

PROPAGATION OF WAVES AT AN IMPERFECTLY BONDED INTERFACE BETWEEN TWO MONOCLINIC THERMOELASTIC HALF-SPACES

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[Received 09 February 2011. Accepted 14 November 2011]

ABSTRACT. An imperfectly bonded interface between two monoclinic thermoelastic half-spaces is chosen to study the reflection and transmission of plane waves in context of generalized thermoelasticity. Six relations between amplitudes of incident, reflected and transmitted quasi- P (qP) waves, quasi thermal (qT) waves and quasi- SV (qSV) waves are obtained. Some particular cases are obtained which agree with earlier well established results. A procedure for computing reflected and refracted angles is derived for a given incident wave, where the angles of reflection are found not equal to the angles of incident waves in a monoclinic thermoelastic medium.

KEY WORDS: generalized thermoelasticity, imperfect boundary, plane waves, reflection and transmission, monoclinic anisotropic.

1. Introduction

Keith and Crampin [1] investigated that three types of body waves with mutually orthogonal particle motion can propagate in an anisotropic elastic solid medium. In general, the particle motion is neither purely longitudinal nor

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purely transverse. Due to this, the three types of body waves in an anisotropic medium are referred to as qP , qSV and qSH , rather than as P , SV and SH , the symbols used for propagation in an isotropic medium. A monoclinic medium has one plane of elastic symmetry. SH motion is decoupled from the $P - SV$ motion for wave propagation in the plane of symmetry. It is neither purely longitudinal nor purely transverse motion in the case of $P - SV$ waves, while the particle motion of SH waves is purely transverse. Chattopadhyay and Choudhury [2] discussed the reflection of qP waves at the plane free boundary of a monoclinic half-space. In a subsequent paper, Chattopadhyay *et al* [3] studied the reflection of qSV waves at a plane free boundary of a monoclinic half-space. In these papers, they assumed that qP waves are purely longitudinal and qSV waves are purely transverse. Singh [4] pointed out the errors in these papers and Singh and Khurana [5] restudied the reflection of P and SV waves at the free surface of a monoclinic half-space.

Biot [6] developed the coupled theory of thermoelasticity, where heat equation being parabolic predicts an infinite speed of propagation for thermal wave, which is not physically relevant. Lord and Shulman [7] introduced the theory of generalized thermoelasticity with one relaxation time by postulating a new law of heat conduction to replace the classical Fourier's law. This law contains the heat flux vector as well as its time derivative. In addition, it contains a new constant which acts as a relaxation time. The heat equation of this theory is of wave type, ensuring finite speeds of propagation for thermal and elastic waves. The remaining governing equations for this theory, namely the equation of motion and the constitutive relation remain the same as those for the coupled and uncoupled theories. Dhaliwal and Sherief [8] extended the Lord and Shulman (L-S) theory for an anisotropic media. The second generalization was developed by Green and Lindsay [9] and is known as G-L theory. This theory contains two constants that act as relaxation times. Many authors contribute towards the wave propagation in isotropic and anisotropic medium with the application of generalized thermoelasticity, for example, Sinha and Sinha [10], Sharma [11], Kumar and Singh [12] and Singh [13–15].

The boundary between the solids may behave as slip, perfect or neither, depending on the properties of the layers, as demonstrated by Rokhlin and Marom [16], and its state significantly affects elastic wave reflection phenomena. Imperfect bonding means that the stress components, heat flux are continuous and the small displacement field and temperature are not continuous. The small vector difference in the displacement is assumed to depend linearly on the traction vector and in temperature it is assumed to depend linearly on the heat flux. In the present paper, the reflection and transmis-

sion of plane waves is studied at an imperfectly bonded interface between two dissimilar thermoelastic half-spaces of monoclinic type in the context of generalized thermoelasticity. Relations between amplitudes of incident, reflected and transmitted quasi- P (qP) waves, quasi thermal (qT) waves and quasi- SV (qSV) waves are obtained, which are reduced to earlier well established relations for some limiting cases. A procedure for computing reflected and refracted angles is also derived for a given incident angle of a striking wave.

