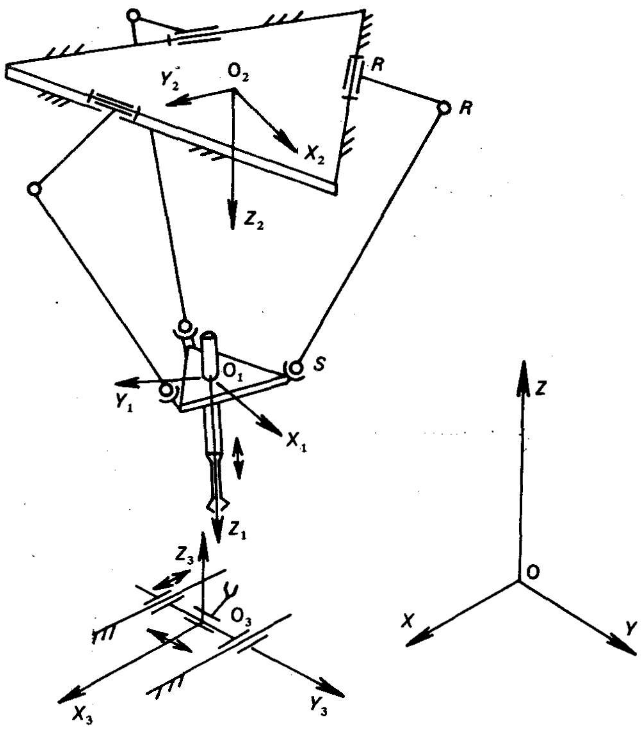

Fig.5

From the inequalities $v_{min} \le v_i \le v_{max}$ and (26) we get a system of six inequalities. The domain produced from the solution of this system for x, y is illustrated at Fig. 4. It is evident that the domain of the possible orientations of the mobile platform ($\mathbf{k} = (x,y,z)^T$) depends on V_{min} and V_{max}, which are design parameters, and on z_0, which can be regarded as a controlled parameter.

The parameter z_0 may be found from the conditions $v_{min} \le v_i \le v_{max}$

$$z_0 \ge \max \begin{cases} V_{min} + R_1\left(x + \sqrt{3}\, y\right)/2\,, \\ V_{min} - R_1 x\,, \\ V_{min} + R_1\left(x - \sqrt{3}\, y\right)/2, \end{cases} \tag{52}$$

$$z_0 \le \min \begin{cases} V_{max} + R_1\left(x + \sqrt{3}\, y\right)/2, \\ V_{max} - R_1 x\,, \\ V_{max} + R_1\left(x - \sqrt{3}\, y\right)/2. \end{cases} \tag{53}$$

5. Applications of the mechanism with parallel topology 3RRS to manipulation systems

The mechanism considered has three degrees of freedom and a closed kinematic chain. For this reason it has a limitted workspace. It is adequate, in order to increase the workspace, for such mechanisms to be used in combination with other mechanisms as links of manipulation systems. Fig. 5 illustrates the diagram of such a manipulation system consisting of two separate kinematic chains. The first kinematic chain has four degrees of freedom, containing a parallel-topology mechanism. The second kinematic chain has two degrees of freedom (two translations, e.g. a two-coordinate table).

Such a manipulation system combines the advantages of the parallel-topology robots with those having a separate kinematic chain and, what is most important, there is no accumulation of errors from the individual drives. This allows to achieve greater accuracy. For this reason, this system can be successfully used in asembly work operations.

In order to make use of the above results we have to connect the $O_1X_1Y_1Z_1$ and $O_2X_2Y_2Z_2$ coordinate systems in the same way. We connect the coordinate system $O_3X_3Y_3Z_3$ with the two-coordinate table.

The angles φ_i (i=1,2,3), translations x^*, y^* along axes OX and OY, respectively, and translation z_1 along the OZ_1 axis are assumed as control parameters.

Denote the transformation matrix between coordinate systems $O_2X_2Y_2Z_2$ and $OXYZ$ by $\mathbf{T}_2$,

$$\mathbf{T}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} ,$$

The direct kinematic problem for the manipulation system (Fig. 5) is described by the equations

(54) $$\mathbf{r}_I = \mathbf{T}_2\left(\mathbf{T}\mathbf{r}_1 + \mathbf{r}_0\right) + \mathbf{r}_c ,$$

(55) $$\mathbf{r}_{II} = \mathbf{r}_m + \mathbf{r}_3 ,$$

where $\mathbf{r}_I$ is the radius-vector related to the first kinematic chain containing the parallel-topology mechanism, $\mathbf{r}_{II}$ is related to the second kinematic chain,

$$\mathbf{r}_c = \mathbf{OO}_2 = \left(x_c, y_c, z_c\right)^T; \qquad \mathbf{r}_m = \mathbf{OO}_3 = \left(x^*, y^*, z^*\right)^T ,$$

and $\mathbf{r}_3 = \left(x_3, y_3, z_3\right)^T$ is the position vector of a selected point in the $O_3X_3Y_3Z_3$ coordinate system.

