Journal of Theoretical and Applied Mechanics, Sofia, 2011, vol. 41, No. 1, pp. 3–18

GENERAL MECHANICS

DYNAMICS OF MECHANISMS IN OVERCONSTRAINED AND SINGULAR CONFIGURATIONS*

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[Received 22 October 2009. Accepted 28 February 2011]

Abstract. The present paper deals with the problems of dynamics simulation of rigid and flexible mechanisms with closed chains falling into singular and self-locking configurations. Dependent constrained equations and singularity of the mass-matrix of the dynamics equations are discussed. A novel approach to decomposition of the closed chains into open branches is applied. The difference to the conventional methods using such decomposition consists in substitution of the constrained equations by stiff forces. The elasticity of certain mechanism links is taken into account for precise dynamics simulation of the singular configurations. The elasticity is modelled using finite element discretization. Generalized Newton-Euler dynamic equations with respect to the quasi-velocities and acceleration of the rigid bodies and flexible nodes are applied. Damping is considered for eliminating of undesirable high level oscillations of the stiff forces in the elastic links. Friction forces in the bearings for self-locking configurations are taken into account. Examples of dynamic analysis of mechanisms in overconstrained and singular configurations are presented.
Key words: dynamics, overconstrained mechanisms, singular configurations, flexibility.

1. Introduction

The theory of the constrained dynamics is very well developed in the mechanics and widely applied in multibody system motion simulation [1–5].

*The authors acknowledge the financial support of Ministry of Education, Youth and Science, contract No. DUNC-01/3, 12.03.2009.

The constraints imposed on the systems have different nature and type. Most often in the mechanism theory kinematic and force constraints are regarded. Joints connecting rigid bodies impose kinematic constraints. On the other hand, if one regards the elastic nature of the contact between the bodies the same joints impose force constraints. Closed chains in the kinematic scheme impose kinematic constraints that additionally diminish the system degree of freedom (dof). Furthermore, the notation "common constraints" is used for systems that have common restrictions imposed on the motion of the links. Such systems are: the plane mechanisms (restrictions for motion in the plane); spherical mechanisms (the joint axes are crossing in a common point); the mechanism of Bennett [6], etc.

The dynamics of constraint system is presented by Differential Algebraic Equations (DAE). Surveys of the existing techniques for solution of DAE may be found in [4, 7]. The classical method to deal with DAE is to express the constraint condition at acceleration level. This leads to replacement of the original system by a system of Ordinary Differential Equations (ODE). Maintaining the acceleration constraints one does not satisfy the position and velocity constraints. Baumgarte's stabilization term [8] is introduced to ensure exponential convergence of the constraint error to zero. The problem with this method is in selection of high gains to keep small constraint errors. A similar approach based on penalty functions is presented in [4, 9]. Implementation of both methods [4, 8, 9] results in inclusion additional terms in the right side of the dynamic equations that could be treated as reaction forces in the joints cut but cannot be compared to the elastic forces in links.

Another group of researchers [10, 11, 12] proposed projection techniques to maintain the constraint conditions without modification of the equations of motion.

A very efficient method, broadly applied in practice, is based on coordinate partitioning [13]. At every step the set of the coordinates is partitioned of dependent and independent coordinates. However, a fixed set of independent coordinates may lead to singular matrix of the derivatives of the constraints [4, 11].

Manipulation of the dynamic equations in the singular configurations is a challenging realm of the investigations. The Augmented Lagrangian formulation proposed in [14, 15] can handle redundant constraints and singular configurations. In [16] an approach for kinematic analysis of mechanisms and their singular configurations using the Moore-Penrose pseudo-inverse matrix is applied. Eich-Soellner and Fuhrer [17] solved the problem of constraint stabilization using optimization algorithms and the pseudo-inverse matrix so derived. In [18] a projection method is applied for simulation of constrained multibody systems. Mechanisms in the vicinity of singular configurations are regarded. Friction is taken into account. In [19] a pseudo-inverse matrix is proposed for effective solution of DAE and its application in singular configurations. However, special singular configurations exist for which there is no general solution and special methods are to be developed taking into account the elasticity of the links.

In the paper an approach to dynamic analysis of multibody systems that provides general solution in singular configurations and self-locking position is proposed. Elasticity of the links is taken into account. Generalized Newton-Euler dynamic equations are applied for rigid bodies and finite element discretization of flexible links. Relative and absolute nodal coordinate formulation is used. The closed kinematic chains are decomposed into open chains substituting the kinematic constraints by stiff forces due to the elasticity of certain links. The open branches are regarded separately and loaded by virtual external forces that are identical to the stiff forces in the link cut. The stiff forces are updated for every configuration of the system. Damping is considered to eliminate the high level oscillations of undesirable stiff forces of the elastic links. Friction in the bearings is taken into account. Special considerations for determination of the friction forces are applied since some or all of the mechanism links in a singular configuration are immobile. Several examples of mechanisms with closed chains in singular configurations and self-locking positions are discussed. The method proposed is applicable for rigid body systems since in singular configurations there are no reliable methods for predicting of the real motion. After overcoming of these configurations conventional methods for rigid body dynamics simulation could be used.

The work presented proposes a method for precise kinematic and dynamic analysis of multibody systems in singular configurations.

