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PROGRESSIVE CRUSHING OF PLASTIC CYLINDRICAL SHELLS UNDER AXIAL IMPACT*

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Introduction

In the present paper the asymmetric and transition crushing form of the deformed shell surface of a cylindrical shell under axial impact load is considered. The method used for obtaining the effective and final axial crushing distance, height and number of complete lobes of the crushed part of the shell is based on the variational principle of Hamilton and its modification by the geometric method for shell stability applied to the crushing problem of a shell under axial impact load. The comparison between the theoretical predictions and experimental results made at the Impact Research Centre at the University of Liverpool is obtained.

Theoretical modelling

Let us consider thin cylindrical shells simply supported at both ends, with a radius R, length L and a thickness h, subjected to an impact force $P(t)$ initiated by a tup mass M with initial velocity v_o.

(i) The form of the crushed shell has an asymmetric (diamond) shape. We suppose that at the beginning of the impact, the shell loses its static stability and enters the postbuckling regime. The shell absorbs the initial kinetic energy of the striking mass and reaches the stable configuration, which significantly differs from the initial perfect cylindrical surface. The approximation of the crushed form of the middle shell surface

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can be obtained by the geometric method given in [1] and [2]. The energetic functional $W = \int(K - U + A)dt$ will be defined on the deformed shell surface Z, isometric to the initial one, constructed in [1].

(ii) The behaviour of the shell material is elasto-plastic, strain rate sensitive in the plastic zones. There for the Cowper-Symonds uniaxial constitutive equations are used:

$$\sigma^d/\sigma_o = 1 + (\dot{\varepsilon}/\dot{\varepsilon}_o)^{1/n}, \quad M^d/M_o = 1 + (\dot{\kappa}/\dot{\kappa}_o)^{1/n} \tag{1}$$

where $\dot{\varepsilon}_o$, $\dot{\kappa}_o$ and n are material constants, σ_o is the uniaxial yield stress, $M_o = \sigma_o h^2/2\sqrt{3}$, σ^d and M^d are the dynamic stress and the bending moment.

(iii) It is assumed that the elastic zones cover almost the entire middle shell surface except for the neighbourhoods of the ribs γ_i generated by the plastic deformation.The ribs γ_i give a stable configuration of the crushed shell and detain the elastic zones in this state.

Geometric method

The application of the geometric method for the crushed shell is developed following the remarks:

As far as the shell loses its stability with a diamond pattern, the deformation energy is determined on the same deformed middle surface Z as in the postcritical (postbuckling) state of a shell. Only one difference arises in comparison with the case of stability loss: the shell has other value of axial shortening, restricted by the geometry of Z. In the case of crushing this shortening is obtainable from the geometry of the diamond pattern.

Now following [1], we approximate the deformed surface by a surface Z isometric to the initial perfect cylindrical one.

Let us consider a regular prism with a given number of sides m, (Fig.1). On one of the lateral faces α_1 a certain regular curve γ_1 with a unique projection on the axis of the prism is drawn. On the face α_2 let the curve γ_2 be the mirror-image of γ_1 from the plane β passing through the lateral edge of α_1 and the axis of the prism. In this way curves γ_i appear on every lateral face α_i of the prism. Through the curves γ_1 and γ_2 we draw a cylindrical surface Z_{12} with perpendicular to β generators. Analogously the cylindrical surfaces Z_{23}, Z_{34} etc. can be constructed. These surfaces form a tube-like surface Z. Z is regular everywhere except on the ribs (curves) γ_i. It is proved by Pogorelov that Z is isometric to a cylinder. When Z deforms, the radius R and height L do not change, i.e. this deformation is a geometric bending. Using such a bending we approximate the main picture of asymmetric deformation of the cylindrical shell. The following assumptions are made for the behaviour of the deformed shell:

(j) the wave number m in the radial direction equals the value of m for the loss of static stability of considered cylindrical shell.