In the inverse kinematic problem the generalized coordinates, i.e. angles $\varphi_i (i=1,2,3)$ and translations x^*, y^*, z_1 are sought, for given end-effector orientation and position. The latter is usually defined by its coordinates in the $OXYZ$ coordinate system. The end-effector orientation is set by the unit vector $\mathbf{n} = \left(n_x, n_y, n_z\right)^T$, $n_x^2 + n_y^2 + n_z^2 = 1$.

The coordinates of the end-effector are set by the vector $\mathbf{r}_3 = (x_3, y_3, z_3)^T$ in the $O_3X_3Y_3Z_3$ coordinate system.

As the orientation of the mobile platform should coincide with that of the end-effector, then

$$\mathbf{T}_0\,\mathbf{T}\,\mathbf{k} = n ,$$

where

$$\mathbf{T}_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix},$$

and $\mathbf{k} = (0,0,1)^T$ is the unit vector connected to the mobile platform along the axis O_1Z_1 .

Hence

(56) $$t_{13} = -n_x \ , \quad t_{23} = n_y \ , \quad t_{33} = n_z \ .$$

As this manipulation system consists of two individual kinematic chains, we have

(57) $$\mathbf{r}_I = \mathbf{r}_{II} \ .$$

It follows from (54), (55) and (57) that

(58) $$\begin{bmatrix} -t_{11}x_1 - t_{12}y_1 - t_{13}z_1 - x_0 \\ t_{21}x_1 + t_{22}y_1 + t_{23}z_1 + y_0 \\ -t_{31}x_1 - t_{32}y_1 - t_{33}z_1 - z_0 \end{bmatrix} + \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} .$$

Here z_0 should also be set since the vector $\mathbf{r}_o = (x_0, y_0, z_0)^T$ depends on the orientation of the mobile platform. Then, from (58) and (56) we get

(59) $$z_1 = \frac{z_c - z_3 - z_0 - n_x x_1 - n_y y_1}{n_z} ,$$

Relations (26) and (56) give

$$v_1 = \frac{1}{2} n_x - \frac{\sqrt{3}}{2} n_y + \frac{z_0}{R_1} ,$$

$$v_2 = n_x + \frac{z_0}{R_1},$$

$$v_3 = \frac{1}{2} n_x + \frac{\sqrt{3}}{2} n_y + \frac{z_0}{R_1} .$$

In a similar way (48) and (56) yield

$$u_1 = 1 - \frac{-\sqrt{3}\, n_x n_y + n_y^2}{1 + n_z} ,$$

$$u_2 = 1 - \frac{3n_x^2 - n_y^2}{2(1 + n_z)} ,$$

$$u_3 = 1 - \frac{\sqrt{3}\, n_x n_y + n_y^2}{1 + n_z} .$$

Hence

$$L_i = R_1 \sqrt{(u_i - \rho)^2 - v_i^2} ,$$

and the angles φ_i are determined from

$$\varphi_i = \arccos \frac{M_i^2 + N_i^2 - L_i^2}{2M_i N_i} \quad ; \; i = 1,2,3 . \tag{60}$$

The remaining generalized coordinates x^*, y^* are obtained after substituting t_{ij} from (49) and (56) and x_0, y_0 from (22):

$$x^* = x_c - \frac{1}{1+n_z} \left[n_x^2 x_1 - n_x n_y y_1 - \frac{R_1}{2} (n_x^2 - n_y^2) \right] - x_3 - x_1 + n_x z_1 , \tag{60}$$

$$y^* = y_c + \frac{1}{1+n_z} \left(n_x n_y x_1 - n_y^2 y_1 + R_1 n_x n_y \right) - y_3 + y_1 + n_y z_1 , \tag{61}$$

The expressions (59)-(62) give the solution of the inverse kinematic problem for this manipulation system.

6. Conclusions

The paper presents a method for kinematic analysis of a 3-DOF mechanism with parallel topology. In contrast to other known results, the solution of the inverse kinematic problem is obtained in a rather simple form free of trigonometric functions. This makes it useful for real time control applications. Explicit bounds are given for the physical-geometric parameters of the mechanism, imposed by the conditions of real motion.

The results of the proposed kinematic analysis for this type of robots are not only of theoretical importance but can also be used in the design of similar types of mechanisms having parallel topology, as well as in real time control applications.

R e f e r e n c e s

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