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AXISYMMETRIC PROBLEMS OF ELASTICITY FOR A TRANSVERSELY ISOTROPIC BODY WITH CRACKS

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The axisymmetric problem of elasticity for a transversely isotropic body, when the plane of isotropy is normal to the axis of revolution, is decomposed into two independent problems of axisymmetric torsion and axisymmetric deformation. In the case of an isotropic elastic body with arbitrary cracks both problems were studied earlier [1],[2]. It is known that the axisymmetric torsion problem of a homogeneous anisotropic body of revolution is reduced to the similar one of an isotropic body by means of a simple change of spatial variables [3]. The case of axisymmetric deformation of a transversely isotropic body requires a special consideration.

We assume that the distribution of stresses in a transversely isotropoc body of revolution is symmetrical about the axis of revolution. Let us introduce a cylindrical coordinate system (r, φ, z) with z-axis coinciding with the axis of the body. Then the problem of the axisymmetric deformation of such bodies is reduced to finding two components of the displacement vector u_r, u_z and four components of the stress tensor σ_{rr}, $\sigma_{\varphi\varphi}$, σ_{zz} and σ_{rz}. Let a system of axisymmetric cuts on smooth surfaces of revolution be located in an elastic medium. Denote by S a set of cut contours in the half-plane $\Pi = \{r \geq 0; -\infty < z < \infty\}$. Assuming that the displacements and stresses are discontinuous on the contour S, we write down the equilibrium equations in displacements in the generalized sense [2]

$$L_{\alpha\beta}u_\beta + X_\alpha = [t_\alpha]_S\delta_S + \Lambda^{+}_{\alpha\beta\gamma}(\partial_x)([u_\beta]_S n_\gamma \delta_S), \tag{1}$$

where

$$L_{rr} = A_{11}(\partial_r^2 + (1/r)\partial_r - 1/r^2) + A_{44}\partial_z^2,$$

$$L_{rz} = (A_{13} + A_{44})\partial_r\partial_z, \qquad L_{zr} = (A_{13} + A_{44})(\partial_r\partial_z + (1/r)\partial_z),$$

$$L_{zz} = A_{44}(\partial_r^2 + (1/r)\partial_r) + A_{33}\partial_z^2;$$

$$\Lambda^{\pm}_{rrr}(\partial_x) = A_{11}\partial_r \pm (A_{11} - A_{12})/r, \qquad \Lambda^{\pm}_{rrz}(\partial_x) = A_{44}\partial_z,$$

$$\Lambda^{\pm}_{rzr}(\partial_x) = A_{44}\partial_z, \qquad \Lambda^{\pm}_{rzz}(\partial_x) = A_{13}\partial_r, \qquad \Lambda^{\pm}_{zrr}(\partial_x) = A_{13}\partial_z,$$

$$\Lambda^{\pm}_{zrz}(\partial_x) = \Lambda^{\pm}_{zzr}(\partial_x) = A_{44}(\partial_r \pm 1/r), \qquad \Lambda^{\pm}_{zzz}(\partial_x) = A_{33}\partial_z;$$

$$t_\alpha(x) = \sigma_{\alpha\beta}(x)n_\beta(x) = \hat{T}_{\alpha\gamma}(n_x, \partial_x)u_\gamma(x),$$

$$\hat{T}_{rr}(n_x, \partial_x) = A_{11}n_r\partial_r + A_{44}n_z\partial_z + A_{12}n_r/r,$$

$$\hat{T}_{rz}(n_x, \partial_x) = A_{13}n_r\partial_z + A_{44}n_z\partial_r,$$

$$\hat{T}_{zr}(n_x, \partial_x) = A_{44}n_r\partial_z + A_{13}n_z\partial_r + A_{13}n_z/r,$$

$$\hat{T}_{zz}(n_x, \partial_x) = A_{44}n_r\partial_r + A_{33}n_z\partial_z.$$

$n_\beta(x) = \cos(n, x_\beta)$ is the direction cosine of the normal n to the contour S; $\partial_\beta = \partial/\partial x_\beta$, $x_r = r$, $x_z = z$; $[f]_S = f^+(\xi) - f^-(\xi)$, $\xi \in S$; the superscripts + and − indicate the limiting values of the function on the contour S; X_α are components of volume forces; A_{11}, A_{12}, ... are elastic constants [3]; δ_S is the surface delta function [2],[4]; the Greek subscripts take the value r and z.

