Journal of Theoretical
and Applied Mechanics
Sofia 1993, Year XXIV No 3

A NUMERICAL SOLUTION OF THE PLANE BOUNDARY VALUE PROBLEM OF THERMOELASTICITY THROUGH NONORTHOGONAL MESHES

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1. Introduction

At present the boundary value problem of the theory of elasticity, as a rule, is solved by FEM, making use of the Lagrange variational principle. However there are problems like the modelling of the plane thermal stress state (TSS) of discretely growing concrete dam blocks which solution with direct determination of stresses by means of the Airy's stress function F, proves to be nearly twice more economical. In this case, because of the second degree of the Castiliano functional, the solution by the biharmonic equation of the strain compatibility in terms of F is more feasible than by the FEM.

2. Problem formulation

We have a heterogeneous domain Ω, growing discretely by homogeneous subdomains ω_l, coinciding with the lifts of block's concreting (Fig. 2. a). In every already cast subdomain "l" the field of the temperature changes $T_l(t,x,y)$ is known. If for more facility in the method exposure the concrete creep and aging are not considered, the plane stress state, caused by T, can be determined through the integration of the following system of equations:

(1) $$\nabla^2\nabla^2 F_l = -\alpha E_l \nabla^2 T_l, \quad l = 1,2,3\dots,N, \quad (x,y) \in \omega_l,$$

(2) $$F''_{ly} - \mu F''_{lx} - \beta(F''_{l-1,y} - \mu_{l-1} F''_{l-1,x}) = \alpha_l E_l (T_{l-1} - T_l) \text{ and}$$

(3) $$F'''_{ly} - \beta F'''_{l-1,y} + [2 + \mu_l - \beta(2 + \mu_{l-1})](F'_y)''_x = \alpha E_l(T'_{l-1,y} - T'_{ly}),$$

where $(x,y) \in AH,\ BG,\ CF, \ldots, \beta = E_l/E_{l-1}$.

Equations (2) and (3) express the conditions of equal displacements along the interfaces between the subregions. There $F_l = F_{l-1}$, $F'_{ly} = F'_{l-1,y} = F'_y$ and $F''_{lx} = F''_{l-1,x}$. On the free boundaries, when only temperature changes are considered, $F_l = \partial F_l/\partial n = 0$. E, μ and α are respectively Young's modulus and Poison's heat expansion coefficients.

3. Description of the method

The numerical integration of this system by the finite difference method (FDM) creates no problems with rectangular domains as the horizontal and vertical mesh lines are not bound with each other. However with nonrectangular domains the standard application of FDM is less efficient because of the inclined boundaries. It is all the more true when in one of the directions, the vertical in the case, mesh ought to be many times thicker than in the other one. For such a case a modified application of FDM with an orthogonal and irregular mesh, with not bound by the inclined boundaries mesh lines, has been developed [1,2]. The method is used in the versions 2 and 3 of the programme "Thermostress" for modelling of thermal regime (TR) and TSS of concrete dam blocks during construction and service life [2]. Anyway for the numerical integration in nonorthogonal domains an arbitrary mesh is more appropriate. In ref. [3] a method for integration of partial differential equations of second order, based on a modifyed colocation principle and nonorthogonal mesh, is exposed. The further development of that method, aiming at the approximation of the biharmonic equation (1) with conditions (2) and (3), is very briefly described in this report.

Considering the most frequent shapes of the massive structures, the method has been developed for a mesh, obtained by deformation of the orthogonal one, keeping the horizontal lines in parallel. The general principles of the method, however, are valid for non-parallel meshes as well. In Unlike [3] the III and IV partial derivatives of F are approximated in two stages using the following presentation:

(4) $$F^{IV}_x = \partial^2/\partial x^2(F''_x),\quad F^{IV}_y = \partial^2/\partial y^2(F''_y),\quad F'''_y = \partial/\partial y(F''_y),\ \text{etc.}$$

In a similar of [3] way the second derivatives F'' are approximated in the vicinity of the mesh node "$r \equiv 13$" (Fig. 1. a) by 9–parametrical incomplete polinomials of 4–th degree, selected in a way to interpolate the values of F in 9 nodes of a simetrical to "r" mesh nodal scheme. It looks as follows:

(5) $$F''_r(x,y) = a_1 + a_2x + a_3y + a_4xy + a_5x^2 + a_6y^2 + a_7x^2y + a_8xy^2 + a_9x^2y^2.$$

The interpolation conditions bring to:

(6) $$F''_r(x,y) = [N_{rs}(x,y)]\{F''_s\},\quad s = 7,8,9,12,13,\ldots,19.$$

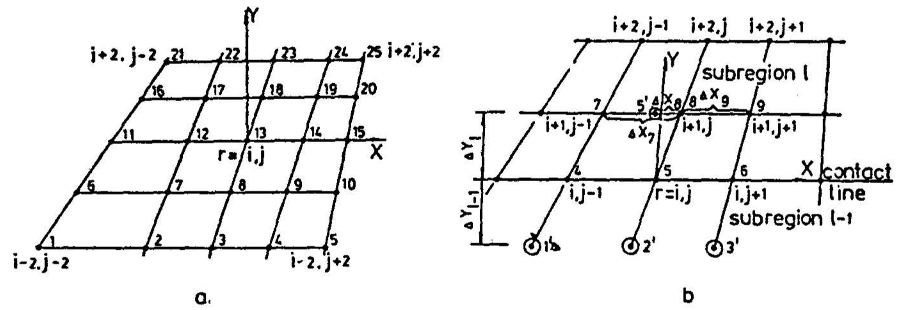

Fig. 1

In (6) the nodal values F''_s are obtained from:

$$F''_s = \left[N''_{sp}(0,0)\right]\{F_p\}, \tag{7}$$

basing on a similar to (5) local approximation of F in the vicinity of every one of the nine nodes with index "s" of the nodal scheme of node "r" and in a co-ordinate system with a center in "s", given by the formula:

$$F_s(x,y) = [N_{sp}(x,y)]\{F_p\}.$$

For example when $s \equiv 7$, $p = 1,2,3,6,7,8,11,12$ and 13 (Fig. 1. a).

Obviously the proposed method for approximation of the fourth partial derivatives requires a 25 points mesh scheme unlike the 13 points one, sufficient for the FDM with an orthogonal mesh.

Accounting for (4),(6),(7) and the colocation principle in every internal nodal point "r" the equation (1) is approximated by the following linear algebraic equation:

$$\sum_{s=1}^{9}[M_{sp}]\{F_p\} = -\alpha E_1[N''_{sx}(0,0) + N''_{sy}(0,0)]\{T_s\}, \tag{8}$$

where $[M_{sp}] = [N''_{sx}(0,0)N''_{spx}(0,0) + (N''_{sx}(0,0) + N''_{sy}(0,0))N''_{spy}(0,0)]$.

The approximation of the contact condition (2) in node $r \equiv 5$ (Fig. 1. b) is obtained by direct use of formulae (7) using extrapolated values of F in fictitious, beyond the subdomain contours, nodes $1'$, $2'$ and $3'$ for subdomain "l" and $7'$, $8'$ and $9'$ for subdomain "$l-1$". Then (2) will look as follows:

$$B_{lr} - \beta B_{l-1,r} = \alpha E_l(T_{l-1,r} - T_{lr}), \quad \text{where } B_{ir} = \left[N''_{isy} - N''_{isx}\right]\{F_{is}\}. \tag{9}$$

The approximation of condition (3) is achieved by presentation of F''' through unilateral finite differences $-F'''_{l5y} = (F''_{l5'y} - F''_{l5y})/\Delta y_l$. The value of $F''_{l5'y}$ is interpolated. Then (3) can be presented as:

$$(10) \qquad C_{lr}/\Delta y_l + C_{l-1,r}\beta/\Delta y_{l-1} + [2+\mu_l - (2+\mu_{l-1})\beta](F'_y)''_x = \alpha E_l(T'_{l-1} - T'_{ly}),$$

where $C_{ir} = F''_{i5'y} - [N''_{isy}]\{F_{is}\}$.

The approximation of $(F'_y)''_x$ with parallel horizontal mesh lines is done by a standard FDM formula.