2. Topology and kinematics of overconstrained mechanisms

Over-constrained mechanisms are systems which dof are less than degrees of mobility. For example, if one analyze a plain mechanism considering the regulations for the spatial mechanism he will obtain negative dof, while it is quite applicable. So, plane mechanisms (Fig. 1(a)) could be also considered overconstrained. Spherical mechanisms (Fig. 1(b)) are also overconstrained and it could be easily observed that if the directions of the axes do not cross each other in a common point the mechanism is unmovable. The length of the links and axes orientation of the Bennett's mechanism also possesses common restrictions shown in figure Fig. 1(c), where the numbers point out the axes, the symbols r and d denote the length of the links and with α and β the angles between the adjacent axes are pointed out. The symbols r and d denote the length of the links that should be orthogonal to their axes, and with these conditions the mechanism is movable.

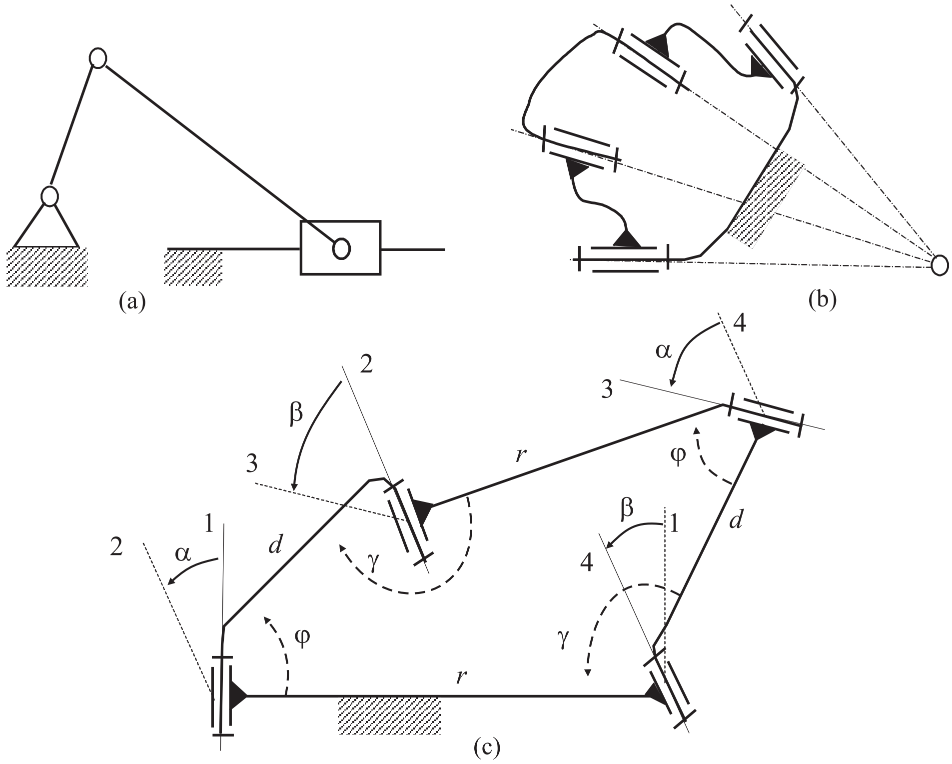

Fig. 1. Linkages regarded as overconstrained mechanisms:
(a) plane; (b) spherical; (c) Bennett's

On the other hand, many mechanisms with closed chain are movable, while the topology analysis shows zero or negative dof. This is because of the kinematic parameters for which the constraints equations are dependent. The simplest examples of such plane mechanisms are shown in Fig. 2. For some mechanisms the proportion between the shape and size of the links is the reason for the increase of mechanism dof in specific positions, called singular configurations. Examples of mechanisms with closed chains in singular configuration for some basic groups of the Assur's classification are presented in Fig. 3.

The different nature and behaviour of the overconstrained mechanisms is the reason for development of specific approaches for the kinematics and dynamic analysis, and simulation for almost every particular case. Even for

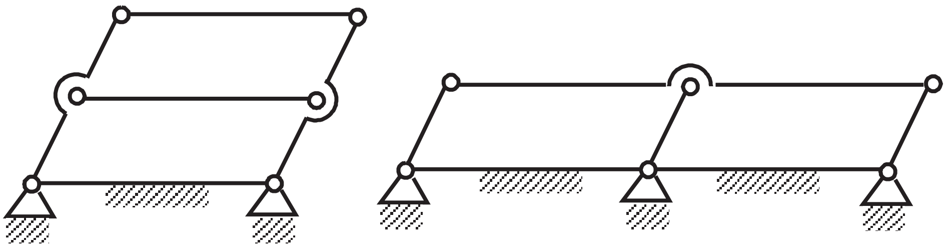

Fig. 2. Plane mechanisms with dependent constraints

a simple mechanism it could happen that different approaches are to be applied during its motion and configurations. That significantly diminishes the effectiveness of the computations. But only for few cases such geometrical considerations depicted in Fig. 3 could be regarded. For more complex plane and space mechanisms as parallel robots and simulation platforms simple regulations cannot be discovered. For the case of singular configurations it is well know that the matrix of the derivatives of the constraint equation system (Jacobean matrix) is singular. This analysis is an onerous task, causes additional branches of the algorithms and most often cannot be implemented effectively in the vicinity of singularities.