(jj) the energy of the postcritical deformation U consists of two parts – U_1 is the elastic energy of almost the entire middle shell surface Z, except for the neighbourhoods of the ribs γ_i, and U_2 – the plastic energy localized in these neighbourhoods.

(jjj) the deformation can be regarded as progressive buckling. Thus, integration of the full energetic functional by time t can be achieved using the theorem of the average value for the integrals. Time t_o is the time for generation of one lobe with a diamond pattern.

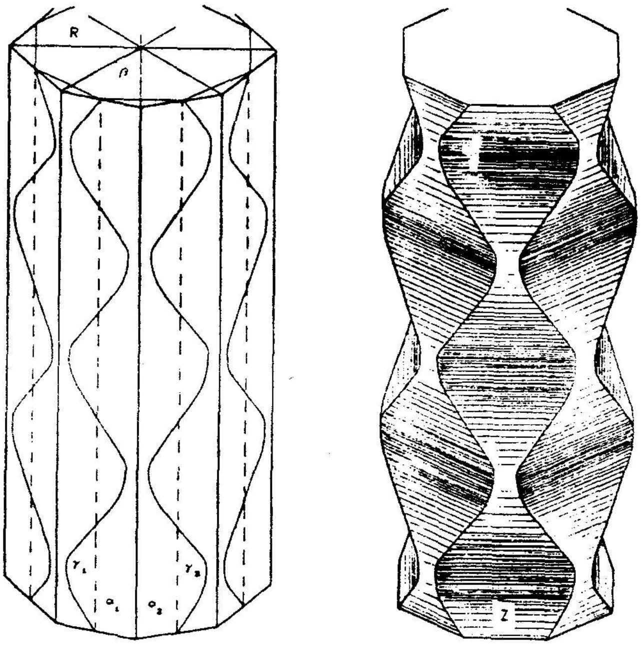

Fig. 1

Introducing in α_1 a Cartesian coordinate system (Oxy) and taking for Ox the line parallel to the lateral edge of the face α_1, the rib γ_1 is given by $\hat{y}(x,t) = y(x)y_1(t)$. The respective curve $\hat{\gamma}$ in the coordinate system, which is a projection of Oxy from α_1 on β will be:

$$\hat{y}_\beta = \sin\frac{\pi}{2m}y(x)y_1(t) \approx \frac{\pi}{2m}y(x)y_1(t),$$

if the number m in the radial direction is large enough.

Following the method in [2], for the time averaged energy of the post-critical deformation it is obtained:

$$\bar{U}(Z) = \bar{U}_1 + \bar{U}_2 =$$

$$\frac{Eh^3 2t_o}{(1-\nu^2)m}\iint_{(Z)}(\Delta\kappa_1^2 + \Delta\kappa_2^2 + 2\nu\Delta\kappa_1\kappa_2)dZ + \tag{2}$$
$$+\frac{C}{n}E_1 h^{5/2} 2t_o \int_{(\gamma)} \frac{\alpha^{5/2}}{\rho^{1/2}}ds + \frac{\sqrt{3}\sigma_o h^3 t_o}{4n\dot{\varepsilon}_o}\int_{(\gamma)}\alpha(-2\kappa_n + \kappa_e + \kappa_i)ds.$$

In order to calculate $\bar{U}_1$ and $\bar{U}_2$ on Z the following characteristics of Z are used in (2):

$$\Delta\kappa_1 = \pm\frac{\pi}{2m}y\prime\prime\,;\ \Delta\kappa_2 = -\frac{1}{R}\,;\ \alpha = \frac{\pi}{2m}(1 + y\prime^2)^{1/2}\,;\ \kappa_n = \frac{1}{R}\sin^2\theta\,;$$
$$ds = (1 + y\prime^2)^{1/2}\,;\ \kappa_e + \kappa_i = 0\,;\ E_1 = \frac{12^{3/4}(1-\nu^2)3^{3/8}\sigma_o}{8^{3/4}\dot{\varepsilon}_o}\,; \tag{3}$$
$$C = 0.19\,;\ \nu = 0.5\,;\ \theta = 1/(1 + y\prime^2)^{1/2}.$$

Let the curve γ be periodical and let k be the number of complete waves (lobs) in the axial (Ox) direction of the cylinder. Both the normal (along γ) and the radial sections of Z divide Z into mk congruent regions Q which are isometric to a rectangle. If the height of the cylinder is L then the height of Q is $b = L/k$, and if the radius of the cylinder is R then the width of Q is $a = \frac{2\pi R}{m}$. To obtain the averaged energy of the post-critical deformation for the whole shell it is enough to calculate the energy of Q and multiply by mk.

