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# GENERAL MECHANICS

## DYNAMICS OF MULTICOMPONENT SYSTEMS BASED ON THE ORTHOGONALITY PRINCIPLE

G. BOJADZIEV AND L. LILOV

The growing interest in complex systems with components of heterogeneous physical nature increases with the development of modern production and contemporary scientific research. A very good example in this aspect are the electromechanical systems widely spread in practice. It is rather curious that there is no general mathematical technique for modelling of such systems from a unit point of view. Both theoretical mechanics and theoretical electrical engineering investigate electromechanical systems with their specific methods directed to only mechanical or only electrical processes and phenomena, trying to transfer them to the hybrid system.

However it turns out, that it is impossible by means of mechanical methods only to describe such electromechanical systems like systems of sliding contacts, volumetric conductors, unlocked currents etc. On the other side, after a profound study of the electrical engineering methods, only systems of simple mechanical parts can be analyzed as containing only mass points, bodies with angular velocity of fixed direction or systems with planner motion. Similar remarks can be made upon other heterogeneous systems like thermomechanical, hydroelectric systems etc.

The subject of this work is to propose a mathematical apparatus based on the works of Koenig, Blackwell [1] and Andrews [2] using the graph theory and the orthogonality principle for modelling of general heterogeneous systems when the space motion of their mechanical components is considered.

The energy as a general measurement of matter motion together with the universal law for conservation of the energy are assumed as basis of the approach, proposed here. The power flow and power potential are taken as fundamental characteristics of energy. Thus the physical system is characterized by the general power space, which combines countable number of power subspaces depending on the different kinds of energy involved in the concrete problem. The parameters of energy of these power subspaces are the specific variables expressing the two basic characteristics of energy-power flow and power potential. The power flow variables are called in [1] "through" variables and the power potential variables — "across" variables. They build the basis for the corresponding subspace and there assembles the basis of the general power space. A quantitative description of the investigated process is given by means of these variables.

An important characteristic of every system is its topology. A general power graph characterizing the energy change in the system is introduced. The system is divided into simple components which allow for a simple description and investigation. The "points" in which the components are connected, building in this way the whole system, are called terminal points or just terminals. A measurement can be accomplished for every two terminals of one and the same physical nature, i. e. the corresponding through and across variables can be determined. This can be interpreted topologically by means of an oriented line segment (edge) called the terminal graph. It characterizes the power, flowing between the two terminals with the corresponding sign. The arrow on the line segment indicates the positive sense for a set of measurements made "across" (parallel to) or "through" (in series with) the "points" of interconnection or "terminals" on the component.

To illustrate a terminal representation consider a spring, which is a very simple example.


Two sets of measurements can be made that completely characterize this element: the displacement $r$ between the two end points of the spring and the force $F$ that acts through it. The displacement and the force are precisely the across and through variables, respectively, as just mentioned (Fig.1). The spring terminal graph is shown in the next Fig.2.

The collection of oriented line segments for each component is defined as its terminal graph. The terminal characteristics of the component are completely described when is given the equation which relates across and through variables. Such an equation, called a terminal equation, together with the terminal graph of the component, forms the terminal representation of the component. The description of the component is not complete without both terminal equation and terminal graph.

The terminal equation of the spring, assuming that the spring is linear, can be

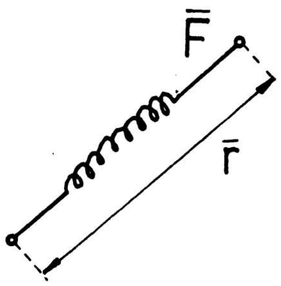


Fig.1

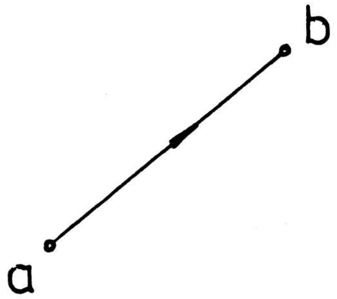


Fig.2

written in the form

$$F = -k(|\bar{r}| - s_0)\frac{\bar{r}}{|\bar{r}|},$$

where $\bar{F}$ is spring force, $k$ is spring constant, $\bar{r}$ – displacement vector, and $s$ – unstrained length of the spring.

Further, by definition, the general system graph is a collection of component terminal graphs, obtained by uniting the vertices of the terminal graphs in correspondence with the union of the component terminals. The projections of this graph in the different power subspaces are connected components of the unconnected general graph and describe the dependence between the power flow and the potential, specific for such subspace, and in accordance with the concrete topology of the system.

