EXACT SOLUTION FOR A BEAM ON OFF-CENTER SPRING SUPPORTS

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ABSTRACT. Exact solutions for the deformed state of a Bernoulli-Euler prismatic beam is presented in the paper using the exact parametric equations of the elastic line with respect to local coordinate systems, derived earlier and valid for every unloaded segment of a plane frame. The beam is supported off-center with helical and rotational springs. Numerical solutions of the governing transcendental equations help to analyze the influence of the springs stiffness and load behaviour upon the beam response.

KEY WORDS: beam, offset spring supports, non-linear analysis.

1. Introduction

Stress and strain analysis of frame structures is most often based on the precondition for centric connection of both their structural members and support devices. In practice however, there are many examples of off-center connection between structural members and off-center supports. As far as the author is aware no adequate analysis has been carried out hitherto to account for these features excepting the works by Trahair [1] and his collaborators mostly related to flexural-torsional buckling. In the same time a few recent studies [2, 3, 4], relating off-center pinned-pinned prismatic beams have shown, that under certain conditions which might appear either purposefully or because of abnormal exploitation, the off-center supports may have advantages over the central ones. Moreover, in some cases the contribution in the total potential energy of the strains due to normal forces is of the same order of magnitude as that of the bending moments.

A prismatic beam bilaterally supported with rotational and horizontal helical springs (Fig. 1) under transversal concentrated load is considered in
the paper. The supports A and B are located off-center the beam axis at a distance \( h_1 \) and \( h_2 \), respectively. In this way are modelled for example beams on elastomeric bearings widely used in bridges and located under the bottom flanges thus giving rise to significant compressive normal forces. Approximate solutions based on first order beam theory [3] revealed that the magnitude of the axial force caused by transversal loading depends highly on the linear strain in the beam axes which is usually negligibly small as compared to the strains from bending moments. As the case stands, a question arises not only about the character of the stressed and strained state but also on the beam stability.

A precise determination of both vertical and horizontal end displacements is indispensable with respect to design of bearings and possible horizontal displacement restraints. The exact large displacement analysis carried out in the paper has provided answers for both the reliability of the first order solutions and possible buckling in the plane of loading.

2. Statement of problem

The Bernoulli–Euler beam AB presented in Fig. 1 of length \( L \) is prismatic from linearly-elastic material with modulus of elasticity \( E \). The beam cross-section is symmetric of area \( A \) and second moment of inertia \( I \). A transverse concentrated load \( P \) is applied at point \( C \) on the beam axis at a distance \( r \) from the support A prior to loading. The load \( P \) is supposed attached to the point \( C \) displacing accordingly in the process of deformation. The supports are elastic modelled by horizontal helical and rotational springs of a spring constant \( k_1,c_1 \) at A and \( k_2,c_2 \) at B. The helical and rotational springs at each point are supposed to respond independently to horizontal displacements and end angle of rotation, respectively. The beam axis is initially horizontal while the supports are located below it at a distance \( h_1 \) and \( h_2 \) at A and B, respectively. The segments \( G_1C \) and \( CG_2 \) of the beam axis being unloaded makes them subjected to the theorem of elastic analogy [4], [5]. On these grounds there are local coordinate systems \( O_i x_i y_i \) \( (i = 1, 2) \), such that the parametric equations of the unloaded segments have the form:

\[
x_i(\varphi_i) = \Omega_i \sqrt{\frac{EI}{F_i}} \left\{ \Psi_i \left[ \Pi \left( n_i; p_i^2, z_i \right) - \Pi \left( n_i, p_i^2 \right) \right] + (1 + \Psi_i) \left[ K \left( p_i^2 \right) - F \left( p_i^2, z_i \right) \right] \right\},
\]

\[
y(\varphi_i) = 2 \sqrt{\frac{EI}{F_i}} \sqrt{1 + \varepsilon_i} \arctg \left( \frac{\varepsilon_i}{1 + q_i^2 \varepsilon_i} \cos z_i \right),
\]
with the arclengths $O_i S_i$

\begin{equation}
  s_i (\varphi_i) = \Omega_i \sqrt{\frac{EI}{F_i}} \left[ K(p_i^2) - F(p_i^2, z_i) \right].
\end{equation}

being solutions to the differential equations \( \frac{d^2 \varphi_i}{ds_i^2} + \frac{F_i}{EI} \left( 1 - \frac{F_i}{EA} \cos \varphi_i \right) \sin \varphi_i = 0. \)