Discretization of the continuum should be implemented for numerical

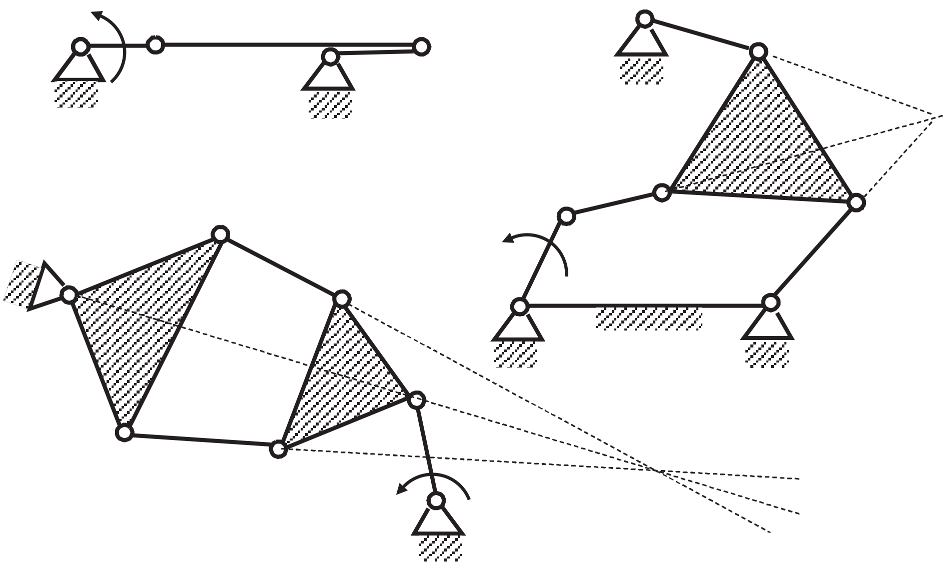

Fig. 3. Singular configurations for some basic groups of the Assur's classification

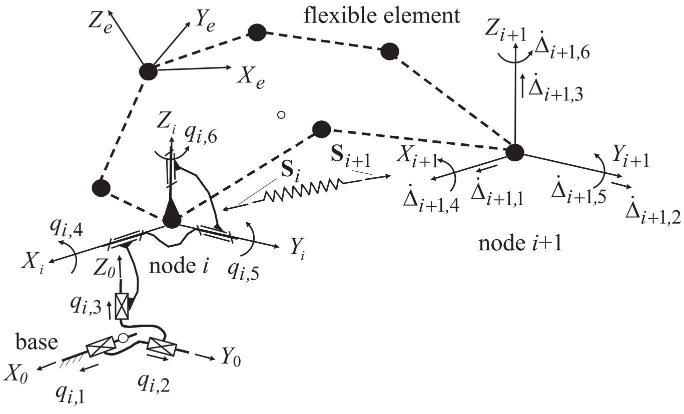

Fig. 4. Node coordinate systems and their motion characteristics

simulation of a flexible system and mass and stiffness properties of the flexible bodies are to be reduced to a finite number of points called nodes. A flexible element with many nodes is shown in Fig. 4, where for the node with index i the coordinates (using translations and Euler angles) of the node coordinate system relative to the absolute reference frame are pointed out, while for the second node $(i+1)$ the linear and angular velocities of the node are shown. ***The node of a flexible element is a free object*** that, in the three dimensional space, possesses six degrees of freedom [20]. The node motion is restricted by the stiff forces acting between the neighbour nodes (S_i and S_{i+1} for nodes i and $i+1$). The coordinates of node i are stored in a 6×1 matrix $\mathbf{q_i} = [q_{i,1}\ q_{i,2}\ \cdots\ q_{i,6}]^{\backslash}$, where the notations $q_{i,m}$, $m = 1, 2, \ldots, 6$ are the elements of the matrix. "\" (backslash) denotes matrix transpose. The small translations and rotations of node i, $\Delta_i = \left[\mathbf{s}_i^{\backslash}\ \theta_i^{\backslash}\right]^{\backslash} = [\Delta_{i,1}\ \Delta_{i,2}\ \cdots\ \Delta_{i,6}]^{\backslash}$, are compiled in a similar matrix with elements $\Delta_{i,m}$, $m = 1, 2, \ldots, 6$. ***The kinematic characteristics of motions of the nodes are mutually independent, which means that*** $\dfrac{\partial \mathbf{q}_i}{\partial \mathbf{q}_j} = {}^{6,6}\mathbf{0}$, $i \neq j$. The left superscripts denote matrix dimension, for example: ${}^{i}\mathbf{A}$, ${}^{i,j}\mathbf{A}$, ${}^{i,j,k}\mathbf{A}$ are $i \times 1$, $i \times j$ and $i \times j \times k$ matrix–vector, two and three dimensional matrices respectively, ${}^{6,6}\mathbf{0}$ is 6×6 zero matrix. If the matrix dimension notation is once defined in the text it could be missed further down. Since the node of a space flexible element has six degrees of freedom either with respect to the element coordinate system $(X_eY_eZ_e)$ or to the absolute $(X_0Y_0Z_0)$ one and its motion could be presented by virtual spatial joint with six dof as described in detail in [20], but it should be made clear difference between the coordinates $\mathbf{q}_i$ and the small possible deflections Δ_i. The definition of the nodes as coordinate systems allows the flexible particles of the multibody systems, similarly to the systems of rigid bodies, to be decomposed into a set of moving coordinate systems connected by joints.

All (of number a) coordinates of the system are compiled in matrix ${}^{a}\mathbf{Q}$. The coordinates $\mathbf{Q}$ are subject to constraints that define the function of $\mathbf{Q}$ with respect to generalized (of number g) coordinates ${}^{g}\mathbf{q}$.

