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**Mathematical Simulation of Weakly Compressible  
Fluid in Hydraulic Drives of Machines**

**1. Introduction**

The solution of the actual problem of the weight decrease of automobiles is connected with the wide application of hydraulic drives in view of their high energy-intensity, speed of response, simplicity of arrangement. Hydraulic drives of machines include a large number of different components: pumps, valves, hydraulic motors, cylinders, pipelines, flexible hoses and so on. Hydraulic drives operation is characterized by significant load variations, pulsation of fluid fuel, change of parameters of drive components. So, in order to improve the reliability of drives operation in the process of their designing it is necessary to investigate fluid flow in hydroline (in pipelines and in aggregate canals) in detail.

In order to describe the fluid flow in a hydraulic system, a mathematical model of weakly compressible fluid, most adequately describing the real process, is used in this paper, [2].

The paper seeks to construct effective unconditionally stable numerical methods for weakly compressible fluid equations, describing processes in machine hydraulic drives and satisfying the following requirements:

- a - fluid flow may be of an oscillating nature;
- b - the domains on which the process is simulating may change their geometry in consequence of the hydraulic aggregate presence (cylinders, accumulators and so on);
- c - both velocity and pressure may be given as boundary conditions, and in the case of unknown moving boundaries the boundary conditions may depend on their positions;

*d* - the presence of hydroelements with a changeable section stimulates the necessity of a throttling accounting.

## 2. Statement of the problem

In order to describe the fluid flow by a hydroline we restrict ourselves considering the onedimensional model that is explained by the factors:

*a* - the influence of constructive parameters of hydraulic drive components upon the fluid may be taken into account within the scope of the onedimensional flow model.

*b* - the research equipment for the rig trial of the hydraulic drive is intended for fixing of the onedimensional flow parameters;

*c* - the onedimensional model is the easiest to realize;

*d* - in the case of a onedimensional flow the movement of a viscous compressible fluid is described by the following system of equations [3]:

$$(1) \quad \rho \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) + \frac{\partial P}{\partial x} = \frac{4}{3} \mu \frac{\partial^2 v}{\partial x^2},$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0,$$

where  $\rho$  is fluid density,  $v$  - velocity,  $P$  - pressure,  $\mu$  - dynamic viscosity coefficient. When we assume weak compressibility  $\rho = \rho_0(1 + \varepsilon P)$ , where  $\varepsilon = (\rho_0 c^2)^{-1}$  is small, and  $c$  is sound velocity in the fluid.

Taking into account the fluid ribbing of the hydroline wall, system (1) may be reduced to the following form [4]:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho_0} \frac{\partial P}{\partial x} + \lambda \frac{v|v|}{2d} = 0,$$

$$(2) \quad \varepsilon \frac{\partial P}{\partial t} + \frac{\partial v}{\partial x} + \varepsilon v \frac{\partial P}{\partial x} = 0,$$

$$(x, t) \in Q_T, \quad \bar{Q}_T = \left\{ (x, t) : 0 \leq x \leq l, \quad 0 \leq t \leq T_0 \right\}.$$

Here  $\lambda = \lambda(\text{Re}, \nabla)$  is the hydraulic resistance coefficient,  $\text{Re}$  - Reynolds number,  $\nabla$  - coefficient of Hydroline walls roughness.

The model of the weakly compressible fluid was investigated in the papers of N. J a n e n k o, O. L a d y z e n s k a j a, P. T e m a m and others.

System (2) is a system of a hyperbolic type with characteristics of the form  $x = (v \pm c) t + c_1$ . Therefore for the subsonic flows  $|v| < c$

boundary conditions are given on both boundaries of the domain  $Q_T$  :

on the left  $v(0,t) = \mu_1(t)$  or  $P(0,t) = \mu_2(t)$   
 and on the right  $v(l,t) = \mu_3(t)$  or  $P(l,t) = \mu_4(t)$  .