After the general graph has been constructed, it can be easily described by an incidence matrix, composed of $+1, -1, 0$ entries with each column corresponding to an oriented line segment (edge) in the graph and each row corresponding to a vertex. Depending on whether a given edge enters, leaves, or does not touch a given vertex, the corresponding entry is given the value $+1, -1$, or $0$, respectively,

$$\begin{bmatrix} \text{Incidence} \\ \text{matrix} \\ v \times l \end{bmatrix} = \left[\begin{array}{cccccc} & 1 & 2 & 3 & \ldots\ldots & e \\ 1 & & & & & \\ 2 & & & & & \\ 3 & & & & +1,-1 \text{ or } 0 & \\ \vdots & & & & \text{entries} & \\ v & & & & & \end{array}\right].$$

Fig.3

where it is assumed that there are $e$ edges and $v$ vertices in the graph.

**Fundamental postulates.** There are two fundamental postulates in the system theory [1].

**Postulate I. The Vertex Postulate.** Let the system graph of a physical system contain $e$ oriented elements (edges), and let $\dot{Y}_j$ represents the fundamental through variable of the $j$-th element; then at the $k$-th vertex of the graph

$$\sum_{j=1}^{l} a_j \dot{Y}_j = 0,$$

where

$$a_j = \begin{cases} 0, & \text{if the } j\text{-th element is not incident at the } k\text{-th vertex} \\ 1, & \text{if the } j\text{-th element is oriented away from the } k\text{-th vertex} \\ -1, & \text{if the } j\text{-th element is oriented toward the } k\text{-th vertex} \end{cases}$$

**Postulate II. The Circuit Postulate.** Let the linear graph of a physical system contain $e$ oriented elements (edges), and let $\dot{X}_j$ represent the fundamental across variable of the $j$ -th element; then for the $k$-th circuit

$$\sum_{j=1}^{e} b_j \dot{X}_j = 0,$$

where

$$b_j = \begin{cases} 0, & \text{if the } j\text{-th element is not included in the } k\text{-th circuit} \\ 1, & \text{if the orientation of the } j\text{-th element is the same as the orientation chosen for the } k\text{-th circuit} \\ -1, & \text{if the orientation of the } j\text{-th element is opposite to that of the } k\text{-th circuit} \end{cases}$$

Postulates I and II supply equations which number equals to the number of all vertices and circuits. Not all of these equations are independent. To establish a set of independent circuit and vertex equations, a tree of a connected graph is introduced as a connected subgraph, containing all $v$ vertices and no circuits. If the graph of the system is not connected and consists of $n$ separate parts, then the term forest is used as a collection of trees — one from each part.

The edges of the graph that belong to the tree are called branches and the remaining edges are called chords and constitute the cotree.

A fundamental cutset can be defined for each branch, such that the branch plus a unique set of chords, when deleted, divide the graph into two parts.

A fundamental circuit can be defined for each chord of the cotree, such that the chord plus a unique set of tree branches form a closed loop.

It is known from the graph theory that there are exactly $(v - 1)$ branches in a tree of a connected graph containing $v$ vertices and in this way the number of the chords is always $e - (v - 1) = e - v + 1$, where $e$ is the number of the edges.

By applying the vertex postulate to the vertices isolated by each fundamental cutset, a set of $(v - 1)$ independent fundamental-cutset equations are obtained

$$[U_t, A] \begin{bmatrix} \dot{Y}_t \\ \dot{Y}_c \end{bmatrix} = 0,$$

where $U_t$ is a $(v - 1) \times (v - 1)$ unit matrix, $\dot{Y}_t$ is the $(v - 1) \times 1$ column matrix of the tree through variables, and $Y_c$ is the $(e - v + 1)$ column matrix of the cotree through variables. The fundamental cutset equations permit the tree through variables to be obtained as explicit of the cotree through variables

$$\dot{Y}_t = -A\dot{Y}_c.$$

By applying the circuit postulate to each fundamental circuit, a set of fundamental circuit equations can be obtained

$$[B, U_c] \begin{bmatrix} \dot{X}_t \\ \dot{X}_c \end{bmatrix} = 0,$$

where $U_c$ is a $(e - v + 1) \times (e - v + 1)$ unit matrix, $\dot{X}_t$ are tree across variables, and $\dot{X}_c$ are cotree across variables.

It can be proved that relation

$$B = -A^T, \qquad A = -B^T$$

holds. It follows from here that the two matrices of cutset and circuit equations are orthogonal, i. e.

$$[U_t, A]\,[B, U_c]^T = [0]$$

or

$$[B_t, U_c]\,[U_t, A]^T = [0]\,.$$

This important property can be generalized in a principle, called Principle of Orthogonality. Its most general formulation can be given in the following way [2]:

"If the scalar products of the through and across variables associated with each edge of a system graph are summed over all edges in the graph, than the sum will be zero."