Fundamental solutions $U_{\gamma\beta}(x, y)$ $(y = (r', z'))$ satisfying equations

$$L_{\alpha\beta}U_{\gamma\beta} = -\delta_{\alpha\gamma}\delta(x - y)$$

are expressed in terms of elliptic integrals [5]. Making use of them we obtain a solution of equations (1) in the form

$$(2) \quad u_\beta(x) = \int_\Pi X_\alpha(y)U_{\alpha\beta}(x, y)dy + \int_S ([u_\alpha(\eta)]_S T_{\alpha\beta}(x, \eta) - [t_\alpha(\eta)]_S U_{\alpha\beta}(x, \eta))dS_\eta$$

with

$$T_{\alpha\beta}(x, y) = T^-_{\gamma\alpha}(n_y, \partial_y)U_{\gamma\beta}(x, y), T^-_{\gamma\alpha}(n_y, \partial_y) = n_\beta(y)\Lambda^-_{\gamma\alpha\beta}(\partial_y).$$

Satisfying the boundary conditions on contour S for the displacements (the first basic problem)

$$u^{\pm}_\alpha(\xi) = v_\alpha(\xi) \pm \gamma_\alpha(\xi), \qquad \xi \in S$$

or the stresses

$$t^{\pm}_\alpha(\xi) = p_\alpha(\xi) \pm \mu_\alpha(\xi), \qquad \xi \in S$$

with the aid of the integral representation of displacements $u_\beta(x)$ (2), we arrive at two systems of boundary integral equations

$$2\int_S (\gamma_\alpha(\eta)T_{\alpha\beta}(\xi,\eta) - \mu_\alpha(\eta)U_{\alpha\beta}(\xi,\eta))dS_\eta =$$

$$= v_\beta(\xi) - \int_\Pi X_\alpha(y)U_{\alpha\beta}(\xi,y)dy, \quad \xi \in S; \tag{3}$$

$$2\int_S (\gamma_\alpha(\eta)S_{\alpha\beta}(\xi,\eta) - \mu_\alpha(\eta)D_{\alpha\beta}(\xi,\eta))dS_\eta =$$

$$= p_\beta(\xi) - \int_\Pi X_\alpha(y)D_{\alpha\beta}(\xi,y)dy, \quad \xi \in S \tag{4}$$

in the unknown jumps of stresses $2\mu_\alpha(\xi)$ (3) (the first basis problem) and of displacements $2\gamma_\alpha(\xi)$ (4) (the second basis problem). Here

$$S_{\alpha\beta}(x,y) = \hat{T}_{\beta\gamma}(n_x,\partial_x)T_{\alpha\gamma}(x,y),$$

$$D_{\alpha\beta}(x,y) = \hat{T}_{\beta\gamma}(n_x,\partial_x)U_{\alpha\gamma}(x,y).$$

In the case of a system of flat coplanar cracks in the plane $z = 0$ the systems (3) and (4) are decomposed into four independent equations which can be written in the form

$$\int_S F_n(r')r'dr' \int_0^\infty J_n(tr')J_n(tr)dt = f_n(r), \qquad r \in S; \tag{5}$$

$$\int_S G_n(r')r'dr' \int_0^\infty t^2 J_n(tr')J_n(tr)dt = g_n(r), \qquad r \in S; \tag{6}$$

where $n = 0,1$. Here $F_n(r)$ and $G_n(r)$ are unknown functions; $f_n(r)$ and $g_n(r)$ are specified functions; $J_n(t)$ is the Bessel function of order n. When the S is an internal $(0 \leq r \leq R)$ or external $(R \leq r < \infty)$ circular region the equations (5) and (6) are solved in closed form for arbitrary $n \geq 0$ [5]. In the case of a penny-shaped cut we have

$$F_n(r) = -\frac{2r^{(n-1)}}{\pi}\frac{d}{dr}\int_r^R \frac{xdx}{x^{2n}\sqrt{x^2-r^2}}\frac{d}{dx}\int_0^x \frac{t^{(n+1)}f_n(t)dt}{\sqrt{x^2-t^2}};$$

$$G_n(r) = \frac{2r^n}{\pi} \int\limits_r^R \frac{dt}{t^{2n}\sqrt{t^2 - r^2}} \int\limits_0^t \frac{x^{n+1} g_n(x)dx}{\sqrt{t^2 - x^2}} \tag{7}$$

with $G_n(r)$ satisfying the condition $G_n(R) = 0$.