If the hydrocylinder is a part of the hydroline then the domain of the definition of the problem may change its geometry due to a displacement of the hydrocylinder piston

$$Q_T = \{(x,t) : 0 < x < \xi(t), \xi(0) = l, 0 \leq t \leq T_0\},$$

where  $\xi(t)$  is unknown and is defined together with the solution of the differential equation. Velocity of boundary movement is connected with velocity of fluid at the boundary by relation

$$(3) \quad \frac{\partial \xi}{\partial t} = k^2 v(\xi, t), \quad k = d/D ,$$

where  $d$  is the inside diameter of the hydroline,  $D$  - diameter of the hydrocylinder. When changing the boundary one obtains the values of pressure depending on its position:

$$(4) \quad P_G = P|_{x=\xi(t)} = P(\xi) ,$$

The hydroline may contain local hydraulic resistances conditioned by the presence of hydroaggregates as well as by the change of a cross-section area of the flow. In the case when the fluid flows across a hydraulic resistance located at a point  $x_k$  of the hydroline, there arises a pressure difference

$$(5) \quad P(x_k - 0, t) = P(x_k + 0, t) + (t)\rho_0 v(x_k, t) |v(x_k, t)| / 2$$

and the flow velocity doesn't change  $v(x_k - 0, t) = v(x_k + 0, t)$ . Here  $\xi(t)$  is a coefficient of resistance.

### 3. Construction of numerical methods

First we construct numerical methods for the weakly compressible fluid (2) without reference to the term  $\epsilon v \frac{\partial P}{\partial x}$  [7] on the domain  $Q_T$  with a fixed boundary. In this case the boundary conditions are the following:

$v(0,t) = v_0(t)$  ,  $v(l,t) = v_N(t)$  . Constructing difference schemes we shall use nets with integer and half-integer steps along space variable, which were widely adopted in Samarskii-Popov schemes for the gas-dynamics equations. By analogous [8] differential function of velocity we shall attribute to the integer nodes, and by differential

functions of pressure - to the half-integer nodes. We shall approximate input problem with the following conservative difference scheme

$$(6) \quad v_t + \hat{b}_{(0.5)}^+ \hat{x}_x + \hat{b}_{(0.5)}^- \hat{x}_x + \frac{1}{\rho_0} \hat{P}_x + \frac{\hat{\lambda} \hat{v} |\hat{v}|}{2d} = 0, \quad i=1, \overline{N-1},$$

$$\varepsilon \bar{P}_t + \hat{v}_x = 0, \quad i=0, \overline{N-1},$$

where

$$\hat{b}_{(0.5)}^+ = 0.5 \left( \hat{v}_{(0.5)}^+ + |\hat{v}_{(0.5)}^+| \right), \quad \hat{b}_{(0.5)}^- = 0.5 \left( \hat{v}_{(0.5)}^- - |\hat{v}_{(0.5)}^-| \right),$$

$$\bar{P} = P_{i+\frac{1}{2}}^j.$$

In order to solve nonlinear scheme (6) we shall construct an iterative process of Newton type

$$(7) \quad v_t^{s+1} + b_{(0.5)}^{s+1} v_x^s + b_{(0.5)}^s v_x^{s+1} + \frac{1}{\rho_0} \bar{P}_x^{s+1} + \frac{\lambda v^s |v^s|}{2d} = 0,$$

$$\varepsilon \bar{P}_t^{s+1} + v_x^s = 0.$$

On each step  $s$  of the iterative process the system of three pointed equations is solved by Thomas algorithm

$$(8) \quad A_1^s v_{i-1}^{s+1} - c_1^s v_i^{s+1} + B_1^s v_{i+1}^{s+1} = -F_1^s, \quad i = 1, \overline{N-1}.$$

The coefficients of this system satisfy sufficient conditions of the Thomas algorithm stability under any net step relations regardless of the sign of the diffetrential functions of velocity. Hence, the scheme proposed allows us to make calculations of the fluid flow being of an oscillating character.