If the through and across variables associated with each and every element are listed in the column matrices $[Y]$ and $[X]$ respectively, then the principle of orthogonality states that

$$\left[\dot{Y}\right]^T\left[\dot{X}\right] = \left[\dot{X}\right]^T\left[\dot{Y}\right] = 0.$$

In fact

$$\left[\dot{X}\right]^T\left[\dot{Y}\right] = \left[\dot{X}_t, \dot{X}_c\right]^T\begin{bmatrix}\dot{Y}_t\\ \dot{Y}_c\end{bmatrix} = \left[\dot{X}_t\right]^T\left[U_t, -B^T\right]\begin{bmatrix}-A\\ U_c\end{bmatrix}\left[\dot{Y}_c\right] = \left[\dot{X}_t\right]^T[0]\left[\dot{Y}_c\right] = [0].$$

The principle of orthogonality is actually a mathematical record of the energy conservation law because the scalar product of each pair of fundamental variables has a dimension of power — the energy, transmitted over each concrete edge in a unit time, and in this way the sum of all scalar products is the general power in the system, i. e. the interchange of energy in the system in a unit time. The principle of orthogonality interprets the condition that in the general energy space the energy remains totally unchanged and can only be transmitted from one subsystem to another subsystem or transformed from one form to another.

The principle of orthogonality in the case of electrical networks is expressed by a formulated in 1952 from Tellegen [3] theorem for such systems. In 1977 Andrews formulated the principle of orthogonality for mechanical system, composed of a point masses, and proved that the principle of virtual work is a corollary of the principle of orthogonality.

Obviously, either through or across variables (or both) may be operated by a "linear operator" such as multiplication, differentiation, integration, etc. and the principle of orthogonality as well as the cutset and circuit equations will be still applied.

$$[Y]^T[X] = 0, \quad \left[\ddot{Y}\right]^T[X] = 0, \quad [Y]^T[\delta X] = 0,$$

$$[U_t, A]\begin{bmatrix}Y_t\\ Y_c\end{bmatrix} = 0, \quad [B, U_c]\begin{bmatrix}\ddot{X}_t\\ \ddot{X}_c\end{bmatrix} = 0.$$

We may emphasize now that for analysis of a given system two groups of equations are necessary: one group expressing the topological properties of the system and the second one expressing the physical properties of the medium. Each of these two groups is again divided into another two groups. The topological properties are given by the equations of the power flow and the power potential expressed respectively as

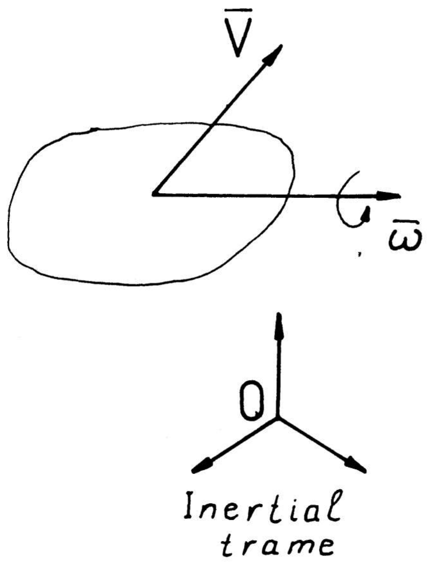

Fig.4

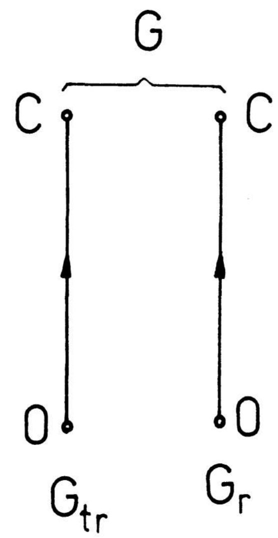

Fig.5

cutset equations and circuit equations. The physical properties are described by means of terminal equations which relate through and across variables of each element and equations for energy transformation which give the connection between fundamental through and across variables from different power subspaces.

Let us consider now the terminal representation of a free rigid body which is a key for spreading the presented approach on arbitrary systems of rigid bodies.

It is well known that the general motion of a free rigid body can be characterized by the velocity $v$ of its center of mass $C$ and rotation about $C$ with angular velocity $\bar{\omega}$ with respect to a given inertial frame. The variables $\bar{v}$ and $\bar{\omega}$ are the fundamental across variables of two power subspaces — translational subspace $H_{tr}$ and rotational one $H_r$. The free rigid body is a multiterminal component, whose terminal graph $G$ has projections $G_{tr}$ and $G_r$ both in $H_{tr}$ and $H_r$ , i. e. the graph is unconnected.

The through variables associated with a free rigid body are D'Alembert inertia forces, i. e. $-m\dot{\bar{v}}$ for $H_{tr}$ and $-(I\dot{\bar{\omega}} + \bar{\omega} \times I\bar{\omega})$ for $H_r$, where the central inertia tensor I can be represented in the dyad form

$$I = \int_m (\bar{\rho}^2 E - \bar{\rho}\bar{\rho})dm$$

with $E$ — unit tensor and $\bar{\rho}$ — radius-vector of mass element $dm$. If we denote these through variables by $\bar{F}$ and $\bar{T}$ respectively,

$$-\bar{F} = m\dot{\bar{V}}, \quad -\bar{T} = I\dot{\bar{\omega}} + \bar{\omega} \times I\bar{\omega},$$

one can consider these notations terminal equations.