The last equation of (7) provides the possibility to write the closed-form expressions of the stress intensity factors $K_{\rm I}$ and $K_{\rm II}$ under the action of volume forces in a solid and arbitrary nonself-equilibrium tractions on faces of the penny-shaped crack. When the volume forces are absent ($X_r(x) = X_z(x) = 0$) these expressions take the form

$$K_{\rm I} = -\frac{2}{\sqrt{\pi R}} \int\limits_0^R \frac{r p_z(r)dr}{\sqrt{R^2 - r^2}},$$

$$K_{\rm II} = -\frac{2}{R\sqrt{\pi R}} \Big(\int\limits_0^R \frac{r^2 p_r(r)dr}{\sqrt{R^2 - r^2}} + b_1 \int\limits_0^R r\mu_z(r)dr \Big)$$

with

$$b_1 = (\pi/2)(A_{11}/A_{33} - c^2)/\sqrt{2(a^2 + c^2)},$$

$$a^2 = (A_{11}A_{13} - 2A_{13}A_{44} - A_{13}^2)/(2A_{33}A_{44}), \qquad c^4 = A_{11}/A_{33}.$$

The case of an external circular crack ($S = \{r \geq R;\ z = 0\}$) can be considered in a similar manner [5]. The corresponding expressions of stress intensity factors are

$$K_{\rm I} = -\frac{2}{\sqrt{\pi R}} \Big(\int\limits_R^\infty \frac{r p_z(r)dr}{\sqrt{r^2 - R^2}} + b_2 \int\limits_R^\infty \mu_r(r)dr \Big),$$

$$K_{\rm II} = -2\sqrt{(R/\pi)} \int\limits_R^\infty \frac{p_r(r)dr}{\sqrt{r^2 - R^2}} \tag{8}$$

with

$$b_2 = 2A_{13}(1/A_{44} - 1/A_{33} - A_{13}/(A_{33}A_{44}) + 1/(A_{33}c^2))/\sqrt{2(a^2 + c^2)}.$$

It should be noted that the solution (8) is obtained under the condition that the displacements are equal to zero at infinity. To avoid such a restriction one should carry out an additional analysis, similar to the isotropic case [6].

We have assumed above that the contour S is open. However the obtained integral representation of the solution (2) of the differential equations (1) remain valid also if S is a closed contour (or an open contour with its origin and end at the axis of revolution) or a set of such contours. Therefore the integral representation constructed above can be used for reducing the boundary value problems for a region G bounded by the contour S to boundary integral equations. Then we carry out an analytic continuation of the solution to the region $G_1 = \Pi \setminus G (G_1$ being the completion of G to the half-plane Π) in such a way that in crossing the contour S, the displacements remain continuous (the first basic problem), or the stresses remain continuous (the second basic problem). As a result we obtain integral equations of the first kind over a closed contour S. Such an approach has been realized, in particular, in two-dimensional problems of elasticity theory [7].

In the theory of boundary integral equations it is common to use integral representation of the general solution of the equilibrium equations in a somewhat different form, when the displacement of internal points is expressed in terms of the boundary values of displacements and stresses. Such integral representations can be easily obtained from the above relationship (2) by assuming that S is a closed surface that bounds the region G, and that the sought function is equal to zero outside G. By assuming that n is the outer normal to the contour S, we obtain

$$[u_\alpha]_S = -u_\alpha^-(\xi) = -u_\alpha(\xi), [t_\alpha]_S = -t_\alpha^-(\xi) = -t_\alpha(\xi), \xi \in S.$$

Now it follows from formula (2)

$$\Delta(x)u_\beta(x) = \int_G X_\alpha(y)U_{\alpha\beta}(x,y)dy+$$

$$+\int_S (t_\alpha(\eta)U_{\alpha\beta}(x,\eta) - u_\alpha(\eta)T_{\alpha\beta}(x,\eta))dS_\eta, \tag{9}$$

where $\Delta(x) = 1$ for $x \in G$, and $\Delta(x) = 0$ for $x \notin (G \cup S)$. The formula (9) is the analogue of Somigliana integral identity for axisymmetric deformation of a transversely isotropic body. Proceeding to the limit as $x \to \xi (x \in G, \xi \in S)$ in (9) we obtain the boundary integral equations having the same form for the first and the second basic problems of axisymmetric deformation of a bounded body of revolution.

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