The following dimensionless parameters have been introduced for convenience in the above equations:

\begin{align}
  \Psi_i &= \frac{2(1 + q_i^2 \varepsilon_i)}{\varepsilon_i}, \quad \Omega_i = \sqrt{\frac{1 + \varepsilon_i}{1 + 2q_i^2 \varepsilon_i}}, \\
  n_i &= \frac{q_i^2 \varepsilon_i}{1 + 2q_i^2 \varepsilon_i}, \quad q_i = \sin \frac{\chi_i}{2}, \quad \varepsilon_i = \frac{F_i}{EA},
\end{align}

\begin{equation}
  z_i = \arcsin \left( \sqrt{\frac{1 + 2q_i^2 \varepsilon_i}{1 + \varepsilon_i \left( q_i^2 + \sin^2 \frac{\varphi_i}{2} \right)}} \sin \frac{\varphi_i}{2} \right), \quad p_i^2 = q_i^2 \left( 1 + \varepsilon_i + q_i^2 \varepsilon_i \right).
\end{equation}

Moreover, \( K(p_i^2) \) and \( F(p_i^2, z_i) \) are the complete and incomplete elliptic integrals of the second kind; \( \Pi(p_i^2, z_i) \) and \( \Pi(n_i; p_i^2, z_i) \) the complete and incomplete elliptic integrals of the third kind [6]. The angle \( \chi_i \) is made by the \( x_i \)-axis and the tangent line to the elastic line at the origin \( O_i \), where the curvature equals zero, so that the current angle \( \varphi_i \) varies within the interval \( \pm \chi_i \). The internal forces at the same point \( O_i \) reduce to the resultants\(^1\) \( F_i \).

The resultants \( F_i \) have to pass through A and B (dashed lines in Fig. 1) for supports without rotational springs and equal to the vector sum of support reactions. Otherwise, the restoring spring moments due to the rotation angles \( \alpha_i \) of the terminal sections, transfer the resultants \( F_i \) in parallel with their support location at distances \( \alpha_i c_i / F_i \) in accordance with the well-known Poinsot’s problem from statics. Simultaneously, the supported points undergo displacements \( H_i / k_i \) outward depending on the magnitude of the horizontal reactions \( H_i \) at A and B.

3. Governing equations

The solution to the problem is based on two sets of equations – equilibrium and geometrical. Consider the deformed state of the beam (Fig. 1),

\(^1\)The non-inflexional case is not commented here because it does not appear.
where the beam axis consists of the segments $G_1O'C$ and $G_2O''C$ which are parts of the elastic curves $O_1C$ and $O_2C$, respectively. Alongside with the horizontal reactions $H_i$ (i=1,2) there are seven more unknown parameters: the angles $\psi_i$ formed by $x_i$-axes and the horizon, the rotation angle $\psi_C$ of the section at point C, the rotation angles $\alpha_i$ of the terminal sections, and the angles $\chi_i$ at the origins of the elastic curves. Bearing in mind that the relationships $H_i + V_i = F_i$ and $H_i \perp V_i$ hold, the magnitudes of the vertical reactions $V_i$ and the resultants $F_i$ will be determined by means of the expressions:

$$V_i = H_i \tan \psi_i, \quad F_i = H_i/\cos \psi_i.$$  \hspace{1cm} (6)

The following nine equations serve to obtain the unknowns:

**The equilibrium equations** for $F_1$, $F_2$ and $P$ taking (6) into account are: $\sum H_i = 0$: $H_1 - H_2 = 0$ and $\sum V_i = 0$: $H_1 \tan \psi_1 + H_2 \tan \psi_2 - P = 0$. With $H_1 = H_2 = H$ the second equation from above yields:

$$H (\tan \psi_1 + \tan \psi_2) - P = 0 \quad \text{or} \quad \Lambda_p^2 (\tan \psi_1 + \tan \psi_2) - \Lambda_p^2 = 0. \hspace{1cm} (7)$$

The set of **geometrical equations** will be started with the arc length $G_1O'C$, which after the deformation equals $r - \delta_1$, where $\delta_1$ is the shortening of the segment $G_1C$, caused by the compressive normal force $N_1 = F_1 \cos \varphi_1$. Accordingly, a differentially small element $ds_1$ prior to deformation shrinks by $N_1 \frac{ds_1}{EA} = F_1 \cos \varphi_1 dx_1 = F_1 \frac{dx_1}{EA}$ so that the shortening of the segment $G_1O'C$ is:

$$\delta_1 = \int_{x_1G_1}^{x_1C} \frac{F_1}{EA} dx_1 = \frac{F_1}{EA} (x_1C - x_1G_1). \hspace{1cm} (8)$$

With the help of (1), (3), (4), (5), (8) and transformations the following equation is arrived at:

$$\Omega_1 \left[1 + (1 + \Psi_1) \frac{\varepsilon_1}{1 + \varepsilon_1}\right] \left[2K (p_1^2) - F (p_1^2, z_{1a}) - F (p_1^2, z_{1C})\right] +$$

$$+ \Omega_1 \Psi_1 \frac{\varepsilon_1}{1 + \varepsilon_1} \left[\Pi (n_1; p_1^2, z_{1a}) + \Pi (n_1; p_1^2, z_{1C}) - 2\Pi (n_1; p_1^2)\right] - \Lambda_1 \rho = 0. \hspace{1cm} (9)$$

It should be noted here that the current angle at point $G_1$ is $\varphi_1 = \psi_1 + \alpha_1$, and at point C is $\varphi_1 = \psi_1 + \psi_C$.

We have by analogy with the equation (9), with current angles $\varphi_2 = \psi_2 + \alpha_2$ at point $G_2$, $\varphi_2 = \psi_2 - \psi_C$ at point C, and the shortening $\delta_2 =$
\[
\frac{F_2}{F} (x_{2C} - x_{2G_2}) \text{ for the length of segment } G_2O''C \text{ the equation:}
\]

\[
\Omega_2 \left[ 1 + (1 + \Psi_2) \frac{\varepsilon_2}{1 + \varepsilon_2} \left[ 2K \left( p_2^2 \right) - F \left( p_2^2, z_{2\alpha} \right) - F \left( p_2^2, z_{2C} \right) \right] + \\
+ \Omega_2 \Psi_2 \frac{\varepsilon_2}{1 + \varepsilon_2} \left[ \Pi \left( n_2; p_2^2, z_{2\alpha} \right) + \Pi \left( n_2; p_2^2, z_{2C} \right) - 2\Pi \left( n_2, p_2^2 \right) \right] - \\
- (1 - \rho) \Lambda_2 = 0
\]

(10)

The next two geometrical equations establish the ordinates of the terminal points \( G_i \) of the deformed axis \( y_{iG_i} = h_i \cos (\psi_i + \alpha_i) + \frac{\alpha_1 c_i}{F_1} \) (Fig. 2). We find upon substitution of (2), (4), (5), (6) and transformations of the ordinate of point \( G_1 \).

\[
\arctg \left( \sqrt{\frac{2}{\Psi_1}} q_1 \cos z_{1\alpha} \right) - \sqrt{\frac{\varepsilon_1}{1 + \varepsilon_1}} \left[ \frac{\beta_1 \Lambda_1}{2} \cos (\psi_1 + \alpha_1) + \frac{\alpha_1 \gamma_1}{2 \Lambda_1} \right] = 0.
\]

