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NONLINEAR VIBRATIONS OF A RIGID BODY WITH COMBINED EXCITEMENT IN THE FIELD OF ELASTIC FORCES

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1. Introduction.

The complication of the functional qualities of machines demands very often a creation of a spatial vibration protection system in the presence of the combined excitations. They may be simultaneous excitations of forced, parametrical and self-excited vibrations. The properties of the spatial movement of a rigid body with excitations of self excited, parametrical and forced vibrations in the field of an elastic hanging forces are described in the article (Fig. 1). The rigid body B with a mass M is under the influence of the elastic forces of the elastic elements with coefficients of elasticity $c_1 \dots c_{24}$ and length r; we are given the main central inertial moments of the body J_x, J_y, J_z in the relative coordinate system, connected with the body. The generalized coefficient of elasticity along the coordinate ξ changes according to the following equation:

$$c_\xi^* = c_\xi(1 + \mu \cos \omega t), \tag{1}$$

where c_ξ =const, ω – frequency of the parametrical excitation with a coefficient μ. The periodical force P^* acts along the axis $C\zeta$:

$$P^* = P_0 + P \cos \omega t, \quad (P_0, P \text{ – coefficients}, \ \omega \text{ – exciting frequency}). \tag{2}$$

The mechanism of a dry friction of self excited vibrations acts along the axis $C\zeta$ too:

(2a) $$T^* = T\text{sign}\dot{\zeta} - D\dot{\zeta}, \quad (T, D \text{ - coefficients}).$$

The system has six degrees of freedom and space nonlinear vibrations can be created with it [1,2,3].

2. A mathematical model of the system.

Having in mind the nonlinear geometrical links between the coordinates of the body in the field of the potential forces [1] and using the methods of mechanics, we can define the differential equations of the movement of the system in the following form:

(3)

a) $$\ddot{\xi} + k_1^2\xi = -\mu(h_1'\dot{\xi} + \sigma_{111}\xi\zeta + \sigma_{118}\eta\varphi + \sigma_{120}\zeta\psi + \sigma_{126}\theta\varphi) - k_1^2\mu\xi(\cos\omega t) + V_1(\xi,\eta,\zeta,\theta,\psi,\varphi),$$

b) $$\ddot{\eta} + k_2^2\eta = -\mu(h_2'\dot{\eta} + \sigma_{214}\xi\varphi + \sigma_{215}\eta\zeta + \sigma_{219}\zeta\theta + \sigma_{227}\psi\varphi) + V_2(\xi,\eta,\zeta,\theta,\psi,\varphi),$$

c) $$\ddot{\zeta} + k_3^2\zeta = -\mu(h_3'\dot{\zeta} + \sigma_{37}\xi^2 + \sigma_{38}\eta^2 + \sigma_{313}\xi\psi + \sigma_{316}\eta\theta + \sigma_{324}\varphi^2) - \mu(T\text{sign}\dot{\zeta} - D\dot{\zeta}) + h_3\cos\omega t + V_3(\xi,\eta,\zeta,\theta,\psi,\varphi),$$

d) $$\ddot{\theta} + k_4^2\theta = -\mu(h_4'\dot{\theta} + \frac{J_x - J_y}{J_x}\ddot{\psi}\varphi + \frac{J_x - J_y + J_z}{J_x}\dot{\psi}\dot{\varphi} + \frac{J_z}{J_x}\ddot{\varphi}\psi + \frac{H_6}{J_x}\dot{\varphi}\psi + \frac{H_3}{J_x}\eta\dot{\zeta} - \frac{H_2}{J_x}\zeta\dot{\eta} + \sigma_{414}\xi\varphi + \sigma_{415}\eta\zeta + \sigma_{427}\psi\varphi) + \frac{\mu M_1}{J_x}(\ddot{\zeta}\eta - \ddot{\eta}\zeta) + V_4(\xi,\eta,\zeta,\theta,\psi,\varphi),$$