Replacing (3) into (2), integrating with respect to y and introducing the following non-dimensional parameters:

$$x = \frac{b}{2}\bar{x}\,;\ y = a\left(\frac{\lambda\mu^2}{2}\right)^{1/2}\bar{y}\,;\ \xi = \frac{b}{a} \tag{4}$$

we obtain in the domain Q:

$$\bar{U}_Q = \frac{\pi^2 E h^3 t_o}{24(1-\nu^2)m^2}\left[\frac{\lambda(1+\mu_1)^2}{\xi^3}\int_{-1}^{1}\bar{y}''^2 d\bar{x} + \frac{0.6\lambda(1+\mu_1)^2}{\xi^2} + \right. \tag{5}$$
$$\left. + \frac{2E_1\lambda^{1/4}(1+\mu_1)^{1/2}}{En}\int_{-1}^{1} |\,\bar{y}''\,|^{1/2}\left(1 + 2\lambda(1+\mu_1)^2\bar{y}\prime^2\right)d\bar{x}\right]$$

Here λ is the dimensionless parameter, which characterizes the deformation in the static case of loading, $\mu = y_1(t_o)$ – the average amplitude of $y_1(t)$; if $y_1(t)$ expands into a series with respect to t, then $\mu = y_1(t_o) = 1 + \mu_1$, where $\mu(0) = y_1(0) = 1$ corresponds to the static case of loading.

Now the problem is reduced to the problem for a selection of a proper class of function $\bar{y}(\bar{x})$, which satisfies some specific conditions for the axially loaded shell and given in [1]. The load-shortening of the deformed shell is $\Delta b = b - b\prime$, where:

$$\Delta b \approx \frac{\pi^2}{8m^2} \int_{(b)} y(x)\prime^2 (1 + \mu_1)^2 dx \tag{6}$$

Finally, we obtain for (5):

$$\begin{aligned} \bar{U}_Q = {} & \frac{\pi^2 E h^3 t_o}{12(1-\nu^2)m^2}\{\frac{3.5\lambda\mu_1^2}{\xi^3} + \frac{0.6\lambda\mu_1^2}{\xi^2} + \\ & + \frac{E_1}{En}[2.24\lambda^{1/4}\mu^{1/2} + \lambda^{5/4}\mu_1^{5/2}]\} \end{aligned} \tag{7}$$

Using (6) we calculate the averaged by time work A done by impact force $P(t_o)$ applied to a cylindrical ring with a height b.

$$\Delta b = \frac{\pi^2 \lambda a^2 (1 + \mu_1)^2}{8m^2 b}.$$

$$\bar{A} = \frac{\pi^4 h^3 E t_o \bar{P}(t_o) \lambda (1 + \mu_1)^2}{3.3\xi m}, \ P(t_o) = \bar{P}(t_o) E \frac{h}{R} \tag{8}$$

The averaged with respect to time kinetic energy K is equal to:

$$\bar{K} = \frac{1}{m} 0.27 t_o \rho_o v_o^2 \lambda^{1/2} (1 + \mu_1) \tag{9}$$

where ρ_o is the density of the shell material.

Considering W as a function of μ_1 , we have to satisfy the condition for extremum of W in order to find the critical value of μ_1 for the case of progressive crushing of a cylindrical shell. Taking the first derivative of W with respect to μ_1 equal to 0 and satisfying the condition for a stationarity of $\bar{P}(t_o)$ we arive at the following equation for μ_1 :

$$\mu_1^{1/2} = 0.52 \frac{\pi^2 h^3 E_1 \lambda^{-1/4}}{\rho_o v_o^2 n} \tag{10}$$

It was calculated that for mild steels $\mu_1 \ll 1$. Thus, for this kind of materials the progressive crushing is a quasi-static process.