2. Nomenclature

c	phase velocity
c_{ij}	isothermal elasticities
C_e	specific heat at constant strain
e_{ij}	components of strain tensor
k	wave number
t	time
τ_0, τ_1	relaxation times
K_2, K_3	thermal conductivities
T	absolute temperature
T_0	the initial uniform temperature
α_2, α_3	coefficients of linear thermal expansion
ρ	density
σ_{ij}	components of stress tensor
ω	circular frequency
$\Theta = T - T_0, \left \frac{\Theta}{T} \right \ll 1$	
$\beta_2 = (c_{12} + c_{22})\alpha_2 + c_{23}\alpha_3, \beta_3 = 2c_{23}\alpha_2 + c_{33}\alpha_3.$	

3. Formulation of the problem and solution

Consider a homogeneous, anisotropic, generalized thermoelastic medium of monoclinic type at a uniform temperature T_0 . The origin is taken on the thermally insulated and stress-free plane surface and z -axis is directed normally into the half-space, which is represented by $z > 0$. Following Lord and Shulman [7] and Green and Lindsay [9] theories, the governing field equations in the $y - z$ plane $\left(\frac{\partial}{\partial x} \equiv 0 \right)$ for generalized monoclinic thermoelasticity, in absence of body forces and heat sources, are:

$$(1) \quad c_{22} \frac{\partial^2 v}{\partial y^2} + c_{44} \frac{\partial^2 v}{\partial z^2} + c_{24} \frac{\partial^2 w}{\partial y^2} + c_{34} \frac{\partial^2 w}{\partial z^2} + 2c_{24} \frac{\partial^2 v}{\partial y \partial z}$$

$$\begin{aligned}
& + (c_{23} + c_{44}) \frac{\partial^2 w}{\partial y \partial z} - \beta_2 \frac{\partial}{\partial y} \left(\Theta + \tau_1 \frac{\partial \Theta}{\partial t} \right) = \rho \frac{\partial^2 v}{\partial t^2}, \\
& c_{24} \frac{\partial^2 v}{\partial y^2} + c_{34} \frac{\partial^2 v}{\partial z^2} + c_{44} \frac{\partial^2 w}{\partial y^2} + c_{33} \frac{\partial^2 w}{\partial z^2} + 2c_{34} \frac{\partial^2 w}{\partial y \partial z} \\
(2) \quad & + (c_{23} + c_{44}) \frac{\partial^2 v}{\partial y \partial z} - \beta_3 \frac{\partial}{\partial z} \left(\Theta + \tau_1 \frac{\partial \Theta}{\partial t} \right) = \rho \frac{\partial^2 w}{\partial t^2}, \\
& K_2 \frac{\partial^2 \Theta}{\partial y^2} + K_3 \frac{\partial^2 \Theta}{\partial z^2} - T_0 \left[\beta_2 \left(\frac{\partial^2 v}{\partial y \partial t} + \tau_0 \Omega \frac{\partial^3 v}{\partial y \partial t^2} \right) \right. \\
(3) \quad & \left. + \beta_3 \left(\frac{\partial^2 w}{\partial z \partial t} + \tau_0 \Omega \frac{\partial^3 w}{\partial z \partial t^2} \right) \right] = \rho C_e \left(\frac{\partial \Theta}{\partial t} + \tau_0 \frac{\partial^2 \Theta}{\partial t^2} \right).
\end{aligned}$$

The use of symbol Ω and τ_1 in equation (3) makes these equations possible for L-S and G-L theories of generalized thermoelasticity. For the L-S theory, $\tau_1 = 0$, $\Omega = 1$ and for the G-L theory $\tau_1 > 0$, $\Omega = 0$. The thermal relations τ_0 and τ_1 satisfy the inequality $\tau_1 > \tau_0 > 0$ for the G-L theory only.

For plane waves of circular frequency ω , wave number k and phase velocity c , incident at the free boundary $z = 0$ at an angle θ with the z -axis, we assume:

$$(4) \quad (v, w, \Theta) = (A, B, C) \exp(\iota P_1),$$

where A, B, C are the amplitude factors and:

$$(5) \quad P_1 = \omega t - k(y \sin \theta - z \cos \theta),$$

is the phase factor. For waves reflected at $z = 0$, we assume:

$$(6) \quad (v, w, \Theta) = (A, B, C) \exp(\iota P_2),$$

where:

$$(7) \quad P_2 = \omega t - k(y \sin \theta + z \cos \theta).$$

Making use of Eq. (4) or (6) in Eqs. (1) to (3), we obtain:

$$(8) \quad (D_1 - \rho c^2) A + L_1 B - \left(\frac{\iota}{k} \right) \tau^1 \beta_2 \sin \theta C = 0,$$

$$(9) \quad L_1 A + (D_2 - \rho c^2) B \pm \left(\frac{l}{k}\right) \tau^1 \beta_3 \cos \theta C = 0,$$

$$(10) \quad \tau T_0 c^2 \beta_2 \sin \theta A \mp \tau T_0 c^2 \beta_3 \cos \theta B - \left(\frac{l}{k}\right) (D_3 - \rho c^2 \tau * C_e) C = 0,$$