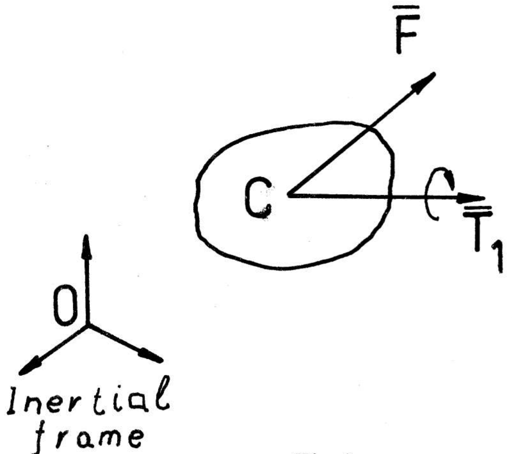


Fig.6

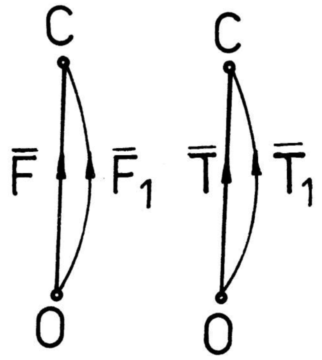


Fig.7

Suppose now that a force $\bar{F}_1$ and a torque $\bar{T}_1$ are acting on the body considered in the inertial frame.

The corresponding graph is shown in the next figure. The circuit equations are now

$$\bar{V} - \bar{V}_1 = 0$$

and

$$\bar{\omega} - \bar{\omega}_1 = 0$$

and the cutset equations

$$\bar{F} + \bar{F}_1 = 0, \quad \bar{T} + \bar{T}_1 = 0$$

or

$$m\dot{\bar{V}} = \bar{F}_1, \quad I\dot{\bar{\omega}} + \bar{\omega} \times I\bar{\omega} = \bar{T}_1.$$

These equations are exactly the axioms of Newton and Euler for a motion of a rigid body in the inertial frame.

The body motion causes an interaction between two power subspaces $H_{tr}$ and $H_r$. This interaction is described by the connecting equations which link the fundamental variables in the form

$$\bar{T} = \bar{\rho} \times \bar{F}, \quad \bar{V} = \bar{\omega} \times \rho,$$

here $\rho$ is the local radius-vector of a body fixed point $P$ at which a force $\bar{F}$ is applied.

The dependence between $H_{tr}$ and $H_r$ is shown schematically in Fig.9. The vector $\bar{\rho}$ plays an important role here. A through variable $\bar{T}$ in $H_r$ appears over it from a through variable $\bar{F}$ in $H_{tr}$. The opposite vector of $\bar{\rho}(-\bar{\rho})$ creates, at the same time, an across variable $\bar{v} = -\bar{\rho} \times \bar{\omega}$ . It is the vector $\bar{\rho}$ through which an exchange of power

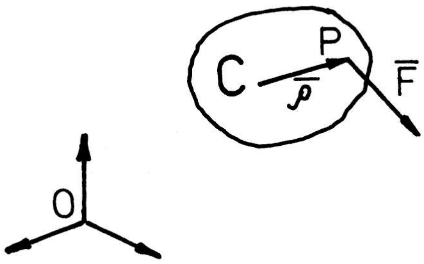


Fig.8

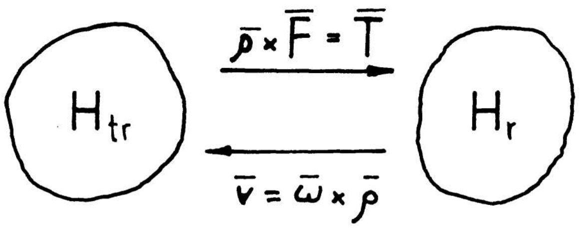


Fig.9

between the two subspaces $H_{tr}$ and $H_r$ is performed. This can be also seen from the expression for the power $P$, considered as an "energy measure" in every one of the two subspaces:

$$P = \bar{V}\bar{F}_{|H_{tr}} = \bar{\omega} \times \bar{\rho}\bar{F} = \bar{\omega}\bar{\rho} \times \bar{F} = \bar{\omega}\bar{T}_{|H_r}.$$

One can point out here the analogy with the electromagnetic space. Although it is possible to consider and investigate isolated electric and magnetic phenomena, as well as separate processes in $H_{tr}$ and $H_r$, they are the particular cases in the more general united spaces — the electromagnetic space $H_{EM}$ and the mechanical space $H_M$. For instance, every change of magnetic flow velocity leads to the appearance of an electric field and (if conductors are presented) to the appearance of a current and vise versa — every change of the current brings a magnetic field. Likewise every action of force $\bar{F}$ in the space $H_{tr}$ results with a help of $\bar{\rho}$ in the appearance of a torque in $H_r$ and vise versa — an across variable is set up in $H_{tr}$ through $\bar{\omega}$ and $\bar{\rho}$.