To find the exact value of the axial shortening, we use the geometry of the crushed form of the ring (see Fig.2,b, the band between $C'C' - D'D'$). This crushed band is generated in the following way: the shell loses stability and takes up the diamond form (pattern $C_1O_1A_1C_2$, see Fig.2, a). The prolongation of the impact action leads to bringing the vertices (A_1, O_1) closer. They move up and down parallel to the axis of the shell (Ox), simultaneously generating the short plastic ribs with midpoints C_1 and C_2, (Fig.2, b). The height of these ribs is equal to $2l$ as in the concertina case (see [5]). Furthermore, we have to take into account the shell thickness in the height between two lobes as well as the manner of bending of crushed surface inwards to the axis of the shell. Now, Δb is just equal to $2A_1A_1'$. Thus the axial shortening of one diamond pattern Δb is equal to:

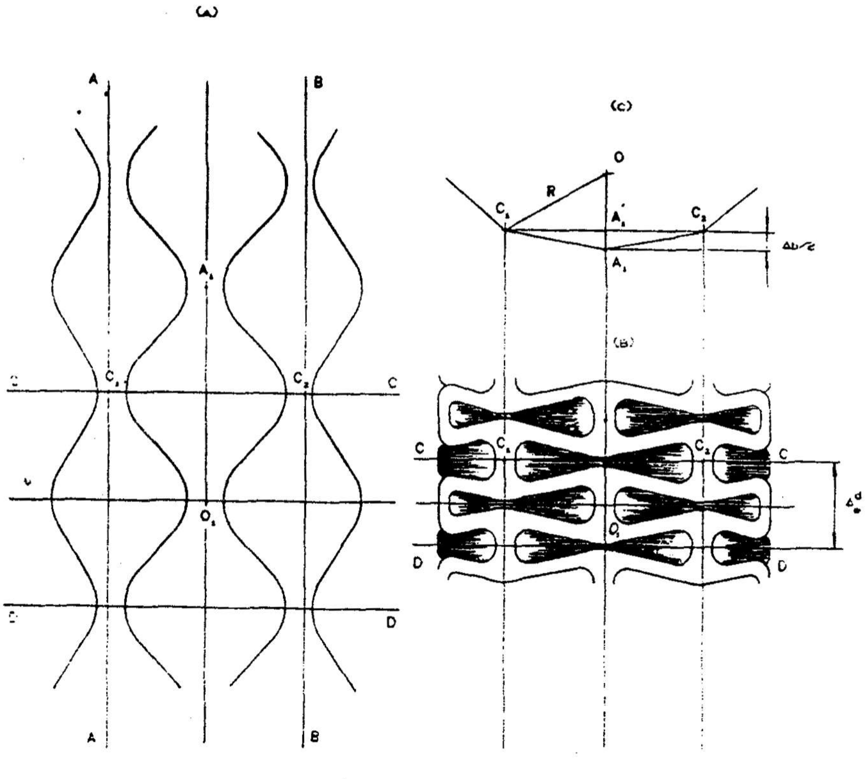

Fig. 2.
a) Collapse diamond mode, b) Crushed diamond mode
c) Cross-section γ and γ_0 M-polygon

(11) $$\Delta b = 4R\sin^2\pi/2m; \text{ if } m \gg 1 \text{ , then } \Delta b = \pi^2 R/m^2$$

(12) $$b = b' + \Delta b = 4R\sin^2\frac{\pi}{2m} + 2l + 4h,$$

where b is this part of the height of the initial cylinder, which has to generate one diamond lobe with final height $b' = 2l + 4h = \Delta_e^d$. Thus, the effective crushed distance $\delta_e^d = \Delta b$.