Here,

$$D_1(\theta) = c_{22} \sin^2 \theta + c_{44} \cos^2 \theta \mp 2c_{24} \sin \theta \cos \theta,$$

$$D_2(\theta) = c_{44} \sin^2 \theta + c_{33} \cos^2 \theta \mp 2c_{34} \sin \theta \cos \theta,$$

$$D_3(\theta) = K_3 \cos^2 \theta + K_2 \sin^2 \theta,$$

$$L_1(\theta) = c_{24} \sin^2 \theta + c_{34} \cos^2 \theta \mp (c_{23} + c_{44}) \sin \theta \cos \theta,$$

$$\tau^* = \tau_0 - \left(\frac{l}{\omega}\right), \quad \tau = \tau_0 \Omega - \left(\frac{l}{\omega}\right), \quad \tau^1 = (1 + i\omega\tau_1),$$

where, the upper sign corresponds to the incident waves and the lower sign corresponds to the reflected waves. Equations (8)–(10) in A , B , C admit a non-trivial solution only if the determinant of their coefficients vanishes, i.e.:

$$(11) \quad A_0 \zeta^3 + A_1 \zeta^2 + A_2 \zeta + A_3 = 0$$

where

$$A_0 = \tau^*,$$

$$A_1 = - [D_1 \tau^* + D_2 \tau^* + D_4 + \varepsilon \eta (\bar{\beta}^2 \cos^2 \theta + \sin^2 \theta)],$$

$$A_2 = (D_1 D_2 - L_1^2) \tau^* + D_1 D_4 + D_2 D_4 + \varepsilon \eta (\bar{\beta}^2 D_1 \cos^2 \theta + D_2 \sin^2 \theta) \pm 2L_1 \varepsilon \eta \bar{\beta} \sin \theta \cos \theta,$$

$$A_3 = -D_4 (D_1 D_2 - L_1^2),$$

$$\zeta = \rho c^2, \quad D_4 = \left(\frac{D_3}{C_e}\right), \quad \varepsilon = \left(\frac{\beta_2^2 T_0}{\rho C_e v_1^2}\right), \quad \eta = \tau \tau^1 v_1^2, \quad v_1^2 = \frac{c_{22}}{\rho}, \quad \bar{\beta} = \left(\frac{\beta_3}{\beta_2}\right).$$

Following Singh [13], the cubic equation (11) with complex coefficients has three roots ζ_1 , ζ_2 , ζ_3 corresponding to phase velocities of three plane waves, namely quasi- P (qP), quasi-thermal (qT) and quasi- SV (qSV) waves, respectively. If we write $c_j^{-1} = V_j^{-1} - i\omega^{-1} q_j$, $j = 1, 2, 3$, then V_j and q_j are the speeds of propagation and the attenuation coefficients of qP , qT and qSV waves, respectively.

4. Reflection and transmission

Consider a homogeneous, monoclinic generalized thermoelastic half-space occupying the region $z > 0$ in imperfectly bonded contact with another homogeneous, monoclinic generalized thermoelastic half-space occupying the region $z < 0$. Incident qP or qSV or qT waves at the interface will generate reflected qP, qSV, qT waves in the half-space $z > 0$ and transmitted qP, qSV, qT waves in the half-space $z < 0$ as shown in Fig. 1.

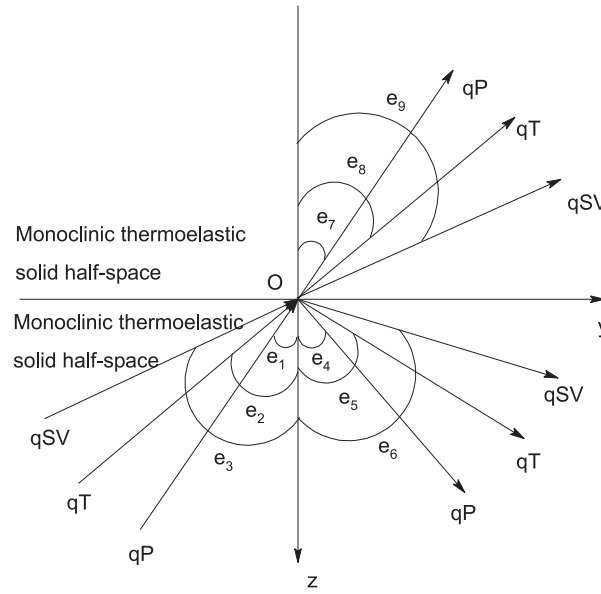


Fig. 1. Geometry of the problem showing incident, reflected and transmitted waves at an interface between two dissimilar thermoelastic half-spaces of monoclinic type