We will now apply the suggested procedure for deriving the motion equation of a tree-like system of interconnected rigid bodies, shown in Fig.10.

The system consists of $n$ bodies, one of them being connected to a body whose motion is preliminary given and which is called a basement. Let $C_i (i = 1, \ldots, n)$ denote the mass centres of the bodies and $C_0$ is a given point of the basement.

Two points are introduced between every two contiguous bodies connected by a kinematic joint. These points have been previously called terminal points or just terminals and the corresponding body is connected by means of them to the remaining bodies, building in this way a certain multiterminal (depending on their number) component. We assume for simplicity that the terminal points for the contiguous bodies coincide and we have one common terminal point between them. Revolute, cardan and spherical joints have this property. The terminal points between bodies $i$ and $j$ are denoted by $B_{ij}$ .

The graph of the system is shown in the next figure

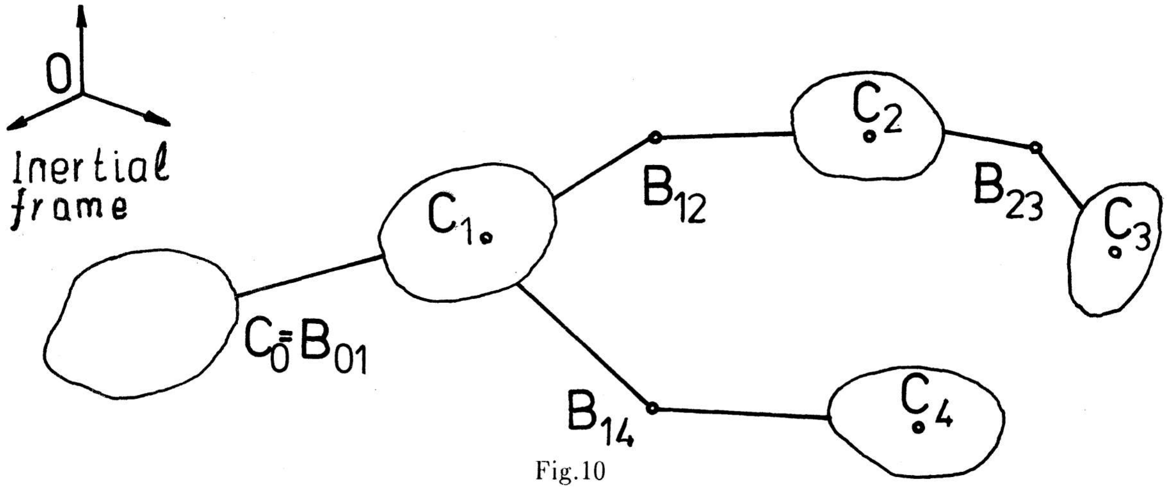


Fig.10

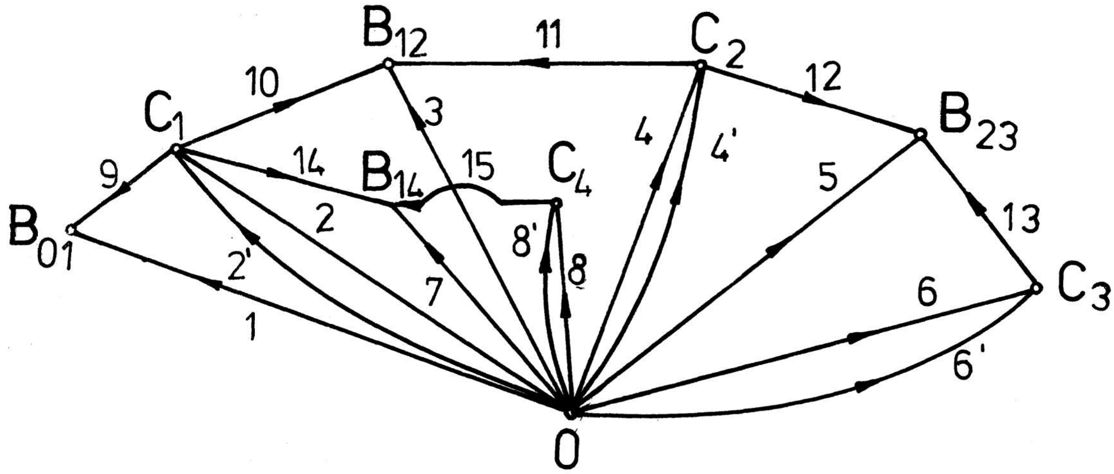


Fig.11

This graph must be considered twice — for translational subspace $H_{tr}$ and for rotational subspace $H_r$ . Vertices of the graph are points $C_i$, $B_{ij}$ and 0. There are edges from vertex 0 to all of the remaining vertices and also from every vertex $C_i$ to vertices $B_{sk}$ when an index $s$ coincides with $i$ or $k$.