e) $$\ddot{\psi} + k_5^2\psi = -\mu(h_5'\dot{\psi} + \frac{J_x - J_y}{J_y}\ddot{\theta}\varphi + \frac{J_x - J_y - J_z}{J_y}\dot{\varphi}\dot{\theta} - \frac{J_z}{J_y}\theta\ddot{\varphi} - \frac{H_6}{J_y}\dot{\varphi}\theta + \frac{H_1}{J_y}\zeta\dot{\xi} - \frac{H_3}{J_y}\dot{\zeta}\xi + \sigma_{511}\xi\zeta + \sigma_{518}\eta\varphi + \sigma_{526}\theta\varphi) + \frac{\mu M_1}{J_y}(\zeta\ddot{\xi} - \xi\ddot{\zeta}) + V_5(\xi,\eta,\zeta,\theta,\psi,\varphi),$$

f) $$\ddot{\varphi} + k_6^2\varphi = -\mu(h_6'\dot{\varphi} + \frac{J_y - J_z}{J_z}\ddot{\psi}\theta + \frac{J_y - J_x - J_z}{J_z}\dot{\psi}\dot{\theta} - \frac{J_x}{J_z}\ddot{\theta}\varphi + \frac{H_2}{J_z}\xi\dot{\eta} - \frac{H_1}{J_z}\eta\dot{\xi} + \frac{H_4}{J_z}\dot{\theta}\psi + \sigma_{610}\xi\eta + \sigma_{612}\xi\theta + \sigma_{617}\eta\psi + \sigma_{621}\zeta\varphi + \sigma_{625}\theta\psi) + \frac{\mu M_1}{J_z}(\xi\ddot{\eta} - \eta\ddot{\xi}) + V_6(\xi,\eta,\zeta,\theta,\psi,\varphi),$$

where: $c_1 = c_7 = c_{19} = c_{20} = c_y$; $d \approx 0$; $c_2 = c_8 = c_{17} = c_{18} = c_x$; $c_3 = c_4 = c_5 = c_6 = c_z$; $c_j = 0$ $(j = 9,\dots,16,21,22,23,24)$; $V_1,\dots,V_6$ – functions, proportional to the third degrees of the coordinates; $\sigma_{ijk}(i,j,k = 0,1,\dots,9)$ – coefficients, functions of the coefficients of elasticity and geometrical dimensions of the system; $\xi,\eta,\zeta,\theta,\psi,\varphi$ – coordinates of the mass centre in the absolute coordinate system and angular coordinates of the body; H_j, h'_j $(j = 1,\dots,6)$ – coefficients of resistance, k_j $(j = 1,\dots,6)$ – natural frequences, μ – a small parameter. We investigate the case, where in the system both main parametrical resonance along the axis $O\xi$ and basically resonance along the axis $O\zeta$ exist, and the following ratios are fulfilled:

$$\omega \neq k_j \quad (j = 2,4,5,6), \tag{4}$$

$$\omega \approx k_3, \quad \frac{\omega}{2} \approx k_1. \tag{5}$$

We create the equations:

$$k_1^2 = \frac{\omega^2}{4} + \mu\omega\varepsilon_1, \quad k_3^2 = \omega^2 + \mu\omega\varepsilon_3, \tag{6}$$

where $\varepsilon_1, \varepsilon_3$ – differences between the frequences k_1, k_3 and ω.

We shall look for a periodical solution of the system (3) in the following form:

$$\begin{aligned} &\xi = C_\xi e^{i\frac{\omega}{2}t} + D_\xi e^{-i\frac{\omega}{2}t}, \quad \eta = C_\eta e^{ik_2 t} + D_\eta e^{-ik_2 t}, \quad \zeta = C_\zeta e^{i\omega t} + D_\zeta e^{-i\omega t}, \\ &\theta = C_\theta e^{ik_4 t} + D_\theta e^{-ik_4 t}, \quad \psi = C_\psi e^{ik_5 t} + D_\psi e^{-ik_5 t}, \quad \varphi = C_\varphi e^{ik_6 t} + D_\varphi e^{-ik_6 t}. \end{aligned} \tag{7}$$