The required boundary conditions at an imperfectly bonded interface $z = 0$ are (Lavrentyev and Rokhlin [17]):

$$\begin{aligned}
 & \text{(i)} \quad \sigma'_{33} = K_n [w - w'], \\
 & \text{(ii)} \quad \sigma'_{23} = K_t [v - v'], \\
 & \text{(iii)} \quad K'_3 \frac{\partial T'}{\partial z} = K_c [T - T'], \\
 & \text{(iv)} \quad \sigma'_{33} = \sigma_{33}, \\
 & \text{(v)} \quad \sigma'_{23} = \sigma_{23}, \\
 & \text{(vi)} \quad K'_3 \frac{\partial T'}{\partial z} = K_3 \frac{\partial T}{\partial z},
 \end{aligned}
 \tag{12}$$

where, K_n, K_t are the normal and transverse stiffness coefficients. The symbols with primes correspond to the upper half-space $z < 0$.

The appropriate components of the displacement vector and the temperature field which satisfy the boundary condition (12) at $z = 0$ are as follows:

For half-space $z > 0$,

$$(13a) \quad v = \sum_{j=1}^6 A_j e^{iP_j}, \quad w = \sum_{j=1}^6 B_j e^{iP_j}, \quad T = \sum_{j=1}^6 C_j e^{iP_j}.$$

For half-space $z < 0$,

$$(13b) \quad v' = \sum_{j=7}^9 A_j e^{iP_j}, \quad w' = \sum_{j=7}^9 B_j e^{iP_j}, \quad T' = \sum_{j=7}^9 C_j e^{iP_j},$$

where

$$P_j = \omega \left[t - \frac{(y \sin e_j - z \cos e_j)}{c_j} \right], \quad (j = 1, 2, 3, 7, 8, 9);$$

$$(14) \quad P_j = \omega \left[t - \frac{(y \sin e_j + z \cos e_j)}{c_j} \right], \quad (j = 4, 5, 6),$$

$$(15) \quad A_j = F_j B_j, \quad C_j = F_j^* B_j,$$

For $j = 1, 2, 3, 7, 8, 9$,

$$F_j = \frac{(D_{2j} - \rho c_j^2) \beta_2 \sin \theta + L_{1j} \beta_3 \cos \theta}{(D_{1j} - \rho c_j^2) \beta_3 \cos \theta + L_{1j} \beta_2 \sin \theta},$$

$$F_j^* = \frac{L_{1j}^2 - (D_{1j} - \rho c_j^2) (D_{2j} - \rho c_j^2)}{\left(\frac{i}{k}\right) \tau' \left[L_{1j} \beta_2 \sin \theta + (D_{1j} - \rho c_j^2) \beta_3 \cos \theta \right]},$$

where,

$$D_{2j} = c_{44} \sin^2 e_j + c_{33} \cos^2 e_j - 2c_{34} \sin e_j \cos e_j,$$

$$D_{1j} = c_{22} \sin^2 e_j + c_{44} \cos^2 e_j - 2c_{24} \sin e_j \cos e_j,$$

$$L_{1j} = c_{24} \sin^2 e_j + c_{34} \cos^2 e_j - (c_{23} + c_{44}) \sin e_j \cos e_j.$$

For $j = 4, 5, 6$,

$$F_j = -\frac{\left(D_{2j} - \rho c_j^2\right) \beta_2 \sin \theta - L_{1j} \beta_3 \cos \theta}{\left(D_{1j} - \rho c_j^2\right) \beta_3 \cos \theta - L_{1j} \beta_2 \sin \theta},$$

$$F_j^* = \frac{\left(D_{1j} - \rho c_j^2\right) \left(D_{2j} - \rho c_j^2\right) - L_{1j}^2}{\left(\frac{i}{k}\right) \tau' \left[\left(D_{1j} - \rho c_j^2\right) \beta_3 \cos \theta - \beta_2 \sin \theta L_{1j}\right]},$$

where,

$$D_{2j} = c_{44} \sin^2 e_j + c_{33} \cos^2 e_j + 2c_{34} \sin e_j \cos e_j,$$

$$D_{1j} = c_{22} \sin^2 e_j + c_{44} \cos^2 e_j + 2c_{24} \sin e_j \cos e_j,$$

$$L_{1j} = c_{24} \sin^2 e_j + c_{34} \cos^2 e_j + (c_{23} + c_{44}) \sin e_j \cos e_j.$$