In the case of moving boundary the solution of input system (2) depends on its unknown position  $x = \xi(t)$  and on the contrary. For the simultaneous location of the  $\xi$  boundary of the domain, and the values of difference functions velocity  $v$  and pressure  $P$ , a special iterative algorithm, founded on the successive correction of the moving boundary position [9] is developed.

We shall present a fluid feed process in the domain with a moving boundary at a time  $\tau = t_{j+1} - t_j$  in a form of two processes: fluid feed in the domain with a known boundary position and changes of a domain at a given quantity of the fluid in it.

In order to simulate the first process we shall assume that the velocity of boundary movement on the  $j+1$ -th time layer coincides with

the corresponding of the preceding layer. Then the velocity of the boundary is  $v_N^0 = v_N^J$ . By the differential equation

$$(9) \quad v_N = (\xi^{J+1} - \xi^J) / (\tau k^2)$$

approximating the equation of boundary movement (3), we shall define a

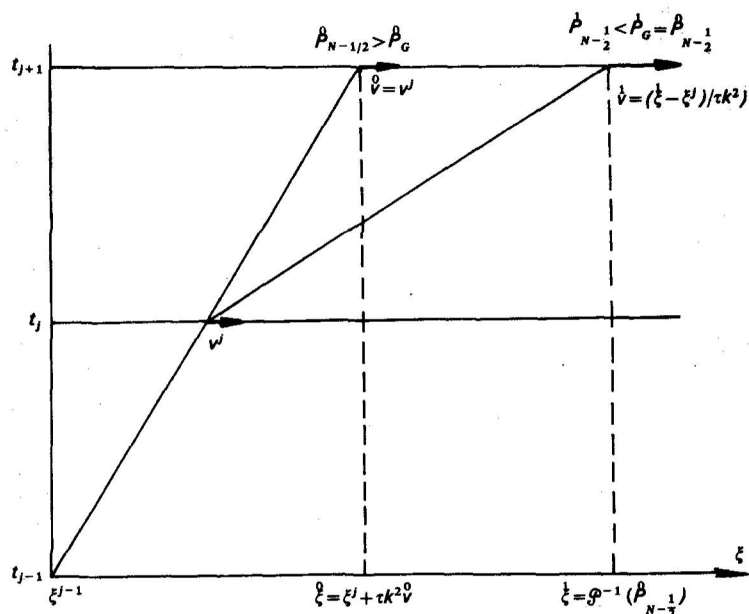


Fig. 1. Calculation scheme of the moving

boundary when  $P_{N-1/2}^0 > P_G^0$ ,  $v_N^0 > 0$

coordinate of moving boundary  $\xi^0 = \xi^J + \tau k^2 v_N^0$ , while by (4) - the corresponding gas pressure  $P_G^0 = \Phi(\xi^0)$ . Solving the system of the differential equations (8) under given boundary conditions,  $v_{N-1}^{J+1} = v_{N-1}^0(t_{j+1}^0)$ ,  $v_N^{J+1} = v_N^0$  we shall define fluid pressure  $P_{N-1/2}^0$  at a boundary.

In order to simulate the process of the boundary movement under a given fluid quantity on a domain, we shall compare calculated fluid pressure  $P_{N-\frac{1}{2}}^0$  at the boundary to external pressure  $P_G$ . The following cases are possible:

1.  $P_{N-\frac{1}{2}}^0 > P_G$ . In this case subsequent moving boundary displacement to the right occurs (increase of calculated domain). On the next step of the iterative process of refinement of boundary position we define its position  $\xi^1 = \Phi^{-1}(P_{N-\frac{1}{2}}^0)$  at which  $P_G = P_{N-\frac{1}{2}}^1$ . Since  $P_G = \Phi(\xi) < P_{N-\frac{1}{2}}^0 = \Phi(\frac{1}{\xi}) = P_G$  and gas pressure increases at displacement to the right then  $\xi^1 > \xi^0$ . Boundary position  $\xi^1$  confirms to the value of fluid velocity at the boundary  $v_N^1 = (\xi^1 - \xi^J)/(\tau k^2)$ . Further solving the system of difference equations (8) for boundary condition  $v_o^{J+1} = v_o(t_{J+1})$ ,  $v_N^{J+1} = v_N^1$  we shall define pressure value  $P_{N-\frac{1}{2}}^1$  of the fluid at the boundary having coordinate  $\xi^1$ . Since at a constant fluid quantity on the domain, an increase of the domain leads to a decrease of pressure, then  $P_{N-\frac{1}{2}}^1 < P_{N-\frac{1}{2}}^0 = P_N^0$ . Therefore, unknown value  $\xi \in (\xi^0, \xi^1)$ , (Fig. 1). In order to correct the position  $\xi$  we shall use subsequent division of interval  $(\xi^0, \xi^1)$  in parts. As next approximation we shall take  $\xi^2 = (\xi^0 + \xi^1)/2$  and define corresponding  $\xi^2$  value of fluid velocity at the boundary  $v_N^2 = (\xi^2 - \xi^J)/(\tau k^2)$ . Solving system (8) and boundary conditions  $v_o^{J+1} = v_o(t_{J+1})$ ,  $v_N^{J+1} = v_N^2$ , we shall define the value of fluid pressure  $P_{N-\frac{1}{2}}^2$  at the boundary  $\xi^2$ . If  $P_{N-\frac{1}{2}}^2 < P_G = \Phi(\xi)$  then  $\xi \in (\xi^2, \xi^1)$ , otherwise  $\xi \in (\xi^0, \xi^2)$ . The process of refinement of the boundary position stops under the fulfilment of the

following inequality at any step of the above defined iterative process.

$$(10) \quad |P_{N-\frac{1}{2}}^m - \Phi(\xi^m)| < \varepsilon_0 \Phi(\xi^m),$$

where  $\varepsilon_0 > 0$ . After that we shall suppose  $\xi^{j+1} = \xi^m$ .

2.  $P_{N-\frac{1}{2}}^0 < P_G^0$ . Unknown value  $\xi \in (\Phi^{-1}(P_{N-\frac{1}{2}}^0), \xi^0)$ . The search algorithm in generally is similar to the above mentioned.

If there is a local resistance  $\zeta(t)$  in the hydroline we shall attribute it to the nearest integer node  $x_k$  of the space net. Approximating the input system of equations of weak compressible fluid with the help of differential scheme (6) in the domains  $\Omega_1 = (0, x_k)$  and  $\Omega_1 = (x_k, l)$  we shall get a system of three - pointed equations

$$(11) \quad A_1^s v_{1-1}^{s+1} - C_1^s v_1^{s+1} + B_1^s v_{1+1}^{s+1} = -F_1^s, \quad i = 1, \dots, k-1, k+1, \dots, N-1$$

coefficients of which are defined as for (8). Considering the second differential equation (6) together with approximation of conjunction (5)

$$\bar{P}_{k+1}^{s+1} = \bar{P}_k^{s+1} + \zeta_k \rho_0 v_k^s |v_k^s| / 2$$

we shall get

$$(12) \quad A_k^s v_{k+1}^{s+1} - C_k^s v_k^{s+1} + B_k^s v_{k+1}^{s+1} = -F_k^s,$$

where

$$A_k^s = \tau \rho_0 c^2 / h_{k-1}, \quad B_k^s = \tau \rho_0 c^2 / h_k,$$

$$C_k^s = \zeta_k \rho_0 |v_k^s| / 2 + A_k^s + B_k^s.$$

Equations (11)-(12) form a system of three-pointed equations solvable by Thomas algorithm. The coefficient of the system satisfy sufficient stability conditions  $|C| > |A| + |B|$  under any net steps relation. Formulas (11) and (12) are easily generalized to the case of presence of several resistances in the hydroline.

