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SOME NUMERICAL METHODS OF LINEAR DESIGN: TIME– INVARIANT CASE*

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1. Introduction

Here the notion of linear design is meant in the following sense: to ensure the asymptotical stability by linear time-invariant approximation of finite-dimension dynamic systems, i.e. to solve the linear stabilization problem. This class of dynamic systems is very important because, first, the deviations from equilibria are usually small and, second, the Kalman filter upon which the state-space design is based has precise meaning in this case only.

There are two main methods of the LQG-design: the Riccati equation method and the Wiener-Hopf one. The first is based on the separation principle: solving the estimation problem and separately the optimal control one. In general the condition of the usage of this approach is the suggestion that the estimation is very precise and the optimal control problem solution does not depended strongly on small errors of estimation. It is the reason that great efforts continue to be produced for obtaining various procedures of design [Einpack, Limpack, Matlab: Control and robust control toolboxes and so on]. However this problem does not lack attractiveness.

Unfortunately the separation principle is motivated only in the case of linear quadratic design and even in this case the usage of estimation proves to be rather problematic because, first, the separation theorem can be not always used [1] and, second, the LQG-design based on the separation theorem is not robust [2].

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The famous Wiener-Hopf method comprises the factoring and the separation usually produced by defining zeros of polynomials and poles of rational functions. Dependence of zeros from polynomial coefficients being very unstable, various approaches to using methods of the linear algebra have been developed for the frequency design without determining zeros of polynomials.

The aim of this talk is to present a linear affine manifold approach [3, 4, 5, 6] and to point out the most effective methods of numerical LQG-design.

2. Linear Systems as Linear Affine Manifolds

Control systems have usually a rather complex structure. They include a control object and a lot of subsystems. In the simplest case, there are input and output devices as well as measuring ones. Let us assume that system description includes interaction between a control object and its functional medium. This medium can be modelled in various ways and defines external uncertainty. In stochastic system theory the influence of functional medium has indeterminate character, and its mathematical representation is called a generating or forming filter [7].

This external uncertainty is not unique one that must be taken in account. Any dynamic system is given under some internal uncertainty which means that having any linear system we work with a ball of linear dynamic systems [8]. That is why we may consider the following system with unbounded input and output operators [2]:

$$(F+\Delta F)u = v \tag{1}$$

$$(A+\Delta A)x+(B+\Delta B)u = \varphi \tag{2}$$

$$(G+\Delta G)y+(C+\Delta C)x+(D+\Delta D)u = \psi \tag{3}$$

where x, y, u, and $v \in \mathcal{H}$ are output, measured, input and auxiliary (indirect) control vector variables; $\varphi, \psi \in \mathcal{H}$ are system disturbances; operators $A, B, \ldots, G$ and their disturbances $\Delta \in \mathcal{B}$ are given. Hereafter $\mathcal{H}$ is an abstract Hilbert space; $\mathcal{B}$ is a Banach space of bounded causal operators [9]. Variables x, y, u, and v are called admissible if they are bounded linear causal maps of the systems disturbances. The map giving admissible variables is called a transfer operator by disturbances.

Now let the operator disturbances are absent. *The first basic problem (P1)* associated with system (1)–(3) is to define the set of all transfer operators.

We may rewrite system (1)–(3) in the following form

$$\mathcal{A}\mathcal{X}+\mathcal{B}\mathcal{U}=\phi, \tag{4}$$

$$\mathcal{Y}+\mathcal{C}\mathcal{X}=0 \tag{5}$$

where $\mathcal{X}=col(y,x,u)$, $\mathcal{U}=v$, $\phi=col(0,\varphi,\psi)$, $\mathcal{Y}=y$; $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$ are known.

Unfortunately, equation (5) does not include a disturbance that is obligatory in the stochastic control [7]. That is why let us consider another problem solution.

Theorem 1. [6] *Let operators*

$$\mathcal{P} = \begin{bmatrix} O & A & B \\ G & C & D \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} O & F \\ A & B \\ C & D \end{bmatrix}, \tag{6}$$

are right and left invertible respectively. Then expression

$$\begin{bmatrix} y \\ x \\ u \\ v \end{bmatrix} = [\alpha_2 + \alpha_3\Phi\gamma_2] \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \tag{7}$$

defines all admissible solutions of system (1)–(3), *and formula*

$$\begin{bmatrix} \zeta \\ y \\ v \end{bmatrix} = [\alpha_1 + (I - \alpha_1 P_1)\Phi](\mathcal{Q}\gamma - I) \begin{bmatrix} 0 \\ \varphi \\ \psi \end{bmatrix} \tag{8}$$

defines the feedback (if the last has sense), where $\alpha_1, \alpha_2, \alpha_3, \gamma, \gamma_2 \in \mathcal{B}$ *are known;* $\Phi \in \mathcal{B}$ *is a parameter;* $v \in \mathcal{H}$ *is an auxiliary variables.*

The set of all admissible solutions is a linear affine manifold [5] that is a reason to employ the projection theorem [10]. Thus we may consider the following main optimal problems for system (1)–(3):

P2. to define a minimal by norm admissible solution (it is analogous to the LQG problem);

P3. to define a minimal by norm mapping of the disturbances which gives us an admissible solution (it is analogous to the $\mathcal{H}_\infty$-design).