Here, the subscript $j = 1$ corresponds to incident qP waves, $j = 2$ corresponds to incident qT waves, $j = 3$ corresponds to incident qSV waves, $j = 4$ corresponds to reflected qP waves, $j = 5$ corresponds to reflected qT waves, $j = 6$ corresponds to reflected qSV waves, $j = 7$ corresponds to transmitted qP waves, $j = 8$ corresponds to transmitted qT waves and $j = 9$ corresponds to transmitted qSV . The boundary conditions at $z = 0$ given by equation (12) must be satisfied for all values of y , then we have:

$$(16) \quad P_1(y, 0, t) = P_2(y, 0, t) = P_3(y, 0, t) = P_4(y, 0, t) = P_5(y, 0, t) \\ = P_6(y, 0, t) = P_7(y, 0, t) = P_8(y, 0, t) = P_9(y, 0, t)$$

which can be written as:

$$(17) \quad \frac{\sin e_1}{c_1(e_1)} = \frac{\sin e_2}{c_2(e_2)} = \frac{\sin e_3}{c_3(e_3)} = \frac{\sin e_4}{c_4(e_4)} = \frac{\sin e_5}{c_5(e_5)} \\ = \frac{\sin e_6}{c_6(e_6)} = \frac{\sin e_7}{c_7(e_7)} = \frac{\sin e_8}{c_8(e_8)} = \frac{\sin e_9}{c_9(e_9)}$$

From equations (12) to (17), we have

$$(18) \quad a_1 B_1 + a_2 B_2 + a_3 B_3 + a_4 B_4 + a_5 B_5 + a_6 B_6 + a_7 B_7 + a_8 B_8 + a_9 B_9 = 0, \\ b_1 B_1 + b_2 B_2 + b_3 B_3 + b_4 B_4 + b_5 B_5 + b_6 B_6 + b_7 B_7 + b_8 B_8 + b_9 B_9 = 0, \\ d_1 B_1 + d_2 B_2 + d_3 B_3 + d_4 B_4 + d_5 B_5 + d_6 B_6 + d_7 B_7 + d_8 B_8 + d_9 B_9 = 0, \\ f_1 B_1 + f_2 B_2 + f_3 B_3 + f_4 B_4 + f_5 B_5 + f_6 B_6 + f_7 B_7 + f_8 B_8 + f_9 B_9 = 0, \\ g_1 B_1 + g_2 B_2 + g_3 B_3 + g_4 B_4 + g_5 B_5 + g_6 B_6 + g_7 B_7 + g_8 B_8 + g_9 B_9 = 0, \\ h_1 B_1 + h_2 B_2 + h_3 B_3 + h_4 B_4 + h_5 B_5 + h_6 B_6 + h_7 B_7 + h_8 B_8 + h_9 B_9 = 0,$$

where the expressions for a_l, b_l, d_l, f_l, g_l and h_l are given in Appendix.

The system of equations (18) can be reduced for different boundaries by taking:

- (i) $K_n \neq 0, K_t \rightarrow \infty, K_c \rightarrow \infty$, for Normal stiffness boundary,
- (ii) $K_n \rightarrow \infty, K_t \neq 0, K_c \rightarrow \infty$, for Transverse stiffness boundary,
- (iii) $K_n \rightarrow \infty, K_t \rightarrow \infty, K_c \neq 0$, for Thermal contact conductance,
- (iv) $K_n \rightarrow \infty, K_t \rightarrow 0, K_c \rightarrow \infty$, for Slip boundary,
- (v) $K_n \rightarrow \infty, K_t \rightarrow \infty, K_c \rightarrow \infty$, for Welded contact.

5. Particular cases

- (a) If we take, $c_{24} = c_{34} = c'_{24} = c'_{34} = 0$, the system of equations (18) reduces for imperfectly bounded interface between two dissimilar orthotropic thermoelastic half-spaces as discussed by Kumar and Singh [12], where we have found minor but significant error in the coefficients of their equations (28) to (33).
- (b) If we take, $K_1 = K_3 = K'_1 = K'_3 = 0, \beta_1 = \beta_3 = \beta'_1 = \beta'_3 = 0$, the system of equations (18) reduces to for imperfectly bounded interface between two dissimilar monoclinic elastic half-spaces. If furthermore, we take $K_n \rightarrow \infty, K_t \rightarrow \infty$, the above results agree with Khurana and Singh [18].
- (c) If we take, $c_{23} = c_{33} - 2c_{44}c'_{23} = c'_{33} - 2c'_{44}, c_{24} = c_{34} = c'_{24} = c'_{34} = 0$, the system of equations (18) reduces for imperfectly bonded interface between two dissimilar transversely isotropic thermoelastic half-spaces.
- (d) If we take, $c_{22} = c_{33} = \lambda + 2\mu, c_{13} = c_{23} = c_{12} = \lambda, c_{44} = c_{55} = c_{66} = \mu, c_{14} = c_{24} = c_{34} = c_{56} = 0, \beta_2 = \beta_3 = \beta = (3\lambda + 2\mu)\alpha_t, \alpha_2 = \alpha_3 = \alpha_t, K_2 = K_3 = K, c'_{22} = c'_{23} = \lambda' + 2\mu', c'_{13} = c'_{23} = c'_{12} = \lambda', c'_{44} = c'_{55} = c'_{66} = \mu', c'_{14} = c'_{24} = c'_{34} = c'_{56} = 0, \beta'_2 = \beta'_3 = \beta' = (3\lambda' + 2\mu')\alpha'_t, \alpha'_2 = \alpha'_3 = \alpha'_t, K'_2 = K'_3 = K'$, the system of equation (18) reduces for imperfectly bonded interface between two dissimilar isotropic thermoelastic half-spaces. If we take further $K_n \rightarrow \infty, K_t \rightarrow \infty, K_c \rightarrow \infty$ the above case agrees with Sinha and Elsibai [19, 20].