Now let us consider the case when at the domain boundaries not only values of velocity may be given, but values of pressure or combination "velocity-pressure".

Note that using scheme (6), some difficulties arise connected with the satisfaction of the boundary conditions. We shall construct free of

defect indicating difference schemes in Riemann invariants. Introducing Riemann invariants  $r = P + \rho_0 cv$ ,  $s = P - \rho_0 cv$  we shall write system (2) in the form:

$$(3) \quad \frac{\partial r}{\partial t} + (v + c) \frac{\partial r}{\partial x} = -\rho_0 c \lambda \frac{v|v|}{2d},$$

$$\frac{\partial s}{\partial t} + (v - c) \frac{\partial s}{\partial x} = \rho_0 c \lambda \frac{v|v|}{2d},$$

where  $v = (r - s)/(2\rho_0 c)$ . Since for subsonic fluid flows  $v + c > 0$ , then we approximate system (13) to the following system of differential equations on the net with integral steps

$$r_t + (\hat{v} + c) \hat{r}_x = -0.25\lambda |v| (\hat{r} - \hat{s}) / d, \quad i = \overline{1, N},$$

$$s_t + (\hat{v} - c) \hat{s}_x = 0.25\lambda |v| (\hat{r} - \hat{s}) / d, \quad i = \overline{0, N-1}.$$

In order to realize the given differential scheme an iterative process of Newton type is used. At each step of the iterative process calculations are lead with the help of the following implicit scheme of "running calculation".

$$(14) \quad r_1^{n+1} = \frac{\chi_{1,1}^{n+1}}{1 + \chi_{1,1}^{n+1} + b_1} r_{1-1}^{n+1} + \frac{r_1^n + b_1 s_1^n}{1 + \chi_{1,1}^{n+1} + b_1}, \quad i = \overline{1, N},$$

$$s_1^{n+1} = \frac{\chi_{2,1}^{n+1}}{1 + \chi_{2,1}^{n+1} + b_1} s_{1+1}^{n+1} + \frac{s_1^n + b_1 r_1^n}{1 + \chi_{2,1}^{n+1} + b_1}, \quad i = \overline{N-1, 0},$$

where

$$\chi_{1,1}^n = \tau(v_1 + c)/h_{1-1}, \quad \chi_{2,1}^n = -\tau(v_1 - c)/h_1, \quad b_1 = 0.25\tau\lambda |v_1| / d.$$

The main difficulty of realization (14) lies in the fact, that values  $r_0^{n+1}$  and  $s_N^{n+1}$  are not given, while some linear combinations of invariants at  $i = 0$  and  $i = N$  are known. In (10) for the acoustic equations the way to satisfy boundary conditions in "running calculations" algorithm was proposed. By this algorithm unknown  $r_0^{n+1}$  and  $s_N^{n+1}$  were defined.

From (14) we have

$$(15) \quad r_N^{n+1} = r_o^{n+1} A_{1,N}^{<1>} + c_N, \quad s_o^{n+1} = s_N^{n+1} A_{o,N-1}^{<1>} + D_N,$$

$$\text{where } c_N = \sum_{k=1}^N A_{k,N}^{<1>} \frac{r_k + b_k s_k}{\chi_{1,k}}, \quad D = \sum_{k=1}^N A_{o,k}^{<2>} \frac{s_k + b_k r_k}{\chi_{2,k}},$$

$$A_{k,l}^{<m>} = \prod_{i=k}^l \frac{\chi_{m,i}}{1 + \chi_{m,i} + b_i}, \quad m = 1, 2$$

Relationship (15) together with the given as boundary conditions linear combinations of invariants  $r_o^{n+1}, s_o^{n+1}, r_N^{n+1}, s_N^{n+1}$  form a system of linear equations for determination of unknown values  $r_o^{n+1}$  and  $s_N^{n+1}$ .