Remark. It is easy to see that problem $P3$ is equivalent to maximizing the ball norm of system (1)–(3). In order to harmonize the relationship between the main two branches of the linear system theory (having their origin in LQG problem and $\mathcal{H}_\infty$-design) the optimal design is considered relative to the sum of norms used in these branches.

It is well-known that dichotomy of spectrum, defining the stabilizing solution of algebraic Riccati equations (ARE), orthogonal projections and polynomial factoring can be produced with the help of an orthogonal transformation to Schur form of matrix [1,5,12,16,18]. This form has more detailed structure than it is necessary. Here it is sufficient to construct the block triangular form, in which its spectrum is separated between these blocks into "stable" and "unstable" parts.

After having defined the matrix sign-function, constructing the orthogonal projections and ARE solution can be produced with the help of numerical stable methods [5] (i.e. Holetzky method) because here the linear equation obtained has the positive defined symmetrical matrix. Together with Larin's balancing procedure it defines more precise ARE solution than by Schur method. Besides, here Newton-Raphson method can be used twice: first – for improving the orthogonal projections, and second – for improving ARE solution [14].

Larin's algorithms of the rational separation and the factoring in which the defining of zeros or poles is not used can be treated as the best ones [3,17]. Unlike his algorithm of the rational separation here the straight separating is produced with the help of orthogonal projection. As a result higher precision of defining orthogonal projection leads to higher precision of separation.

With the help of constructing orthogonal projections there is obtained a transform of a matrix into the block triangular form separating its spectrum into the "stable" and "unstable" parts.

Larin's separation procedure employs constructing non-orthogonal projections and solving Diophantine equations. At the same time it is clear that separation of a scalar proper rational function leads to use its part which defines a completely controllable and observable stable system. Any control system determined with the help of a scalar proper transfer function W can be reduced to its state-space form [4].

Larin's algorithms were realized in Basic with matrix operations (*PC* "Iskra-226") and in Fortran (IBM 360/370) [8]. With the help of several classic test examples [8] it was demonstrated that they are more effective than Schur method [5,16] used for constructing the stabilizing solution of ARE. One of the main goals of the present paper is to show the possibilities of the Larin's algorithm computer realization, created using Turbo C with long double precision operator (Copyright^R, Borland 1987, 1988) on the IBM *PC* family of computers, including the *XT*, AT, and *PS*/2, along with all true IBM compatibles. Our experience of the computer realization of Larin's algorithm with the balancing procedure (4.2) has confirmed its higher precision than that of traditional methods, included in Eispack or Linpack [16]. As an example we may point out Table 1 containing the results of computing the known model ARE used for characterizing the algorithms of solving ARE [8,16].

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REFERENCES

[1] Curtain, R. A synthesis of time and frequency domain methods for the control of infinite–dimensional systems: A system theoretic approach, to appear in the series. *SIAM Frontiers in Applied Mathematics.*

[2] Curtain, R. A synthesis of time and frequency domain methods for the control of infinite–dimensional systems: A system theoretic approach, to appear in the series. *SIAM Frontiers in Applied Mathematics.*

[3] Aliev, F.A., V.B.Larin, K.I.Naumenko. Optimizing Linear Time-Invariant Control Systems. Kiev, Naukova Dumka, 1978 (in Russian).

[4] Kucera, V. Discrete Linear Control: the Polynomial Equation Approach. Praga, Academia, 1979.

[5] Fomin, V.N. Discrete Linear Control Systems. Dordrecht – Boston – London, Kluwer Acad. Publishers, 1991.

[6] Cheremensky, A.G. Robust frequency design of linear stationary systems in aeroautoelastics. *J. Sci. Comp.*, **6(2)** (1991), 211–227.

[7] Kalman, R.E., P. Falb, M. Arbib. Topics in Mathematical System Theory. New York, McGraw–Hill, 1966.

[8] Zames, G. Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Trans. Autom. Control*, **26** (1981), 301–320.

[9] Feintuch, A., R. Saeks. System Theory: a Hilbert Space Approach. New York, Acad. Press, 1982.

[10] Porter, W.A. Modern Foundations of System Theory. New York, Macmillan, 1966.

[11] Cheremensky, A., N. Dakev. Matrix "Stable-Unstable" Bloc Triangular and Diagonal Forms. *Proc. of the Sixth European Conference on Math. in Industry*, Limerick, 1991.

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