The system is subject to d constraints ($d = a - g$), i.e.:

$$(1) \qquad {}^{d}\mathbf{\Phi} = \mathbf{\Phi} = \mathbf{\Phi}(\mathbf{Q}) = {}^{d}\mathbf{0}$$

The time derivatives of Eq. 1 in the configuration space are as follows:

$$(2) \qquad \dot{\mathbf{\Phi}} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{Q}} \cdot \dot{\mathbf{Q}} = {}_{\partial \mathbf{Q}}\mathbf{\Phi} \cdot \dot{\mathbf{Q}} = {}^{d}\mathbf{0}$$

$$(3) \qquad \ddot{\mathbf{\Phi}} = {}_{\partial \mathbf{Q}}\mathbf{\Phi} \cdot \ddot{\mathbf{Q}} + {}_{\partial^2 \mathbf{Q}}\mathbf{\Phi} \underset{1\backslash 3}{\otimes} \dot{\mathbf{Q}} \cdot \dot{\mathbf{Q}} = {}^{d}\mathbf{0}$$

where ${}_{\partial \mathbf{Q}}\mathbf{\Phi} = {}^{d,a}_{\partial \mathbf{Q}}\mathbf{\Phi}$ and ${}_{\partial^2 \mathbf{Q}}\mathbf{\Phi} = {}^{d,a,a}_{\partial^2 \mathbf{Q}}\mathbf{\Phi}$ are matrices of the first and second order partial derivatives (the left subscripts denote the differentiating variables). Notation "$\underset{1\backslash 3}{\otimes}$" presents matrix multiplication of three dimensional matrix and the numbers point out the indices of the corresponding elements [20]. Eq. 1 defines two sets of dependent ${}^{d}\underline{\mathbf{Q}}$ and independent ${}^{g}\mathbf{q}$ coordinates, i.e., $\mathbf{Q} = \left[\underline{\mathbf{Q}}^{\backslash}\ \mathbf{q}^{\backslash}\right]^{\backslash}$. The term ${}_{\partial^2 \mathbf{Q}}\mathbf{\Phi} \underset{1\backslash 3}{\otimes} \dot{\mathbf{Q}} \cdot \dot{\mathbf{Q}}$ in Eq. 3 written in the configuration space is the well know term $\dot{\mathbf{\Phi}} \cdot \dot{\mathbf{Q}}$ in the space of the velocities.

The velocity equation, Eq. 2, could be transformed to [4]:

$$(4) \qquad \dot{\mathbf{Q}} = \mathbf{R} \cdot \dot{\mathbf{q}},$$

and the time derivatives of Eq. 4 are as follows:

$$(5) \qquad \ddot{\mathbf{Q}} = \mathbf{R} \cdot \ddot{\mathbf{q}} + {}_{\partial \mathbf{q}}\mathbf{R} \underset{1\backslash 3}{\otimes} \dot{\mathbf{q}} \cdot \dot{\mathbf{q}}$$

The matrices $\mathbf{R}$ and ${}_{\partial \mathbf{q}}\mathbf{R}$ are computed from the partial derivatives ${}_{\partial \mathbf{Q}}\mathbf{\Phi}$ and ${}_{\partial^2 \mathbf{Q}}\mathbf{\Phi}$ as it is presented in [20]. Most often, the equality constraints represent the connectivity of the links in closed chains. The equation constraints of closed chains contain dependent coordinates that is the reason for the singularity of the Jacobean matrix. This singularity could be observed in certain system configurations. Equation constraints of open branches could be directly transformed with respect to the independent coordinates. This approach is widely applied in constraint dynamics [1–4] and consists in virtual disconnections of joints that pertain to a closed chain. The entire system of constraint equations is then compiled including additional kinematic constraints that present the connectivity in the joints cut.

The principle proposed in the paper consists in transformation of a closed chain into open branch cutting not the joints but some of the links assuming them flexible. Application of the finite element approach allows the kinematic constraints to be substituted by force constraints, i.e. by stiff forces in the nodes. Illustration of this approach applied for the mechanisms in Fig. 3 is presented in Fig. 5. Such transformation of a six-link mechanism with three closed chains depicted in Fig. 5 results in an open chain with four branches and eight links. The stiff forces S_1 and S_2 in Fig. 5 depend on the shape, size and stiffness of the links cut. The first step is the transformation of the closed chain into open branches and substitution of the missing links by stiff forces. These forces represent the connectivity between the coordinate systems of the nodes of the flexible elements. For example, for the four-bar mechanism in Fig. 3 and its transformation (Fig. 5) the orientation of the link coordinate systems and of the nodes of the flexible shuttle are shown in Fig. 6. The coordinate systems of the flexible nodes 2' and 2" of link 2 are $X_2'Y_2'$ and $X_2''Y_2''$, respectively. The resultant two open chains have degree of freedom four and the stiff force S_2 keeps both nodes 2' and 2" in a common link that prevents the open branches so obtain to implement arbitrary motion. The kinematic analysis of these two branches is a trivial task for every computer

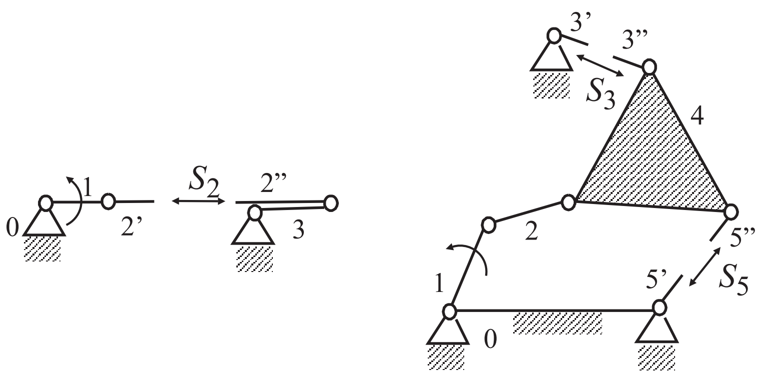

Fig. 5. Transformation of a closed chain into open branches substituting flexile links by elastic forces