The above exposed approximation is convergent and of I degree, as it is with an orthogonal and irregular mesh. But unlike the latter from the error expression of the proposed method only a part of the mixed derivatives is eliminated. This deficiency is compensated with a more rationally distributed mesh. With an orthogonal mesh method's equations authomaticaly transform into FDM equations.

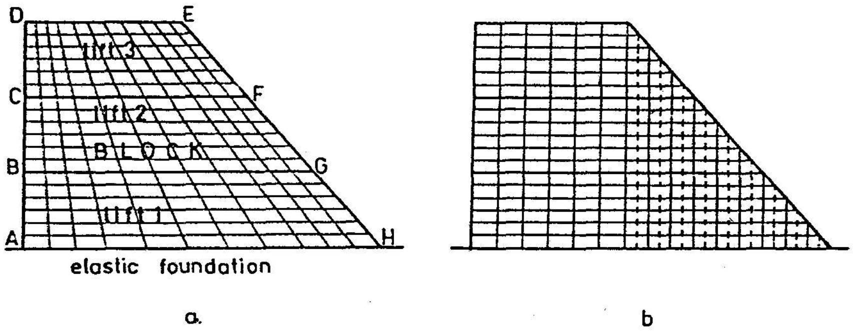

Fig. 2
— with nonorthonogal mesh
-·-- with northonogal thick mesh
- - - with orthonogal thick mesh

The described solution is used in version 4 of the programme "Thermostress" for dam blocks' TR and TSS modelling.

The method has been tested by various computations. Hereby, an example of a block, constructed with three successively laid no creeping and aging lifts with constant and uniform thermal changes of $\Delta T = 10^oC$, occurred after casting, is illustrated in Fig. 2, 3. The computation is done by a nonorthogonal mesh (NM) with total numbers of unknowns (TNU) 280 (Fig. 2. a), an orthogonal thick mesh (Fig. 2. b with broken lines) with TNU 478 and a thinner one (Fig. 2. b without broken lines) with TNU 307. If the results by the orthogonal thick mesh are assumed as refference values the NM yields the next by accuracy and close to them stresses. The investigations show that in this particular case the results of the method are good when $\Delta x/\Delta y < 1$ and mesh line inclination is less than 45^o.

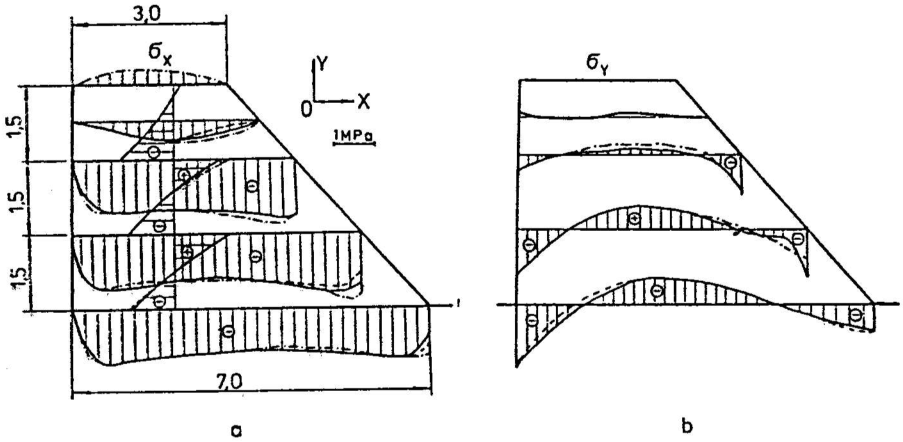

Fig. 3
— with nonorthonogal mesh
-·-·- with northonogal thick mesh
- - - with orthonogal thick mesh

4. Conclusions

Based on the colocation principle, by means of local approximation of the Airy's stress function and its II derivatives through incomplete algebraic polynomials of IV degree, developed is a method for numerical solution of the boundary value problem of the theory of elasticity through nonorthogonal meshes.The method is appropriate for some special problems like the numerical modelling of TR and TSS of growing by lifts elastic-creeping and aging concrete dam blocks.

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Presented at the 7th Congress on Theoretical and Applied Mechanics, Sofia, August 30 – September 1, 1993
section: Solid Mechanics