The following fundamental variables associated with the edges are introduced

**Through variables**

1. D'Alembert forces $\bar{F}^d_{2k}(k = 1, \ldots, n)$ associated with edges $2k$
2. External force $\bar{F}_{(2k)'}(k = 1, \ldots, n)$ acting from inertial frame on body $k$
3. Forces $\bar{F}_{2k-1}(k = 1, \ldots, n)$ acting between terminal points and reference point 0. Only $\bar{F}_1$ is different from zero.

4. Forces $F_l^c(l = 2n+1, \ldots, 4n-1)$ which express the interaction between contiguous bodies.

**Across variables**

Radius-vectors of mass centres and terminal points $B_{ij}$ are associated as across variables for all edges starting from 0 and local radius-vectors of terminal points for the remaining edges.

The tree with branches of edges from 1 to $2n$ is selected as a formulation tree. All remaining edges are chords.

The projection $G_r$ of the general graph in the subspace $H_r$ has the same form as $G_{tr}$ but other fundamental variables are now associated with the edges.

**Through variables**

1. D'Alembert torques $\bar{T}_{2k}^d$ associated with edges $2k(k = 1, \ldots, n)$.
2. External torque $\bar{T}_{(2k)'}, (k = 1, \ldots, n)$ applied to the body $k$ from the external frame.
3. Torques $\bar{T}_{2k-1}(k = 1, \ldots, n)$ of interaction between terminal points and reference s stem. Only $\bar{T}_1$ is different from zero.
4. Torques $\bar{T}_l^c(l = 2n+1, \ldots, 4n-1)$ expressing the interaction torques between contiguous bodies.

**Across variables**

All edges starting from vertex 0 have, as associated across variables, the absolute angular velocities of the bodies to whose points they are appointed. The terminal points $B_{sk}$ are considered to belong to body number $s$. The across variables for the edges with numbers from $(2n+1)$ to $(4n-1)$ express the relative angular velocities and all even across variables are zeros according to the assumed agreement for points $B_{sk}$.

The following notations are introduced further: the variables of the branches are denoted by $X_b^{tr}$, $\dot{Y}_b^{tr}$ (the superscript shows their relation to $G_{tr}$, while the chords variables are denoted by subscript "$c$" — $X_c^{tr}, \dot{Y}_c^{tr}, X_{c'}^{tr}, \dot{Y}_{c'}^{tr}$. The same notations but with superscript "$r$" are used for the corresponding variables in $H_r$.

The topological equations for the graph $G_{tr}$ are

a) Cutset equations:

$$\bar{F}_1 + \bar{F}_{2n+1} = 0 \tag{1}$$

$$U\dot{Y}_b^{tr} + P\dot{Y}_c^{tr} + \hat{E}\dot{Y}_{c'}^{tr} = 0, \tag{2}$$

where $U$ is unit matrix and $P$ is nondegenerate matrix whose inverse matrix is denoted by $H = P^{-1}$.

b) Circuit equations:

$$UX_c^{tr} - 1_{2n-1}X_{b1}^{tr} + QX_b^{tr} = 0, \tag{3}$$

$$UX_{c'}^{tr} - \hat{E}X_b^{tr} = 0. \tag{4}$$

The one dimensional matrix $1_{2n-1}$ has all elements zeroes except for the first one which is 1. $X_{b1}$ is the preliminary given across variable associated to the first branch.

The following relations expressing the connection between cutset and circuit matrices hold

$$Q = -P^T, \quad Q^{-1} = H^T. \tag{5}$$

Relations (1)-(5) are true for $H_r$ if we replace $\dot{X}^{tr}, \dot{Y}^{tr}$ by $\dot{X}^r, \dot{Y}^r$.

The terminal equations have the form

a) for $G_{tr}$

$$M_T X_b^{tr} = -\dot{Y}_b^{tr}, \tag{6}$$

where $M_T$ is an operator matrix

$$M_T = \begin{bmatrix} m_1\frac{d^2}{dt^2} & & & 0 \\ & 0 & & \\ & & m_2\frac{d^2}{dt^2} & \\ & & & 0 \\ & 0 & & \\ & & & m_n\frac{d^2}{dt^2} \end{bmatrix}.$$

and $m_i$ are masses of the bodies.

b) for $G_r$

$$I_T\dot{X}_b^r = -\dot{Y}_b^r, \tag{7}$$

where $I_T$ is an operator matrix

$$I_T = \begin{bmatrix} I_1\frac{d}{dt} + \bar{\omega}_2 \times I_1\cdot & & & & \\ & 0 & & & \\ & & I_2\frac{d}{dt} + \bar{\omega}_4 \times I_2\cdot & & \\ & & & 0 & \\ & & & & I_n\frac{d}{dt} + \bar{\omega}_{2n} \times I_n\cdot \end{bmatrix}.$$

and $I_i$ is central inertia tensor of the $i$-the body.