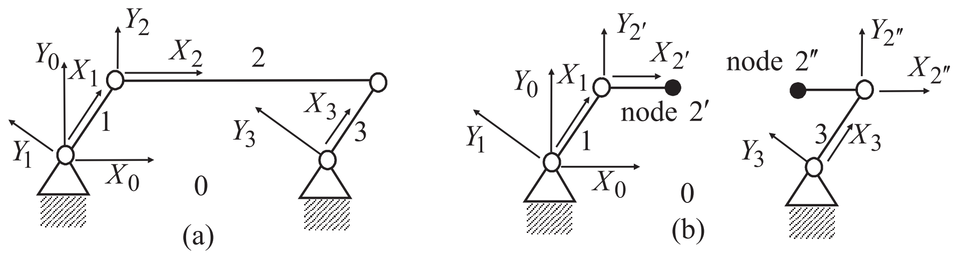

Fig. 6. Orientation of the link coordinate systems:
(a) rigid four-bar mechanism; (b) rigid input and output shafts and flexible shuttle

code generation program. For the next stage, the dynamic analysis, the main activities will consist in mass distribution of the flexible elements to the node coordinate system (estimation of the mass matrix), as well as, in estimation of the stiff forces (the stiffness matrix). No other method will be effective and the elasticity of a link, i.e., the method proposed here, should be applied if one tries to analyze the motion of the four-bar mechanism in this singular configuration with zero initial velocities.

3. Dynamics of rigid and flexible mechanisms

The application of the method proposed requires specific approach to dynamics of the decomposed system. As it was said above, the nodes of the flexible elements are kinematically independent but the force distribution has quite different nature. For example, accelerations in one of the nodes exaggerate inertia forces not only in this node but in the adjacent to it. The same could be concluded for the deflections that cause the stiff and damping forces. In the paper generalized Newton-Euler dynamic equations [20] for rigid and flexible bodies are applied, as well as, a method of finite elements in relative coordinates [23].

3.1. Inertia forces in rigid bodies and flexible elements

The vector of the small translation $\mathbf{s}_{C_i}$ of an object (point or node C_i of a body or a node of flexible element), and the vector of the small rotations θ_i compile the matrix of the small displacements $\Delta_{C_i} = \left[\mathbf{s}^{\backslash}_{C_i}\ \theta^{\backslash}_i\right]^{\backslash}$. The coordinate transformation matrix, ${}^{6,6}\boldsymbol{\tau}\Delta_i$, for vector Δ_i is:

$$\boldsymbol{\tau}\Delta_i(\mathbf{q}) = \begin{bmatrix} \tau_i & {}^{3,3}\mathbf{0} \\ {}^{3,3}\mathbf{0} & \tau_i \end{bmatrix}, \tag{6}$$

where ${}^{3,3}\boldsymbol{\tau}_i$ is the matrix transformation of the coordinate system i. Similarly to Δ_i and Eqs. 4, 5, the quasi velocities and accelerations $\dot{\Delta}_i$, $\ddot{\Delta}_i$ are expressed with respect to $\mathbf{q}$, as well as, $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ [20], i.e.:

$$\dot{\Delta}_i(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} \mathrm{v}_{C_i} \\ \omega_i \end{bmatrix} = \mathbf{R}\Delta_i(\mathbf{q}) \cdot \dot{\mathbf{q}}, \tag{7}$$

$$\ddot{\Delta}_i(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \begin{bmatrix} \dot{\mathbf{v}}_{C_i} \\ \dot{\omega}_i \end{bmatrix} = \mathbf{R}\Delta_i(\mathbf{q}) \cdot \ddot{\mathbf{q}} + {}_{\partial\mathbf{q}}\mathbf{R}\Delta_i(\mathbf{q}) \underset{1\backslash 3}{\otimes} \dot{\mathbf{q}} \cdot \dot{\mathbf{q}}, \tag{8}$$

where $\mathbf{R}\Delta_i$ and ${}_{\partial\mathbf{q}}\mathbf{R}\Delta_i$ are compiled from the firs and second order partial derivatives of Δ_i with respect to $\mathbf{q}$ [20].

For the rigid body i the velocities that define its motion are stored in 6×1 matrix $\dot{\Delta}_{C_i} = \begin{bmatrix} \mathbf{v}^{\backslash}_{C_i} & \omega^{\backslash}_i \end{bmatrix}^{\backslash}$ of the velocity of its centre of gravity C_i and the body angular velocity. For a flexible element with n nodes the $6.n \times 1$ matrix of the velocities is $\dot{\Delta}_i = \begin{bmatrix} \dot{\Delta}^{\backslash}_{i,1} & \dot{\Delta}^{\backslash}_{i,2} & \cdots & \dot{\Delta}^{\backslash}_{i,n} \end{bmatrix}^{\backslash}$, where $\dot{\Delta}_{i,j} = \begin{bmatrix} \mathbf{v}^{\backslash}_{i,j} & \omega^{\backslash}_{i,j} \end{bmatrix}^{\backslash}$, $j = 1, 2, \ldots, n$ are 6×1 matrices of the node velocities. The Newton-Euler equations define the inertia forces and moments loading the body. The 6×1 matrix of the inertia forces and moments for rigid body i in the centre of gravity and relative to the inertial frame are:

$$\mathbf{F}_{C_i} = \mathbf{M}_{C_i} \cdot \ddot{\Delta}_{C_i} + \begin{bmatrix} {}^{3,3}\mathbf{0} & {}^{3,3}\mathbf{0} \\ {}^{3,3}\mathbf{0} & \omega_i^{\times} \end{bmatrix} \cdot \mathrm{M}_{C_i} \cdot \begin{bmatrix} {}^{3}\mathbf{0} \\ \omega_i \end{bmatrix} = \mathbf{M}_{C_i} \cdot \ddot{\Delta}_{C_i} + \underline{\mathbf{F}}_{C_i}(\dot{\mathbf{q}}, \omega_i), \tag{9}$$

where

$$\mathbf{M}_{C_i} = \begin{bmatrix} \mathrm{diag}(m_i, m_i, m_i) & {}^{3,3}\mathbf{0} \\ {}^{3,3}\mathbf{0} & \tau_i \cdot \underline{\mathbf{J}}_i \cdot \tau_i^{\backslash} \end{bmatrix}, \tag{10}$$

and m_i, $\underline{\mathbf{J}}_i$ are the body mass and inertia tensor, respectively. The underlined notations point out reference to body fixed coordinate system.

The mass matrix of a flexible element i with n nodes is $6.n \times 6.n$ symmetric positive defined dense matrix $\underline{\mathbf{M}}_i$. The mass matrices are computed assuming the equivalence of the kinetic energy of the deformable particles to the energy of the masses reduced to the nodes. These matrices are considered constant if small relative deflections within the elements are assumed. $\mathbf{M}_i$ is the $6.n \times 6.n$ element mass matrix relative to inertial reference frame. The inertia forces ($6.n \times 1$ matrix $\mathbf{F}_i$) in the nodes are defined as follows [20]:

$$\mathbf{F}_i = \mathbf{M}_i \cdot \ddot{\Delta}_i + \dot{\Delta}_i^{\otimes} \cdot \mathbf{M}_i \cdot \dot{\Delta}_i - \mathbf{M}_i \cdot \dot{\boldsymbol{\Theta}}_i = \mathbf{M}_i \cdot \ddot{\Delta}_i + \underline{\mathbf{F}}_i(\dot{\mathbf{q}}, \dot{\Delta}_i), \tag{11}$$

where, $\dot{\Delta}_i^{\otimes} = \mathrm{diag}\left(\dot{\Delta}_{i,1}^{\otimes}, \dot{\Delta}_{i,2}^{\otimes}, \cdots, \dot{\Delta}_{i,n}^{\otimes}\right)$; $\dot{\Delta}_{i,j}^{\otimes} = \begin{bmatrix} \omega_{i,j}^{\times} & {}^{3,3}\mathbf{0} \\ \mathbf{v}_{i,j}^{\times} & \omega_{i,j}^{\times} \end{bmatrix}$ is generalized skew-symmetric matrix of the linear and angular velocities of node j from element i; $\dot{\mathbf{\Theta}}_i = \begin{bmatrix} \dot{\mathbf{\Theta}}_{i,1}^{\setminus} & \dot{\mathbf{\Theta}}_{i,2}^{\setminus} & \cdots & \dot{\mathbf{\Theta}}_{i,n}^{\setminus} \end{bmatrix}^{\setminus}$; $\dot{\mathbf{\Theta}}_{i,j} = \begin{bmatrix} \omega_{i,j}^{\times} \cdot \mathbf{v}_{i,j} \\ {}^{3}\mathbf{0} \end{bmatrix}$.

3.2. Stiff forces in flexible elements

The nodes of flexible elements of moving multibody systems achieve large displacements with respect to the inertial reference frame, while the deflections relative to the element coordinate system are small. In order the finite element methodology [21, 22] to be accurately applied the stiff forces loading the nodes are to be computed using only the small relative deflections of the nodes with respect to the element coordinate system. The small relative deflections of an element i with n nodes are compiled, similarly to the velocities in Sec. 3.1, in a $6.n \times 1$ matrix $\underline{\Delta}_i = \begin{bmatrix} \underline{\Delta}_{i,1}^{\setminus} & \underline{\Delta}_{i,2}^{\setminus} & \cdots & \underline{\Delta}_{i,n}^{\setminus} \end{bmatrix}^{\setminus}$. The element stiffness properties are presented by $6.n \times 6.n$ sized stiffness matrix $\underline{\mathbf{K}}_i$. The matrix $\underline{\mathbf{K}}_i$ of the finite element is to be transformed to the absolute reference frame to compile the global stiffness matrix $\mathbf{K}_i$. The stiff forces $\mathbf{S}_i = \begin{bmatrix} \mathbf{S}_{i,1}^{\setminus} & \mathbf{S}_{i,2}^{\setminus} & \cdots & \mathbf{S}_{i,n}^{\setminus} \end{bmatrix}^{\setminus}$ with respect to the absolute reference frame are computed using the relation $\mathbf{S}_i = -\mathbf{K}_i \cdot \Delta_i$. The stiff forces relative to the coordinate system of a moving element are calculated taking into account the regulations for selection of the reference coordinate systems of the flexible elements [23]. In this case, the element coordinate system is located in a node numbered with index 1. For example, the stiff forces relative to the beam element coordinate system are calculated using the relation:

$$\underline{\mathbf{S}}_i = -\underline{\mathbf{K}}_i \cdot \underline{\Delta}_i = -\underline{\mathbf{K}}_i \cdot \begin{bmatrix} {}^{1,6}\mathbf{0} & \underline{\Delta}_{i,2}^{\setminus} & \cdots & \underline{\Delta}_{i,n}^{\setminus} \end{bmatrix}^{\setminus}. \tag{12}$$

It could be seen that the relative deflections of the node with index 1 relative to the local (of the element) coordinate system are zero.