6. Procedure for computing reflected and refracted angles

The reflection coefficients depend on the velocities $c_i (e_i), i = 1, 2, 3, 4, 5, 6$, which are the functions of reflected and refracted angles. They are

supposed to be known for an incident qP wave, e_1 and, therefore $c_1(e_1)$. One has to compute e_4 , e_5 and e_6 for given e_1 . The velocities $c_4(e_4)$, $c_5(e_5)$ and $c_6(e_6)$ can then be computed from explicit algebraic formulae. The procedure is given below for computing e_4 , e_5 and e_6 for given e_1 in the case of incident qP wave, for given e_2 in the case of incident qT wave, and for given e_3 in the case of incident qSV wave.

Equation (11) can be written as

$$(19) \quad A_0 c^6 + A_1 c^4 + A_2 c^2 + A_3 = 0.$$

Here,

$$\begin{aligned} A_0 &= \rho^3 \tau^*, \quad A_1 = \rho^2 \left[U \tau^* + Z \tau^* + \frac{W}{C_e} + \varepsilon \eta (\bar{\beta}^2 p_3^2 + p_2^2) \right], \\ A_2 &= \rho \left[(UZ - V^2) \tau^* + \frac{UW}{C_e} + \frac{ZW}{C_e} + \varepsilon \eta (\bar{\beta}^2 U p_3^2 + Z p_2^2) - 2V \in \eta \bar{\beta} p_2 p_3 \right], \\ A_3 &= -\frac{W}{C_e} (UZ - V^2), \end{aligned}$$

where,

$$\begin{aligned} U &= c_{22} p_2^2 + c_{44} p_3^2 + 2c_{24} p_2 p_3, & Z &= c_{44} p_2^2 + c_{33} p_3^2 + 2c_{34} p_2 p_3, \\ V &= c_{44} p_2^2 + c_{34} p_3^2 + (c_{23} + c_{24}) p_2 p_3, & W &= K_3 p_3^2 + K_2 p_2^2. \end{aligned}$$

We define dimensionless apparent velocity \bar{c} by the relation:

$$(20) \quad \bar{c} = c_a / \beta = c / p_2 \beta,$$

where,

$$\beta = (c_{44} / \rho)^{\frac{1}{2}}, \quad \rho c^2 = p_2^2 c_{44} \bar{c}^2.$$

With the help of relation (20), the equation (19) becomes

$$(21) \quad \begin{aligned} \bar{c}^6 &- \left[(\bar{U} + \bar{Z}) \tau^* + \frac{\bar{W}}{C_e} + \frac{\varepsilon \eta}{c_{44}^2} (\bar{\beta}^2 p^2 + 1) \right] \frac{\bar{c}^4}{\tau^*} \\ &+ \left[(\bar{U} \bar{Z} - V^2) \tau^* + (\bar{U} + \bar{Z}) \frac{\bar{W}}{C_e} + \frac{\varepsilon \eta}{c_{44}^2} (\bar{\beta}^2 \bar{U} p^2 + \bar{Z}) - \frac{2\varepsilon \eta \bar{\beta}}{c_{44}^2} \bar{V} p \right] \frac{\bar{c}^2}{\tau^*} \\ &- \frac{\bar{W}}{C_e} (\bar{U} \bar{Z} - \bar{V}^2) \frac{1}{\tau^*} = 0. \end{aligned}$$

Here,

$$(22) \quad \begin{aligned} \bar{U} &= \bar{c}_{22} + p^2 + 2\bar{c}_{24}p, \quad \bar{Z} = 1 + \bar{c}_{33}p^2 + 2\bar{c}_{34}p, \\ \bar{V} &= \bar{c}_{24} + \bar{c}_{34}p^2 + (\bar{c}_{23} + 1)p, \quad \bar{W} = \bar{K}_3p^2 + \bar{K}_2, \end{aligned}$$

where,

$$\bar{c}_{ij} = \frac{c_{ij}}{c_{44}}, \quad \bar{K}_3 = \frac{K_3}{c_{44}}, \quad \bar{K}_2 = \frac{K_2}{c_{44}}, \quad p = \frac{p_3}{p_2}.$$