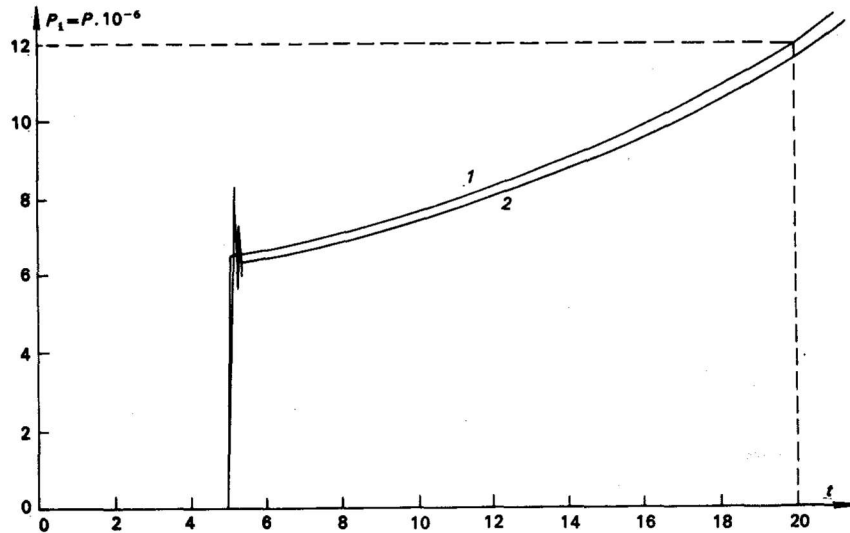


Fig.2 Pressure change on the moving boundary of the hydroline

### 3. Calculating experiment

Calculating experiment on simulation of the process of charging of a hydraulic accumulator with fluid was carried out. At the left boundary of the hydroline, the length of which was 7 m, a hydraulic pump feeding fluid into the hydroline was placed, and on the right - a hydroaccumulator was placed. The length of its gas cavity was 0.63 m. The pipeline diameter was  $d = 0.013$  m, and the diameter of the gas cavity of the hydroaccumulator was  $D = 0.18$  m. In the hydroaccumulator, the gas pressure depends on piston position  $\xi(t)$  by the following relation:

$$P|_{x=\xi(t)} = 6.5 \times 10^6 \left( 1 - \frac{\xi - 7}{0.63} \right)^{-1.17}$$

The hydroaccumulator charging process consists of two stages. At first an engine was powered, the hydraulic pump was started, the velocity of fluid feed increased to  $v = 0.647 t$ . At an instant of time  $t = 5$  s the flow was directed into the hydroaccumulator. Later on fluid velocity remained constant  $v = 3.235$  m/s. When at the right boundary of the hydroline fluid pressure exceed the pressure of hydroaccumulator charging pressure  $P = 6.5 \times 10^6$  Pa the hydroaccumulator piston movement was started.

Mathematical simulation was developed by using differential scheme (14) in Riemann invariants. The first stage of the charging process was simulated by defining the boundary conditions in the form:  $v(0, t) = 0.647t$ ,  $P(l, t) = 0$ . Further iterative process of successive correction of moving the boundary position described above was used.

In Fig. 2 gas pressure in hydraulic accumulator to time ratios obtained as results of the calculating experiment (curve 1) and as a result of the hydraulic rig trials (curve 2) are represented. Insignificant deviation of calculated pressure from measured one as a result of rig trials may be rather a consequence of the smaller sound velocity in the fluid adopted in calculations ( $c = 1400$  m/s) since there was a suction by the pump of small quantity of insoluble in fluid air. Attenuated as time goes on pressure variants (curve 2) are explained by the inertial properties of the right hydroline boundary: the presence of mass of the hydraulic accumulator charging ( $P = 12 \cdot 10^6$  Pa) without reference to the first stage of charging was 15 s, but charging time under the rig trial was 15.6 s. Distinction accounted for 4% is completely acceptable for the design of hydraulic drives.

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*Received on February 2, 1991*