3.3. Damping in flexible elements

Similarly to the stiff forces, the damping forces loading the nodes $\mathbf{D}_i = \begin{bmatrix} \mathbf{D}_{i,1}^{\setminus} & \mathbf{D}_{i,2}^{\setminus} & \cdots & \mathbf{D}_{i,n}^{\setminus} \end{bmatrix}^{\setminus}$ with respect to the absolute reference frame are computed using the relation $\mathbf{D}_i = -\mathbf{C}_i \cdot \dot{\Delta}_i$, where matrix $\mathbf{C}_i$ is the damping matrix [21, 22] of the element i. The damping forces, loading the nodes with respect to the element coordinate system are calculated using the relation:

$$\underline{\mathbf{D}}_i = -\underline{\mathbf{C}}_i \cdot \underline{\dot{\Delta}}_i = -\underline{\mathbf{C}}_i \cdot \begin{bmatrix} {}^{1,6}\mathbf{0} & \underline{\dot{\Delta}}_{i,2}^{\setminus} & \cdots & \underline{\dot{\Delta}}_{i,n}^{\setminus} \end{bmatrix}^{\setminus}. \tag{13}$$

Damping presented in the paper is of great importance for application of the method proposed, as well as, for simulation of reliable motion. The stiff forces that appear as a result of the mutual penetration of the contact surfaces are very high and cause abrupt changes of the values and signs of these forces, as it is well know form the examples of contact simulation. The same phenomenon could be observed using the approach to substitution a flexible link by the stiff forces. In this case, the mechanism motion based on the flexible body simulation could be corrupted, because of large stiff forces that cause divergence of the numerical procedures.

3.4. Dynamic equations

The process of computation of the node stiff and damping forces in case of space motion for which the node displacements with respect to the inertial frame incorporate large transportation displacements and flexible deformations goes trough the following steps:

– computation of the node position in the space;

– computation of the small relative node deflections within the element;

– computation of the stiff and damping forces in the nodes (Eqs. 12, 13) relative to the element coordinate systems;

– transformation of the stiff and damping, as well as, the external forces to the absolute reference frame.

The final form of the dynamic equations is derived summing up the reduced inertia forces (with respect to the generalized coordinates $\mathbf{q}$) for l rigid bodies and m flexible elements (Eqs. 9, 11), as well as, the reduced external forces, stiff and damping forces in the nodes (with the common notation $\mathbf{G}_{M_i}$, $i = 1, 2, \ldots, n$), i.e.:

$$\sum_{i=1}^{l} \Big[\mathbf{R}\Delta_{C_i}^{\backslash} \cdot \mathbf{M}_{C_i} \cdot \mathbf{R}\Delta_{C_i} \cdot \ddot{\mathbf{q}} + \mathbf{R}\Delta_{C_i}^{\backslash} \cdot \underline{\mathbf{F}}_{C_i}(\dot{\mathbf{q}}, \omega_i)\Big] \tag{14}$$

$$+ \sum_{i=1}^{m} \Big[\mathbf{R}\Delta_{i}^{\backslash} \cdot \mathbf{M}_{i} \cdot \mathbf{R}\Delta_{i} \cdot \ddot{\mathbf{q}} + \mathbf{R}\Delta_{i}^{\backslash} \cdot \underline{\mathbf{F}}_{i}(\dot{\mathbf{q}}, \dot{\Delta}_i)\Big] - \sum_{i=1}^{n} \mathbf{R}\Delta_{M_i}^{\backslash} \cdot \mathbf{G}_{M_i} = {}^{g}\mathbf{0}.$$

Equation 14 is $g \times 1$ linear system of ordinary differential equations for the generalized accelerations.

4. Examples

An example of application of the approach proposed to motion simulation of the six-link mechanism in Fig. 5 in case of singular configuration with no initial velocity is presented. The mechanism is of three chains and is decomposed of four branches. The kinematics scheme and elastic forces in the nodes of flexible element, link 3, ($f_{x_{3'}}$, $f_{y_{3'}} m_{3'}$, $f_{x_{3''}}$, $f_{y3''}$, $m_{3''}$) are presented in Fig. 7. Singular configuration for this mechanism is when the directrixes of the links 2, 3 and 5 are crossing in a common point. For this case, the Jacobean matrix, as well as, the mass-matrix of the dynamic equations are singular.