Results

The way to find all needed characteristics of the diamond crushed shell comprises of the following steps:

- a) to find the value of the crushing time t_1 ;
- b) to calculate $l = 0.28978(hR)^{1/2}$;
- c) to calculate $m = 0.8(hR)^{1/2}$ and $\delta_e^d = \Delta b = 4R\sin^2 \pi p/2m$;
- d) to calculate $b = 4R\sin \pi^2/2m + 2l + 4h = \delta_e^d + \Delta_e^d$;
- e) the value of the final crushed distance $\delta_f^d = v_o t_1/2 = k\delta_e^d$, thus $k = \delta_f^d/\delta_e^d$;
- f) for obtaining P_d^{th} – the average value of the crushing force during the time of crushing process t_1 – the equation of motion of the crushing mass (13) is used:

$$P_d^{th} = K_o/\delta_f + Mg\,,\ K_o = Mv_o^2/2 \tag{13}$$

Let us consider the transition mode – from concertina to diamond. Using the results, obtained in [5] we have:

Pure concertina mode – mild steel:

$$\varepsilon_o = (2+\sqrt{3})v_o/10\pi R = 0.356 v_o/R,\ n = 5,\ l = 0.28978(hR)^{1/2};$$

$$\delta_e^c = 8.42l - h;\ \delta_f = v_o t_1/2;\ c = \delta_f/\delta_e^c;\ b = 10\pi l/3;\ \Delta_e^c = b' = 2l + h;$$

Pure diamond mode – mild steel:

$$m = 0.8(R/h)^{1/2} \text{– for } m \text{ we take the entire part of } m \to [m];$$

$$l = 0.28978(hR)^{1/2};\ \delta_e^d = 4R\sin^2 \pi/2m \approx R\pi^2/m^2;$$

$$\delta_f = v_o t_1/2;\ d = \delta_f/\delta_e^d;\ \Delta_e^d = b' = 2l + 4h;\ b = \delta_e^d + \Delta_e^d.$$

The final crushed height of the shell for pure concertina, pure diamond and transition crushing modes is:

$$\Delta^c = c\Delta_e^c;\ \Delta^d = d\Delta_e^d;\ \Delta^t = k^c\Delta_e^c + k^d\Delta_e^d \text{ and } \delta_f = k^c\delta_e^c + k^d\delta_e^d.$$

where k^c and k^d are numbers of concerina and diamond modes, respectively when a transition (t) mode is generated. All the possible transition modes can be obtained following the scheme:

Let $k^c = 1$. Calculating $l = 0.28798(Rh)^{1/2}$ and $m = 0.8(R/h)^{1/2}$ we obtain $\delta_e^c = 8.42l + h$; $\Delta_e^c = 2l + h$ and $\delta_e^d = 4R\sin^2 \pi/2m$; $\Delta_e^d = 2l + 4h$. Now from the equation $\delta_f = v_o t_1/2 = \delta_e^c + k_1^d \delta_e^d$, we have $k_1^d = (v_o t_1/2 - \delta_e^c)\delta_e^d$. Thus the first possible transition mode is $S(1)Dm(k_1^d)$. The second mode can be obtain as $k^c = 2$. The scheme ends with the restriction $k_n^d \geq 0.8$.

In this way all possible transition modes for the considered specimens are obtainable, (Table 2, [4]). The results of the comparison are shown in Table 1.

Table 1. Comparison between experimental data from Table 2 [4] and theoretical predictions for final crushing distance (δ_f), mode of deformation (MD) and final crushing height (Δ).