We apply now the principle of orthogonality

$$[X]^T\left[\dot{Y}\right]=0 \tag{8}$$

in an equivalent form.

$$[\delta X]^T\left[\dot{Y}\right]=0, \tag{9}$$

where $\delta X$ are virtual variations of across variables. Relation (9) implies

$$\begin{gathered}\left[\delta X_b^{tr}\right]^T\dot{Y}_b^{tr}+\left[\delta X_c^{tr}\right]^T\dot{Y}_c^{tr}+\left[\delta X_{c'}^{tr}\right]^T\dot{Y}_{c'}^{tr}+\\ \left[\delta X_b^{r}\right]^T\dot{Y}_b^{r}+\left[\delta X_c^{r}\right]^T\dot{Y}_c^{r}+\left[\delta X_{c'}^{r}\right]^T\dot{Y}_{c'}=0.\end{gathered} \tag{10}$$

Substituting cutset equations (4) and terminal equations (6) and (7) in (10) we get

$$\begin{gathered}\left(\dot{Y}_{c'}^{tr}-M_T X_b^{tr}\right)^T\delta X_b^{tr}+\left(\dot{Y}_{c'}^{r}-I_T X_b^{r}\right)^T\delta X_b^{r}+\\ \left(\dot{Y}_c^{tr}\right)^T\delta X_c^{tr}+\left(\dot{Y}_c^{r}\right)^T\delta X_c^{r}=0.\end{gathered} \tag{11}$$

In order to see what (11) means, we take into account that $\delta X_b^{tr}=\{\delta\bar{r}_i\}$ and $\delta X_b^{r}=\{\delta\bar{\pi}_i\},(i=1,\dots n)$ are the variations of linear and angular positions of the bodies in the internal frame; that $\dot{Y}_{c'}^{tr}=\{\bar{F}_i\},\dot{Y}_{c'}^{r}=\{\bar{T}_i\}$ are vectors and moments of external forces applied to mass centers $C_i$ and finally that the term

$$\left(\dot{Y}_c^{tr}\right)^T\delta X_b^{tr}+\left(\dot{Y}_c^{r}\right)^T\delta X_b^{r}=\delta W$$

expresses the total virtual work done in the system.

Relation (11) is, in this way, nothing but D'Alembert principle, written for the system of $n$ interconnected bodies

$$\sum_{i=1}^{n}\left[\left(\bar{F}_i-m_i\ddot{\bar{r}}_i\right)\delta\bar{r}_i+\left(\bar{T}_i-I_i\dot{\bar{w}}_i-\bar{w}_i\times I\bar{w}_i\right)\delta\bar{\pi}_i\right]+\delta W=0.$$

The work in subspaces $H_{tr}$ and $H_r$ is expressed by means of two different bases: $(\delta X^{tr},\dot{Y}^{tr})$ for $H_{tr}$ and $(\delta X^{r},\dot{Y}^{r})$ for $H_r$. In order for the final equations to be written in generalized coordinates, it is necessary to "equalize" the two bases. This "equalization" is given by the connecting equations which have the form

$$\delta X_c^{tr}=diag X_c^{tr}\times\left(-P^*\right)^T\delta X_b^{r},$$

where matrix $P^*$ is derived from matrix $P$.

The final equations have the form

$$A\ddot{q} = B, \tag{12}$$

where $q$ is joint variables matrix, describing the relative positions of the contiguous bodies and matrices A and $B$ have the form (13) and (14) respectively

$$A = (\bar{S}H)\, I\, (\bar{S}H)^T + (\bar{S}HP^* \times diagX_c^{tr})\, HmH^T\, (\bar{S}HP^* \times diagX_c^{tr})^T. \tag{13}$$

Matrix $\bar{S}$ here contains the unit vectors along the rotational axes of joints.

$$B = \bar{S}\,(\bar{B}_3 + \bar{Z}_1 + HP^* \times diagX_c^{tr}\bar{Z}_2) + \bar{S}HP^* \times diagX_c^{tr} HmH^T \tag{14}$$
$$\left[diagX_c^{tr} \times P^{*T}H^T\bar{W}^T\right] - \bar{S}HIH^T\bar{W} + \bar{S}\dot{Y}_c^r,$$

where

$$\bar{B}_3 = HP^* \times diagX_c^{tr}\bar{B}_1 + HIH^T 1_{2n-1}\ddot{X}_{b_1} + HP^* \times diagX_c^{tr} H\hat{E}\dot{Y}_{c'}^{tr} + \bar{B}_2,$$

$$\bar{B}_2 = H\hat{E}\dot{Y}_c^r - HIH^T\left(\dot{X}_c^r - 1_{2n-1}\dot{\bar{X}}_{b_1}^r\right),$$