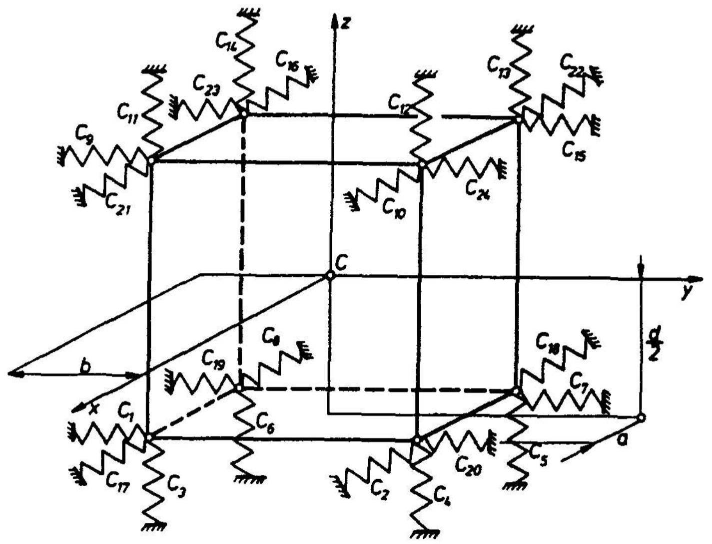

Fig. 1.

We use the methods of average and harmonical linearization (for the force of the dry friction) and for defining the coefficients $C_\xi, D_\xi, C_\zeta, D_\zeta$ we obtain the system equations in a stationary regime:

$$
(8)\quad
\begin{aligned}
a)\quad & i\omega\frac{dC_\xi}{dt} = 0 = -i\frac{\omega}{2}h_1' C_\xi - \sigma_{111} D_\xi C_\zeta - \frac{k_1^2}{2} D_\xi - \omega\varepsilon_1 C_\xi \\
& -3\sigma_{128} C_\xi^2 D_\xi - 2\sigma_{144} C_\zeta D_\zeta C_\xi, \\
b)\quad & i\omega\frac{dD_\xi}{dt} = 0 = -(i\frac{\omega}{2}h_1' D_\xi - \sigma_{111} C_\xi D_\zeta - \frac{k_1^2}{2} C_\xi - \omega\varepsilon_1 D_\xi) \\
& -3\sigma_{128} C_\xi D_\xi^2 - 2\sigma_{144} C_\zeta D_\zeta D_\xi, \\
c)\quad & 2i\omega\frac{dC_\zeta}{dt} = 0 = -i\omega(h_3' - D)C_\zeta - \sigma_{37} C_\xi^2 + \frac{h_3^*}{2} - \omega\varepsilon_3 C_\zeta - 3\sigma_{330} C_\zeta^2 D_\zeta \\
& -2\sigma_{335} C_\xi D_\xi C_\zeta - \frac{\Gamma i}{\rho_3} C_\zeta, \\
d)\quad & 2i\omega\frac{dD_\zeta}{dt} = 0 = -[i\omega(h_3' - D)D_\zeta - \sigma_{37} D_\xi^2 + \frac{h_3^*}{2} - \omega\varepsilon_3 D_\zeta - 3\sigma_{330} C_\zeta D_\zeta^2 \\
& -2\sigma_{335} C_\xi D_\xi D_\zeta] - \frac{\Gamma i}{\rho_3} D_\zeta,
\end{aligned}
$$

where ρ_3 is the amplitude of the vibrations along the axes $C\zeta$ and $\Gamma = \frac{4T}{\pi}$. We make the substitution:

$$
(9)\qquad C_\xi = \rho_1 e^{-\delta_1 i},\quad D_\xi = \rho_1 e^{\delta_1 i},\quad C_\zeta = \rho_3 e^{-\delta_3 i},\quad D_\zeta = \rho_3 e^{\delta_3 i}
$$

and represent the system (8) by means of Eqs.(9) in the form:

$$
(10)\quad
\begin{aligned}
a)\quad & h_1' i\frac{\omega}{2}\rho_1 e^{-\delta_1 i} + \sigma_{111}\rho_1 e^{\delta_1 i}\rho_3 e^{-\delta_3 i} + \frac{k_1^2}{2}\rho_1 e^{\delta_1 i} + \omega\varepsilon_1\rho_1 e^{-\delta_1 i} \\
& +3\sigma_{128}\rho_1^3 e^{-\delta_1 i} + 2\sigma_{144}\rho_3^2\rho_1 e^{-\delta_1 i} = 0, \\
b)\quad & h_1' i\frac{\omega}{2}\rho_1 e^{\delta_1 i} - \sigma_{111}\rho_1 e^{-\delta_1 i}\rho_3 e^{\delta_3 i} - \frac{k_1^2}{2}\rho_1 e^{-\delta_1 i} - \omega\varepsilon_1\rho_1 e^{\delta_1 i} \\
& -3\sigma_{128}\rho_1^3 e^{\delta_1 i} - 2\sigma_{144}\rho_3^2\rho_1 e^{\delta_1 i} = 0, \\
c)\quad & -h_3' i\omega\rho_3 e^{-\delta_3 i} - \sigma_{37}\rho_1^2 e^{-2\delta_1 i} + \frac{h_3^*}{2} - \omega\varepsilon_3\rho_3 e^{-\delta_3 i} - 3\sigma_{330}\rho_3^3 e^{-\delta_3 i} \\
& -2\sigma_{335}\rho_1^2\rho_3 e^{-\delta_3 i} - \Gamma i e^{-\delta_3 i} = 0, \\
d)\quad & h_3' i\omega\rho_3 e^{\delta_3 i} - \sigma_{37}\rho_1^2 e^{2\delta_1 i} + \frac{h_3^*}{2} - \omega\varepsilon_3\rho_3 e^{\delta_3 i} - 3\sigma_{330}\rho_3^3 e^{\delta_3 i} \\
& -2\sigma_{335}\rho_1^2\rho_3 e^{\delta_3 i} + \Gamma i e^{\delta_3 i} = 0.
\end{aligned}
$$

After some calculations in system (10), we obtain the following expressions for the unknowns $\rho_1, \rho_3, \delta_1, \delta_3$:

$$(11)\quad \begin{aligned} &a)\ f_1(\rho_1,\rho_1)=0,\\ &b)\ f_2(\rho_1,\rho_1)=0,\\ &c)\ e^{(2\delta_1-\delta_3)i}+e^{(-2\delta_1+\delta_3)i}=\frac{1}{\sigma_{37}\sigma_{111}\rho_1^2\rho_3}\cdot\left[h_3^*A\sigma_{37}\rho_1^2-k_1^2(-CE+\frac{h_3^{*2}}{4}+\sigma_{37}^2\rho_1^4)\right],\\ &d)\ e^{2\delta_1 i}+e^{-2\delta_1 i}=\frac{2(-CE+\frac{h_3^{*2}}{4}+\sigma_{37}^2\rho_1^4)}{\sigma_{37}\rho_1^2h_3^*}, \end{aligned}$$

where: $A=-2(\omega\varepsilon_1+3\sigma_{128}\rho_1^2+2\sigma_{144}\rho_3^2)$,
$C=-[(h_3'-D)i\omega\rho_3+\omega\varepsilon_3\rho_3+3\sigma_{330}\rho_3^3+2\sigma_{335}\rho_1^2\rho_3+\Gamma i]$,
$E=(h_3'-D)i\omega\rho_3-\omega\varepsilon_3\rho_3-3\sigma_{330}\rho_3^3-2\sigma_{335}\rho_1^2\rho_3+\Gamma i$.

The first two equations (11*a*) and (11*b*) of the system give solutions for ρ_1 and ρ_3. In the generalized case it is not possible to obtain the solutions of this system of two equations in the radical form from the parameters of the given vibration system, because the equations are from 6^{th} degree for ρ_1 and ρ_3. Using Eqs. (11*a*) and (11*b*) we may express the values of the amplitudes ρ_1 and ρ_3 by means of a computer experiment, for the concrete system. If we have the amplitudes ρ_1 and ρ_3, the unknowns δ_1 and δ_3 can be obtained from (11c) and (11d). The solution (11) shows that when there exist simultaneously combined (parametrical, forced and self excited) vibrations, then the amplitudes ρ_1 and ρ_3 of the main parametrical and basic resonance are connected in a complicated manner and both of them depend on the amplitude h_3^* of the external force and parameters T, D of a mechanism of self excited vibrations. We multiply the first two equations (10a) and (10b) of the system (10) and we have the expression:

$$(12)\quad \rho_3^2\sigma_{111}^2+\left(\frac{k_1^2}{2}\right)^2+\sigma_{111}\rho_3\frac{k_1^2}{2}(e^{-\delta_3 i}+e^{\delta_3 i})=(\alpha_1+\rho_3^2b_1+\rho_1^2d_1)^2+\beta_1^2$$

where $d_1=3\sigma_{128}$, $\beta_1=\frac{1}{2}h_1'\omega$, $\alpha_1=\omega\varepsilon_1$, $b_1=2\sigma_{144}$.