EXPERIMENT			THEORY	
Specimen No	δ_f (mm)	MD, Δ(mm)	δ_f(mm)	MD, Δ(mm)
AT14	14.2	S(1)D2(1.5); Δ —	13.82	S (3.33);S (2)D2(0.61);S (1)D2(1);D2(1.54); Δ=7.35; Δ=7.6; Δ=7.82 Δ=8.02
AT17	9.8	S(2); Δ —	10.3	S (2.48);S (2)D2(0.22);S (1)D2(0.68);D2(1.15); Δ=5.48; Δ=5.56 Δ=5.78; Δ=5.98
BT1	68	S(1)D3(3); Δ —	66.27	D2(2.74);S (1)D2(2.39);S (2)D2(2.04);S (8.14); Δ=19.28; Δ=20.26 Δ=21.18; Δ=27.28
BT2	31.4	S(1)D2(1); Δ —	31.88	D2(1.32);S (1)D2(0.93);S (3.4); Δ=9.35; Δ=10.02 Δ=11.67;
BT7	34.5	S(1)D2(1); Δ —	29.3	D2(1.2);S (1)D2(0.87);S (3.58); Δ=8.62; Δ=9.65 Δ=12.31
BT8	41	S(1)D2(2); Δ —	40.3	D2(1.7);S (1)D2(1.32);S (2)D2(0,98);S (4.93); Δ=11.73;Δ=12.77 Δ=13.08; Δ=16.89
BT9	27.72	S(2); Δ —	29.34	D2(1.2);S (1)D2(0.87);S (2)D2(0.5);S (3.58); Δ=8.5; Δ=9.55 Δ=8.52; Δ=12.24
BT10	41	S(2)D2(1); Δ —	40.3	D2(1.67);S (1)D2(1.3);S (2)D2(0.99);S (4.92); Δ=11.8; Δ=12.87 Δ=13.9; Δ=16.9
BT11	30.3	S(3); Δ —	32.7	D2(1.36);S (1)D2(1.02);S (2)D2(0.6);...S (4); Δ=9.57; Δ=10.6 Δ=11.64;.......Δ=13.7
BT12	37.7	S91)D2(1.5); Δ —	40.3	D2(1.67);S (1)D2(1.32);...S (4.9); Δ=11.9; Δ=12.9 Δ=16.96;
CT2	49.9	S(4); Δ —	50.8	S (5.13);S (4)D3(0.6);S (3)D3(1.2);...D3(3); Δ=19.7; Δ=20.35; Δ=20.9;........Δ=22.58
CT3	71.8	S(2)D3(5); Δ —	62.9	S (6.4);..S (3)D3(1.97);S (2)D3(2.5);..D3(4.5) Δ=24.5;..Δ=26.3; Δ=26.9;.......Δ=28.05
CT5	74.4	S(2)D3(4); Δ —	75.9	S (7.7);...S (2)D3(3.3);S (1)D3(3.9);D3(4.5); Δ=29.5;...Δ=32.66; Δ=33.21; Δ=33.76;
CT6	30.3	S(1)D2(1); Δ —	33.86	S (3.4);S (2)D3(0.8);S (1)D3(1.4);D3(2); Δ=13.1;Δ=13.9; Δ=14.5; Δ=15.06;
CT8	102.1	S(2)D (5); Δ —	98.03	S (10);...S (3)D3(4);S (2)D3(4.6);..D3(5.8); Δ=37.9;..Δ=41.44; Δ=41.94;......Δ=42.97;
DT1	44.9	S(3); Δ —	45.3	S (3.34);..S (2)D3(0.7);S (1)D3(1.3);D3(1.74); Δ=17;.....Δ=17.34; Δ=17.6; Δ=17.8;
DT2	60.3	S(4); Δ —	58.5	S (4.3);..S (2)D3(1.27);S (1)D3(1.8);D3(2.37); Δ=21.9;..Δ=22.52; Δ=22.77; Δ=13.08;
DT5	30.1	S(2); Δ —	32.9	S (2.4);S(2)D3(0.23);S (1)D3(0.8);D3(1.34); Δ=12.43; Δ=12.54; Δ=12.8; Δ=13.08;
DT6	62.3	S(1)D2(2); Δ —	58.5	S (4.3);...S (2)D3(1.3);S (1)D3(1.8);D3(2.4); Δ=21.9;...Δ=22.6; Δ=22.83; Δ=23.07;
DT7	120.8	S(1)D3(3.5); Δ —	112.1	S (8.28);..S (2)D3(3.4);S (1)D3(4);D3(4.54); Δ=42.24;..Δ=43.54; Δ=43.75; Δ=43.96;
DT8	53.4	S(3); Δ —	56.7	S (4.2);...S (2)D3(1.2);S (1)D3(1.7);D3(2.3); Δ=21.4;...Δ=28.6; Δ=22.31; Δ=25.6;
D1T1	77.4	S(5); Δ —	79.2	D3(2.9);...S (3)D3(1.3);S (4)D3(0.8);S (5.44); Δ=28.4;....Δ=28.75; Δ=28.9; Δ=29.04;
D1T2,3,4		have the same theoretical predictions with D1T1		