$$\bar{B}_1 = HmH^T\Big[1_{2n-1}\ddot{\bar{X}}_{b_1}^{tr} + diagX_c^{tr} \times (-HP^*)^T\ddot{\bar{X}}_{b_1}^r -$$
$$-diag\left[diagX_c^{tr} \times (-HP^*)^T\left(\dot{X}_c^r - 1_{2n-1}\dot{\bar{X}}_{b_1}^r\right)\right] \times (-HP^*)^T\left(\dot{X}_c^r - 1_{2n-1}\dot{\bar{X}}_{b_1}^r\right)\Big],$$

$$\bar{Z}_1 = -\left(HP^* \times diagX_c^{tr}\right) HmH^T \left(HP^* \times diagX_c^{tr}\right)^T \bar{V},$$

$$\bar{Z}_2 = -HmH^T\left(HP^* \times X_c^{tr}\right)\bar{V},$$

$$\bar{V} = diag\left[(HP^*)^T\left(\dot{X}_c^r - 1_{2n-1}\dot{\bar{X}}_{b_1}^r\right)\right] \times \dot{X}_c^r,$$

$$\bar{W} = \left[\bar{W}_1, \dots, \bar{W}_n\right]^T, \quad \bar{W}_a = \sum_{i=1}^{n_a}\sum_{j=1}^{n_a}\frac{\partial\bar{S}_{a_i}}{\partial q_{a_i}}\dot{q}_{a_i}\dot{q}_{a_j}.$$

Quantities $\dot{Y}_c^{tr}$ and $\dot{Y}_c^r$ included in equations (12) realize a change of energy between the considered system and its general power space. An interaction with other power subspaces can be described in a similar way, using the corresponding power graphs.

The approach proposed here is extended for systems where translation motion takes place. The part of the graph which corresponds to the system in such a case is shown in Fig.12.

Two contiguous bodies with translation joint between them are considered. The final equations in the same way, like the one represented up to now, have the form (12) again but in this case matrices A and $B$ become more complex. For instance, matrix A has the form

$$A = (\bar{S}H)\, I\, (\bar{S}H)^T + (\bar{S}HP^* \times diag\,(X_c^{tr} + X_{ct}^{tr}) + \bar{K})\,. \tag{15}$$

$$.HmH^T\,(\bar{S}HP^* \times diag\,(X_c^{tr} + X_{ct}^{tr}) + \bar{K})^T\,,$$

where $X_{ct}^{tr}$ is the matrix of variables associated with the new edges denoted in Fig.12 by "$\alpha$", "$\beta$" and "$\gamma$"; $\bar{K}$ is the matrix, whose elements are the unit vectors along the translational joints axes. These matrices have appeared in the expressions for matrix $B$, too.

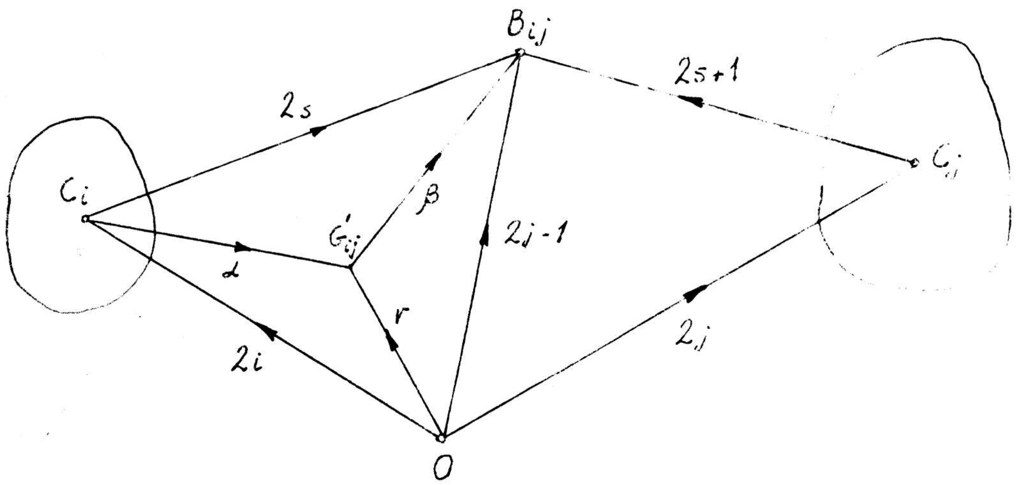


Fig.12

Summarizing we can conclude that the principle of orthogonality is a powerful device for deriving the state equation of complex systems. The extension of the principle is carried out through four basic groups of equations: terminal and connecting equations, expressing the physical laws in the general power space and the cutset, and circuit equations, expressing the topology of the system. State equations in generalized coordinates are won as a result, while all variables characterizing the system can be determined using the inverse schemes.

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