For incident qP waves, $p = -\cot e_1$; for incident qT waves, $p = -\cot e_2$; for incident qSV waves, $p = -\cot e_3$; for reflected qP waves, $p = -\cot e_4$; for reflected qT waves, $p = \cot e_5$; for reflected qSV waves, $p = -\cot e_6$. For a given p , equation (21) can be solved for \bar{c}^3 , the three roots corresponding to qP , qT and qSV waves. However, for a given \bar{c} , equation (21) is a six degree equation in p , corresponding to incident qP , qT and qSV , reflected qP , qT and qSV . The positive roots corresponding to the reflected waves and the negative roots corresponding to the incident waves. On inserting the expressions for \bar{U} , \bar{Z} , \bar{V} and \bar{W} from equation (22) into equation (21), we obtain the six degree equation in p as:

$$(23) \quad G_0p^6 + G_1p^5 + G_2p^4 + G_3p^3 + G_4p^2 + G_5p + G_6 = 0.$$

The expressions for G_0 , G_1 , G_2 , G_3 , G_4 , G_5 and G_6 are given in Appendix. If we neglect thermal effects, the equation (23) reduces to the equation (48) obtained by Singh and Khurana [5]. Now, we define $q = \frac{1}{p}$, The equation (23) transforms into

$$(24) \quad G_6q^6 + G_5q^5 + G_4q^4 + G_3q^3 + G_2q^2 + G_1q + G_0 = 0.$$

Equation will possess three positive roots for angles of incidence, for which all three reflected qP , qSV and qT waves exist, the smaller positive root (say q_6) corresponding to reflected qT and the root (q_5) corresponding to reflected qSV and the larger positive root (q_4) corresponding to reflected qP . Further, $e_4 = \tan^{-1}(q_4)$, $e_5 = \tan^{-1}(q_5)$, $e_6 = \tan^{-1}(q_6)$.

For an isotropic thermoelastic medium, if we take:

$$\begin{aligned} c_{14} = c_{24} = c_{34} = 0, \quad c_{11} = c_{22} = c_{33} = \lambda + 2\mu, \quad c_{44} = c_{55} = c_{66} = \mu, \\ c_{12} = c_{13} = c_{23} = \lambda, \quad \bar{K}_2 = \bar{K}_3 = \bar{K}, \quad \beta_2 = \beta_3 = \beta \end{aligned}$$

Then,

$$\begin{aligned}
G_0 &= -\frac{\bar{K}}{C_e\tau^*}\gamma, \quad G_1 = 0, \\
G_2 &= \left\{ \gamma + \frac{\bar{K}}{C_e\tau^*}(1 + \gamma) + \frac{\varepsilon\eta}{\mu\tau^*} \right\} \bar{c}^2 + \frac{\bar{K}}{C_e\tau^*}(-3\gamma), \quad G_3 = 0, \\
G_4 &= -\left\{ 1 + \gamma + \frac{\bar{K}}{C_e\tau^*} + \frac{\varepsilon\eta}{\mu\tau^*} \right\} \bar{c}^4 + \left\{ 2\gamma + \frac{2\bar{K}}{C_e\tau^*}(1 + \gamma) + \frac{2\varepsilon\eta}{\mu\tau^*} \right\} \bar{c}^2 - \frac{3\bar{K}}{C_e\tau^*}\gamma, \\
G_5 &= 0, \\
G_6 &= \bar{c}^6 - \left(1 + \gamma + \frac{\bar{K}}{C_e\tau^*} + \frac{\varepsilon\eta}{\mu\tau^*} \right) \bar{c}^4 + \left(\gamma + \frac{\bar{K}}{C_e\tau^*}(1 + \gamma) + \frac{\varepsilon\eta}{\mu\tau^*} \right) \bar{c}^2 - \frac{\bar{K}}{C_e\tau^*}\gamma,
\end{aligned}$$

and, the equation (23) reduces to

$$(25) \quad -s\gamma(p^2 - \bar{c}^2 + 1) \left(p^2 - \frac{\alpha_1\bar{c}^2}{\gamma} + 1 \right) \left(p^2 - \frac{\beta_1\bar{c}^2}{\gamma} + 1 \right) = 0,$$

where

$$\alpha_1 = \frac{r + \sqrt{r^2 - 4s\gamma}}{2s}, \quad \beta_1 = \frac{r - \sqrt{r^2 - 4s\gamma}}{2s}, \quad r = \gamma + \frac{\bar{K}}{C_e\tau^*} + \frac{\varepsilon\eta}{\mu\tau^*}, \quad s = \frac{\bar{K}}{C_e\tau^*}.$$