From (12) the amplitude of vibration ρ_1^2 along the coordinate ξ can be expressed:

$$\rho_1^2=\frac{1}{d_1}\left[\pm\sqrt{\rho_3^2\sigma_{111}^2+\left(\frac{k_1^2}{2}\right)^2+\sigma_{111}\rho_3k_1^2\cos\delta_3-\beta_1^2}-\alpha_1-b_1\rho_3^2\right].$$

Equation (13) gives the amplitude ρ_1, depending on the amplitude ρ_3 along the coordinate ζ and the parameters of the system (including the parameters T and D

of self excited vibrations). It is shown, that the coefficient of resistance β_1 decreases the amplitude and the expression $\left(\frac{k_1^2}{2}\right)^2$ (obtained under the influence of the external parametrical excitation) acts to increase the amplitude ρ_1.

If in Eqs. (10c) and (10d) we take out the small expressions, containing ρ_1(without taking in mind the influence of the amplitude ρ_1 and ρ_3), we obtain the next equation for ρ_3:

$$(14)\quad \omega^2(\varepsilon_3^2 + (h_3' - D)^2)\rho_3^2 + 2\omega\rho_3(3\varepsilon_3\sigma_{330}\rho_3^3 + \Gamma(h_3' - D)) = \frac{h_3^{*2}}{4} - \Gamma^2 - 9\sigma_{330}^2\rho_3^6.$$

The equation (14) gives the possibility to express the function of frequency $\omega = \omega(\rho_3)$:

$$(15)\quad \omega_{1,2} = \frac{-3\varepsilon_3\sigma_{330}\rho_3^3 + \Gamma(h_3' - D)) \pm \sqrt{F}}{(\varepsilon_3^2 + (h_3' - D)^2)\rho_3},$$

where: $F = 6\varepsilon_3\sigma_{330}\rho_3^3\Gamma(h_3' - D) - 9(h_3' - D)^2\sigma_{330}^2\rho_3^6 - \varepsilon_3^2\Gamma^2 + \frac{h_3^{*2}}{4}(\varepsilon_3^2 + (h_3' - D)^2)$.
From (15), the condition for a real $\omega_{1,2}$ is:

$$(16)\quad 0 \le h_3^{*2}(\varepsilon_3^2 + (h_3' - D)^2) + 24\varepsilon_3\sigma_{330}\rho_3^3\Gamma(h_3' - D) - 36(h_3' - D)^2\sigma_{330}^2\rho_3^6 - 4\varepsilon_3^2\Gamma^2.$$

Condition (16) gives the possible amplitudes for a real ω, as:

$$(17)\quad (u - u_1)(u - u_2) \le 0,$$

where $u = \rho_3^3$, $u_{1,2} = \dfrac{2\varepsilon_3\Gamma \pm h^*\sqrt{\varepsilon_3^2 + (h_3' - D)^2}}{6(h_3' - D)\sigma_{330}}$.

If condition (16) is realized, the condition for positive values of frequency ω is:

$$(18)\quad 0 < -(3\varepsilon_3\sigma_{330}\rho_3^3 + \Gamma(h_3' - D)) \pm \sqrt{F}.$$

The conditions (16) and (18) give the different possibilities of realizing the vibration amplitudes along the axes $O\xi$ and $O\zeta$, depending on the stability of the movement and on the parameters of the system (including the parameters T and D of the mechanism of self excited vibrations).

3. Computer experiment

It is evident from the previous results that the system of differential equations (3), describing the movement of a body with a space elastic suspension and different ways of excitation, is very complicated for the purpose of investigation using analytical methods. The analytical solutions are obtained with a definite precision because different approximations are used. Therefore, to obtain greater precision and a full investigation on the movement, computers are used. A procedure (programme) for the solution of the system (3) is performed by using the method of TUTSIM.