(11)

The analogous equation for the end \( G_2 \) is

\[
\arctg \left( \sqrt{\frac{2}{\Psi_2}} q_2 \cos z_{2\alpha} \right) - \sqrt{\frac{\varepsilon_1}{1 + \varepsilon_1}} \left[ \frac{\beta_2 \Lambda_2}{2} \cos (\psi_2 + \alpha_2) + \frac{\alpha_2 \gamma_2}{2 \Lambda_2} \right] = 0.
\]

(12)

Next, the distance \( AB \) is equal to the span \( L \) before deformation plus the shortenings of the helical springs (Fig. 1 and Fig. 2), i.e. \( A''C' \cos \psi_1 + C'C \sin \psi_1 + CC'' \sin \psi_2 + C''B'' \cos \psi_2 = L + \Delta_1 + \Delta_2 \). By this, \( A'' \) and \( B'' \) are the intersection points of the local \( x_i \)-axes with vertical lines through the supports after deformation (Fig. 2). On the other hand, the relationships

\[
C'C'' = y_{1C}, \ C'C'' = y_{2C}, \ A''C' = \frac{\alpha_1 c_1 \tan \psi_1}{F_1} + h_1 \sin (\psi_1 + \alpha_1) + x_{1G_1} + x_{1C'}, \nC''B'' = \frac{\alpha_2 c_2 \tan \psi_2}{F_2} + h_2 \sin (\psi_2 + \alpha_2) + x_{2G_2} + x_{2C'}, \text{ and } \Delta_i = \frac{H}{k_i} \text{ hold leading}.
\]
after transformations to the equation:

\[
\left\{ \begin{array}{l}
\beta_1 \sin(\psi_1 + \alpha_1) + \frac{\Omega_1}{\Lambda_1} [\Psi_1(\Pi(n_1, p_1^2; z_{1\alpha}) + \Pi(n_1, p_1^2; z_{1C}) - 2\Pi(n_1, p_1^2))] + \\
+ \frac{\alpha_1 \gamma_1 \sin \psi_1}{\Lambda_2^2} + (1 + \Psi_1) \frac{\Omega_1}{\Lambda_1} [2K(p_1^2) - F(p_1^2, z_{1\alpha}) - F(p_1^2, z_{1C})] \right\} \cos \psi_1 + \\

\left\{ \begin{array}{l}
\beta_2 \sin(\psi_2 + \alpha_2) + \frac{\Omega_2}{\Lambda_2} [\Psi_2(\Pi(n_2, p_2^2; z_{2\alpha}) + \Pi(n_2, p_2^2; z_{2C}) - 2\Pi(n_2, p_2^2))] + \\
+ \frac{\alpha_2 \gamma_2 \sin \psi_2}{\Lambda_2^2} + (1 + \Psi_2) \frac{\Omega_2}{\Lambda_2} [2K(p_2^2) - F(p_2^2, z_{2\alpha}) - F(p_2^2, z_{2C})] \right\} \cos \psi_2 + \\

\frac{2}{\Lambda_1} \sqrt{1 + \frac{\varepsilon_1}{\varepsilon_1}} \arctg \left( \sqrt{\frac{2}{\Psi_1} q_1 \cos z_{1C}} \right) \sin \psi_1 \\
+ \frac{2}{\Lambda_2} \sqrt{1 + \frac{\varepsilon_2}{\varepsilon_2}} \arctg \left( \sqrt{\frac{2}{\Psi_2} q_2 \cos z_{2C}} \right) \sin \psi_2 - 1 - \frac{\Lambda_2^2}{\lambda^2} \left( \frac{1}{\zeta_1} + \frac{1}{\zeta_2} \right) = 0.
\end{array} \right.
\]
Since the point C is common for segments $G_1C$ and $CG_2$ (Fig. 1, Fig. 2), its location in the vertical plane must satisfy the equation:

\[ A''C' \sin \psi_1 - C'C \cos \psi_1 + C''C' \sin \psi_2 - A' A''' + B' B''' = h_1 - h_2, \]

or after apparent substitutions and rearrangements

\[ \left\{ \frac{\alpha_1 \gamma_1 \sin \psi_1}{\Lambda_H^2} + (1 + \frac{\Omega_1}{\Lambda_1}[2K(p_1^2) - F(p_1^2, z_1) - F(p_1^2, z_1)]) \right\} \sin \psi_1 - \\
\left\{ \frac{\alpha_2 \gamma_2 \sin \psi_2}{\Lambda_H^2} + (1 + \frac{\Omega_2}{\Lambda_2}[2K(p_2^2) - F(p_2^2, z_2) - F(p_2^2, z_2)]) \right\} \sin \psi_2 - \\
- \frac{2}{\Lambda_1} \sqrt{1 + \varepsilon_1} \arctg \left( \sqrt{\frac{2}{\Psi_1} q_1 \cos z_1} \right) \cos \psi_1 + \\
+ \frac{2}{\Lambda_2} \sqrt{1 + \varepsilon_2} \arctg \left( \sqrt{\frac{2}{\Psi_2} q_2 \cos z_2} \right) \cos \psi_2 - \beta_1 + \beta_2 - \frac{1}{\Lambda_H^2} (\alpha_2 \gamma_2 - \alpha_1 \gamma_1) = 0. \]

The continuity of the beam at point C necessitates one and the same curvature on the two sides of the section which is equivalent to equal bending moments. In analytical parlance the equation (Fig. 1) $F_1 y_{1C} = F_2 y_{2C}$ must hold or after some transformations we get:

\[ \sqrt{1 + \varepsilon_1} \cos \psi_2 \arctg \left( \sqrt{\frac{2}{\Psi_1} q_1 \cos z_1} \right) - \sqrt{1 + \varepsilon_2} \cos \psi_1 \arctg \left( \sqrt{\frac{2}{\Psi_2} q_2 \cos z_2} \right) = 0. \]

The new notations introduced in the equations (7), (9)-(15) in addition to those of (4) and (5) are:

\[ \Lambda_H = L \sqrt{\frac{H}{EI}}, \quad \Lambda_P = L \sqrt{\frac{P}{EI}}, \quad L \rho = r, \quad \lambda^2 = \frac{L^2 A}{I}, \quad \Lambda_i = L \sqrt{\frac{F_i}{EI}}. \]
\( L/\beta_i = h_i; \quad i = 1, 2. \)

Moreover, the spring constants \( k_i \) and \( c_i \) have been defined in proportion to the specific beam tension-compression and uniaxial bending stiffness, respectively:

\[
\frac{E}{l} \rightarrow k_i = \zeta_i \frac{E}{l}, \quad c_i = \gamma_i \frac{E}{l}.
\]

The equations (9) and (10) do not hold when the position of the concentrated load \( P \) is fixed, for example a distance \( r \) from the support \( A \) prior to deformation. One of the two new equations stands for the length of the entire beam after deformation, i.e. \( G_1 CG_2 = L - \delta_1 - \delta_2 \), or in terms of the notations from above:

\[
\frac{\Omega_1}{\lambda_1} \left\{ \left[ 1 + (1 + \Psi_1) \frac{\varepsilon_1}{1 + \varepsilon_1} \left[ 2K \left( p_1^2 \right) - F \left( p_1^2, z_{10} \right) - F \left( p_1^2, z_{1C} \right) \right] \right] + \right. \\
+ \Psi_1 \frac{\varepsilon_1}{1 + \varepsilon_1} \left[ \Pi \left( n_1; p_1^2, z_{10} \right) + \Pi \left( n_1; p_1^2, z_{1C} \right) - 2\Pi \left( n_1, p_1^2 \right) \right] \right\} + \\
\frac{\Omega_2}{\lambda_2} \left\{ \left[ 1 + (1 + \Psi_2) \frac{\varepsilon_2}{1 + \varepsilon_2} \left[ 2K \left( p_2^2 \right) - F \left( p_2^2, z_{20} \right) - F \left( p_2^2, z_{2C} \right) \right] \right] + \right. \\
+ \Psi_2 \frac{\varepsilon_2}{1 + \varepsilon_2} \left[ \Pi \left( n_2; p_2^2, z_{20} \right) + \Pi \left( n_2; p_2^2, z_{2C} \right) - 2\Pi \left( n_2, p_2^2 \right) \right] \right\} - 1 = 0.
\]

The second new equation comes from the condition \( A''C' \cos \psi_1 + C'C \sin \psi_1 - \Delta_1 = r \). This is transformed by means of (4), (5), (16) and rearrangements to the form

\[
\left\{ \beta_1 \sin(\psi_1 + \alpha_1) + \frac{\Omega_1}{\lambda_1} [\Psi_1 (\Pi (n_1, p_1^2; z_{10}) + \Pi (n_1, p_1^2; z_{1C}) - 2\Pi (n_1, p_1^2))] + \\
+ \frac{\alpha_1 \gamma_1 \sin \psi_1}{\lambda_2} + (1 + \Psi_1) \frac{\Omega_1}{\lambda_1} [2K (p_1^2) - F (p_1^2, z_{10}) - F (p_1^2, z_{1C})] \right\} \cos \psi_1 + \\
+ \frac{2}{\lambda_1} \sqrt{\frac{1 + \varepsilon_1}{\varepsilon_1}} \arctg \left( \sqrt{\frac{2}{\Psi_1 q_1} \cos \left( \frac{z_{1C}}{\lambda_1} \right)} \right) \sin \psi_1 - \frac{\lambda_2^2}{\zeta_1 \lambda_1^2} - \rho = 0.
\]

In this case, the governing set of equations consists of (7), (9'), (10'), (11)–(15). The numerical results obtained for particular cases considered do
not show considerable differences depending on the load behaviour. Table 1 contains results for the two sets of unknown parameters derived for \( \rho = 0.4; \beta_1 = \beta_2 = 0.05; \gamma_1 = \gamma_2 = 0, \Lambda_P = 0.786; \zeta_1 = \zeta_2 = 1 \), and witnesses in favour of the above statement.

<table>
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<th>Unknowns</th>
<th>Load type</th>
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<tr>
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<tr>
<td>( \sigma_2 )</td>
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</tr>
<tr>
<td>( \Lambda_H )</td>
<td>0.6973</td>
</tr>
</tbody>
</table>

4. Numerical results

The two sets of transcendental equations (7), (9)–(15) and (7), (9'), (10'), (11)–(15) have been solved numerically using MATLAB [7]. Figure 3 shows variation of the parameter \( \Lambda_H \) of the horizontal reaction \( H \) in the symmetric case \( (\rho = 0.5; \beta_1 = \beta_2 = 0.05; \gamma_1 = \gamma_2 = 0; \zeta_1 = \zeta_2 = \zeta) \) depending on the loading parameter \( \Lambda_P \) for three different values \( (\infty; 10; 1) \) of the helical spring parameter \( \zeta \). It is seen that \( \Lambda_H \) increases almost linearly up to some maximum value for load parameter \( \Lambda_P \approx 1.8 \), where after \( \Lambda_H \) decreases.

Closer inspection of the deflection of point C reveals that the transition from increasing to decreasing branch for \( \Lambda_H \) appears with point C approaching the supports level. Another particularity of the diagrams in the same figure is the closeness of the lines for \( \zeta = 10 \) and \( \zeta = \infty \), which gives idea of the order of bearings stiffness magnitude above which they may be regarded as rigid.