EXPERIMENT			THEORY	
Specimen No	δ_f(mm)	MD, Δ(mm)	δ_f(mm)	MD, Δ(mm)
D1T5	74.3	S(1)D3(2.5); Δ —	65.5	D3(2.35);S(1)D3(1.82);.....S(4.44); Δ=23.4; Δ=23.5; Δ=23.84;
D1T8	94.9	S(6); Δ —	92.0	D3(3.3);S(1)D3(2.8);..S(4)D3(1.23);S(6.38); Δ=32.45;Δ=32.65; Δ=33.23; Δ=33.68;
GT2	45.2	S(1)D4(2.5); Δ —	42.9	S(3);S(2)D4(0.7);S(1)D4(1.4);D4(2.05); Δ=15.03;Δ=15.8; Δ=16.65; Δ=17.5
GT5	77.6	S(2)D4(2); Δ —	77.0	S(5.1);...S(2)D4(2.2);S(1)D4(2.8);D4(3.5); Δ=25.5;...Δ=28.3; Δ=28.9; Δ=29.83;
GT6	107.9	S(3)D4(3); Δ —	107.13	S(7);..S(3)D4(2.8);..S(1)D4(4.2);D4(4.9); Δ=35.6;Δ=39.2; Δ=40.86; Δ=41.8;
GT7	141.9	S(4)D4(3.5); Δ —	148.0	S(9.9);..S(4)D4(4);..S(1)D4(6);D4(6.5); Δ=49.2;..Δ=53.96;.. Δ=56.67; Δ=57.54;
GT8	73.2	D4(3); Δ —	72.5	S(4.85);...S(1)D4(2.6);...D4(3.3); Δ=24;......Δ=26.87; Δ=27.7;
FT1	81.4	S(3)D3(4.5); Δ —	80.6	S(8.15);..S(5)D4(2.4);..S(3)D4(4);D4(6.3); Δ=27.8;...Δ=31.6; Δ=33.99; Δ=37.62;
FT2	59.1	S(6); Δ —	59.65	S(6);...S(3)D4(2.3);... D4(4.63); Δ=20.4;..Δ=23.99; Δ=27.5;
IT2	52.2	D3(3); Δ —	52.2	D3(1.9);S(1)D3(1.4);S(2)D3(0.9);...S(4.07); Δ=14.8; Δ=15.65; Δ=16.84; Δ=18.15;
IT3	61.4	S(3)D3(1); Δ —	69.0	D3(2.5);..S(3)D3(1.9);S(4)D3(0.6);S(5.3); Δ=19.87;..Δ=22.22; Δ=23; Δ=24.05;
IT5	98.1	D3(4); Δ —	93.7	D3(3.4);...S(2)D3(1.9);S(3)D3(1.9);S(7.28); Δ=26.8;....Δ=28.36; Δ=29.1; Δ=32.6;
IT6	143.8	D3(6); Δ —	134.8	D3(3.26);S(1)D3(2.8);S(2)D3(3.9);..S(10.5); Δ=38.7; Δ=39.52; Δ=43.32;......Δ=47.11
IT8	92.3	S(1)D3(3.5); Δ —	91.1	D3(3.3);S(1)D3(2.8);...S(2)D3(2.3);...S(7); Δ=26.4; Δ=27.2;....... Δ=27.94;.......31.7;

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