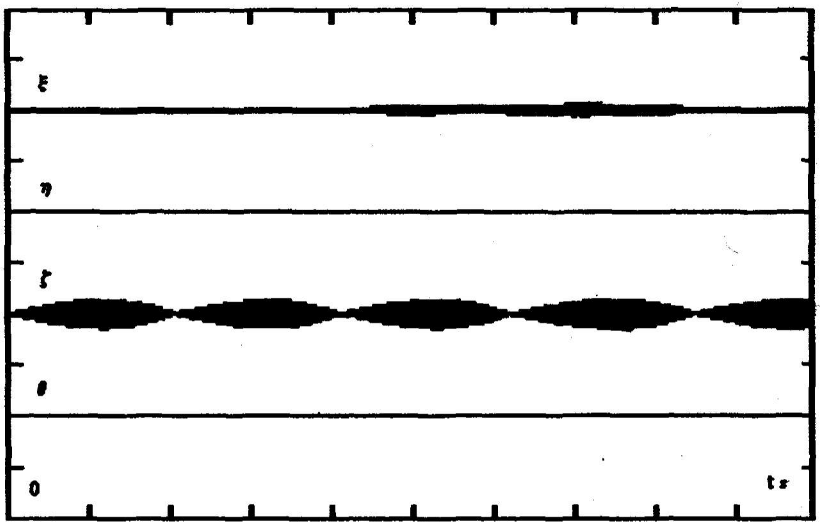

Fig. 2 $\mu = 0.04$, $\xi_0 = 0.00005$.

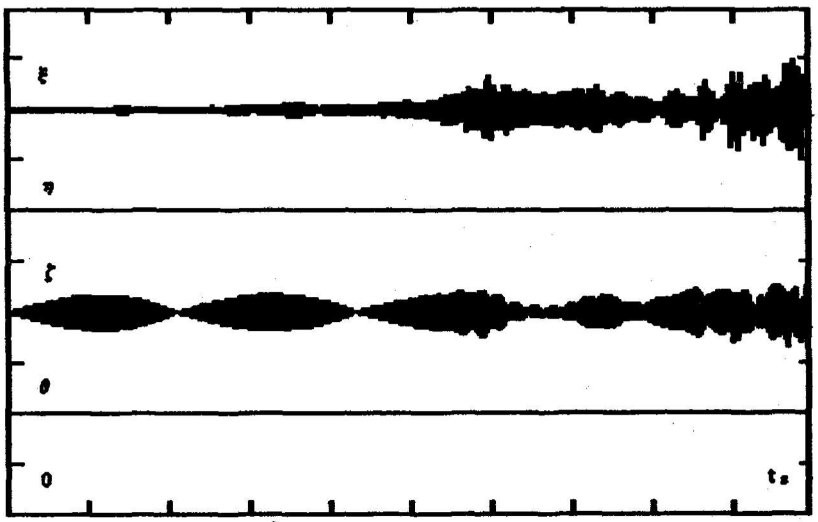

Fig. 3 $k_3^* = 0.47$

Movement is investigated numerically in cases of different resonance ratios in the system, for example the frequency ratios (5). In Fig. 2 the resonance regimes along the two coordinates ξ and ζ and zero initial conditions along the rest coordinates are presented ($\xi_0 = 0.00005$, $\mu = 0.04$). The development of a parametrical resonance is traced along the coordinate ξ and a basic resonance along the coordinate ζ, where is applied the mechanism of self excited vibrations. When the parameter h_3^* increases (for example, when $T = 0.1$, Fig. 3), the parametrical resonance regime along the axes 0ξ decreases and disappears.

4. Conclusions

The application of the model is, for example, in the case of the railway vehicle system. The mutual interaction with the railway generates parametrical, forced and self excited vibrations simultaneously. The resonance correlations are created in different regimes of movement, accidentally. The investigation of the dynamics of this model is a condition for the creation of active vibroisolation of transport vehicles [2].

This model provides a great variety of different vibrational regimes - stationary and transient resonance and nonresonance, periodical and nonperiodical, stable and unstable, with mutual influence between the coordinates.

The model realizes different forms of the resonance zones with transmission of energy from one coordinate to the other, exciting and declining of the resonance and others.

The great variety of the frequency correlations and the nonlinear links between the coordinates ensures many versions for the trajectories in space.

Movement (by frequency and by amplitude) is defined to a great degree by the initial conditions.

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