We once again consider the symmetric case and the same parameters as for Fig. 3 but with \( \zeta = \infty \). Figure 4 traces out the variations of some characteristic deflections depending on the loading parameter \( \Lambda_P \). Here \( W_{max} \) is the maximum deflection of the point C with respect to undeformed beam axis, \( U_{max} \) is the horizontal displacement and \( W_{end} \) is the vertical displacement of the end points. The actual numbers for the last two quantities have been scaled quintuple and tenfold thus, making them observable together with \( W_{max} \).
The calculation of $U_{\text{max}}$ and $W_{\text{end}}$ might be of interest for the design of bridge restrainers for girders on common pier. The curve shapes in Fig. 4 impress with their points of inflection and change of curvature near $\Lambda_P \approx 1.8$, where $\Lambda_H$ attains its maximum. The decrease of the horizontal reaction together with its arm with respect to the middle section implies a weaker resistance against the moments from the vertical reactions, whose arms have gained length with the spring deformation. This is the reason for a relatively greater change rate of displacements. In authors opinion, however, this state should not be looked at as critical one, because even in comparison with $H_{\text{max}}$ the Euler’s buckling load is twice greater. Also, the displacements continue to rise smoothly further with the loading parameter $\Lambda_P$, whereas $\Lambda_H$ approaches zero. This statement
conforms to the conclusion made in [3] under similar circumstances.

Figure 5 presents graphics for an asymmetric case with \( \rho = 0.4; \beta_1 = \beta_2 = 0.05; \gamma_1 = \gamma_2 = 0 \) and different helical spring constants for three loading parameters \( \Lambda_p \). Apparently, the horizontal reaction increases rapidly until the spring constants reach the corresponding specific beam stiffness \( (\zeta_1 = \zeta_2 = 1) \), then the effect of the springs gradually dies out.

The influence of the rotational springs on the horizontal reaction is not so strongly pronounced. The relationships \( \Lambda_H \) vs. \( \gamma_1 = \gamma_2 \) in Fig. 6 drawn for bilaterally pinned beam \( (\zeta_1 = \zeta_2 = \infty; \rho = 0, 4) \) are almost linear while \( \Lambda_H \) decreases slightly with increasing rotational spring stiffness.

Figure 7, on the other hand, shows that a considerable horizontal reaction exists when one of the supported ends tends to be clamped. The corresponding numerical data has been obtained for \( \rho = 0.4; \beta_1 = \beta_2 = 0.05; \gamma_2 = 0 \) and \( \zeta_1 = \zeta_2 = \infty \). The parameter \( \Lambda_H \) is seen to decrease asymptotically while for \( \gamma_1 > 15 \) the effect of a further rotational spring hardening becomes immaterial.
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Figure 7. $\Lambda_H$ vs. $\gamma_1$

Figure 8. $\Lambda_H$ vs. $\zeta_1$

Figure 8 has been constructed for $\rho = 0.4; \beta_1 = \beta_2 = 0.05; \zeta_2 = \infty; \gamma_1 = 20; \gamma_2 = 0$ and variable parameter of the helical spring $\zeta_1$. In this way, as of limit transition one can model from movably to fully fixed left beam end and pinned right end. It is found that the horizontal reaction is mostly due to off center rigid and/or spring restraints preventing the terminal cross-sections from rotating freely about beam axis.

It is worthwhile to note that in the nonsymmetrical cases the point with maximum deflection is closer to the beam midpoint than that of the load application.
Conclusions

A Bernoulli-Euler prismatic beam is considered based on the exact differential equation of the elastic line accounting to both the flexure moment and axial force the stressed and strained state. The beam is supported at the extremities with helical and rotational springs and is acted upon by a concentrated transverse load. The numerical results derived show that the deformed state is almost independent on whether the load is attached to a particular point or preserves its position during deformation. Moreover, variations of springs stiffness illustrate their effect on the horizontal support reaction which is due to the restraints imposed on the beam bottom edge.

REFERENCES