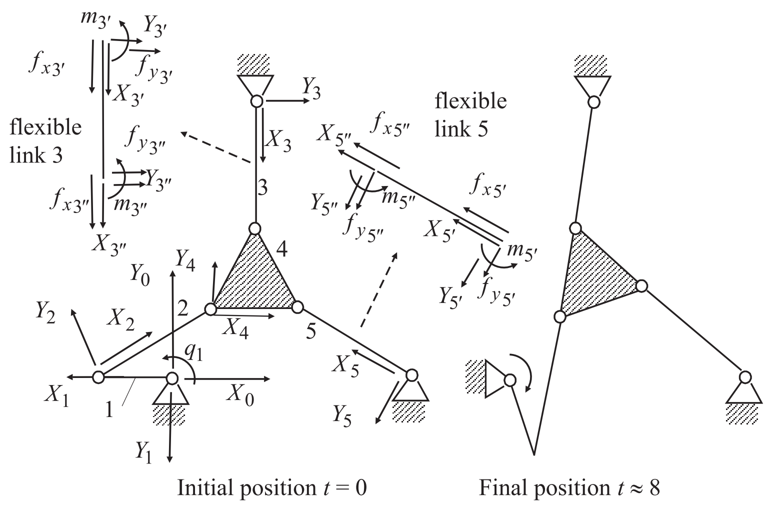

Fig. 7. Six link mechanism in its initial and final configuration

No kinematics constraints that describe the connectivity of nodes $3' - 3''$ (of link 3) and $5' - 5''$ (of link 5) are applied using the approach proposed. These constraints are the reason for the singularity and it is avoided substituting the links by external elastic forces. On every step, the relative position of the coordinate systems of links $3' - 3''$ and $5' - 5''$ is estimated and the small relative node deflections are computed. The elastic forces are calculated (Eq. 12) and used as external forces in the dynamic equations. The kinematic parameters, the mass and stiffness properties of mechanism links are as follows (all measures are in SI UNITS): ternary link – size $1\times1\times1$, mass $m_4 = 3$, inertia moment $J_4 = 0.3$; link 1 – length $l_1 = 1$, $m_1 = 1$, $J_1 = 0.1$; link 2 – $l_2 = 2$, $m_2 = 2$, $J_2 = 0.2$; flexible links – $l_3 = l_5 = 2$, mass density $\rho = 3000$, modulus of elasticity $E = 0.7 \times 10^{11}$, cross section area $A = 4 \times 10^{-4}$, second moment of area $I_z = 2 \times 10^{-7}$. The initial configuration of the mechanism is defined by the coordinates q_i, $i = 1, 3, 5$ as it is shown in Fig. 7, i.e.: $q_1 = \pi$; $q_3 = -\pi/2$; $q_5 = 5\pi/6$. The prescribed motion is realized as a reonomic constraint for q_1, i.e.: $q_1 = \pi.\cos\left(\dfrac{t}{4}\right)$ for $0 \le t \le 4$; $\dot{q}_1 = \text{const}$ for $t > 4$.

The time histories of the mechanism motion, links 1, 3 and 5, are presented in Fig. 8. The time histories of the elastic longitudinal forces in links 3 and 5 are depicted in Fig. 9. The longitudinal elastic forces that arise as a result of the compulsive motion of the crank 1 do not cause significant influence of the transfer functions q_3 and q_5, and these motion fully coincides with the exact rigid body motion. The integration process starts from the initial singular configuration and stops when the mechanism reaches another singular configuration at $t \approx 8$. It could be seen that longitudinal elastic forces are exaggerated in the initial stage for going out from the singular configuration. The stiff forces become extremely high at the end of the motion and cause the interruption of the integration process when the singular final configuration is obtained. After that, destruction of the system is observed.

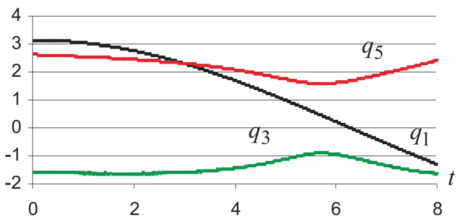

Fig. 8. Time history of the characteristics of motion of links 1, 3, 5

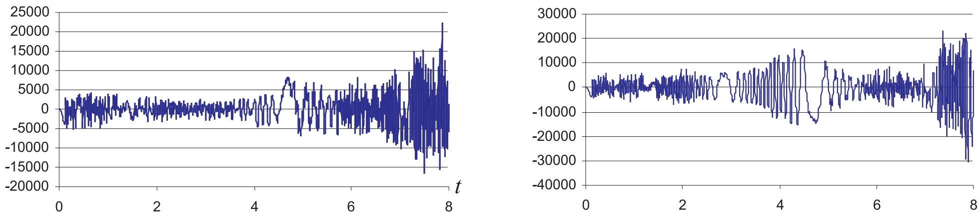

Fig. 9. Time history of the longitudinal elastic forces in link 3 and 5

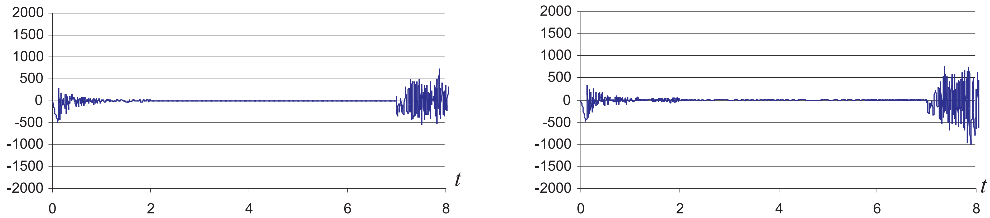

Fig. 10. Time history of the longitudinal elastic forces in link 3 and 5 taking into account damping in the flexible links

Obviously, the high level oscillations of the elastic forces are undesirable and unreal. That is why damping forces should be taken into account that will diminish the velocities of the internal deflections in the flexible link. That will help the mechanism motion to be similar to the rigid body motion and will facilitate the integration procedure. The time histories of the elastic longitudinal forces in links 3 and 5 are shown in Fig. 10 taking into account the damping in the flexible links. It could be observed that the longitudinal stiff forces are in the admissible values.

5. Conclusions

An approach to simulation of rigid and flexible multibody system is proposed for which the kinematic constraints are substituted by elastic forces in the flexible links. The method provides general solution for every kind of closed chains including overconstrained mechanisms and singular configurations.

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