In this case, the Snell's law becomes:

$$(26) \quad \frac{\sin e_1}{v_{qP}} = \frac{\sin e_2}{v_{qSV}} = \frac{\sin e_3}{v_{qT}}.$$

Therefore, the roots of equation (25) are given by $p^2 = \bar{c}^2 - 1 = \cot^2 e_3$ corresponding to qSV waves, $p^2 = \frac{\bar{c}^2}{\gamma}\alpha_1 - 1 = \cot^2 e_1$, corresponding to qP wave and $p^2 = \frac{\bar{c}^2}{\gamma}\beta_1 - 1 = \cot^2 e_2$ corresponding to qT wave. Thus, we may choose $q = \frac{1}{p}$:

$$(27) \quad q_1 = -\tan e_1, \quad q_2 = -\tan e_2, \quad q_3 = -\tan e_3, \\ q_4 = \tan e_1, \quad q_5 = \tan e_2, \quad q_6 = \tan e_3, \dots$$

as the six roots of the equation (24) for an isotropic medium. This choice will act as a guiding factor in computing the angles of reflection of qP , qT and qSV

waves in a monoclinic thermoelastic medium. For an orthotropic medium, it can be shown that $g_1 = g_3 = g_5 = 0$ in equation (24). Therefore, equation (24) reduces to a cubic equation in q^2 . Thus we may choose

$$q_1 = -q_4, \quad q_2 = -q_5, \quad q_3 = -q_6,$$

so that the angle of reflection of qP , qT and qSV waves is equal to the angle of incidence of qP , qT and qSV waves, respectively. This is not true for a monoclinic thermoelastic medium. A similar analysis can be derived to compute the angles of refraction of qP , qSV and qT waves in the upper monoclinic half-space for given incident angle.

7. Conclusion

A problem of reflection and transmission is considered at an imperfectly bonded interface between two dissimilar thermoelastic solid half-spaces of monoclinic type, where we have obtained the six relations between amplitudes of incident, reflected and transmitted waves. These relations are verified theoretically for some limiting cases. A procedure for computing the reflected and refracted angles for a given incident angle of a striking wave is explained in detail. It is concluded that the angle of particular reflected wave is not equal to the angle of incidence of that wave in a monoclinic medium. Numerical verification of present theoretical results can be attempted by following the procedure given in the previous section.

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Appendix

 The expressions for a_l , b_l , d_l , f_l , g_l and h_l are

$$a_\ell = -iK_n, \quad (\ell = 1, \dots, 6)$$

$$a_\ell = - \left[-iK_n - c'_{23} F_\ell \frac{\omega \sin e_\ell}{c_\ell} + c'_{33} \frac{\omega \cos e_\ell}{c_\ell} + c_{34} \frac{\omega \cos e_\ell}{c_\ell} F_\ell - c'_{34} \frac{\omega \sin e_\ell}{c_\ell} + i\beta'_3 \tau'_1 F_\ell^* \right],$$

$$(\ell = 7, \dots, 9)$$

$$b_\ell = -iK_t F_\ell, \quad (\ell = 1, \dots, 6)$$

$$b_\ell = - \left[-iK_t F_\ell - c'_{24} \frac{\omega \sin e_\ell}{c_\ell} F_\ell + c'_{34} \frac{\omega \cos e_\ell}{c_\ell} + c'_{44} \frac{\omega \cos e_\ell}{c_\ell} F_\ell - c'_{44} \frac{\omega \sin e_\ell}{c_\ell} \right],$$

$$(\ell = 7, \dots, 9)$$

$$d_\ell = -iK_c F_\ell^*, \quad (\ell = 1, \dots, 6), \quad d_\ell = - \left[-iK_c F_\ell^* + K'_3 \frac{\omega \cos e_\ell}{c_\ell} F_\ell^* \right], \quad (\ell = 7, \dots, 9)$$

$$f_\ell = -c_{23} \frac{\omega \sin e_\ell}{c_\ell} F_\ell + c_{33} \frac{\omega \cos e_\ell}{c_\ell} + c_{34} \frac{\omega \cos e_\ell}{c_\ell} F_\ell - c_{34} \frac{\omega \sin e_\ell}{c_\ell} + i\beta_3 \tau_1 F_\ell^*,$$

$$(\ell = 1, \dots, 3)$$

$$f_\ell = -c_{23} \frac{\omega \sin e_\ell}{c_\ell} F_\ell - c_{33} \frac{\omega \cos e_\ell}{c_\ell} - c_{34} \frac{\omega \cos e_\ell}{c_\ell} F_\ell - c_{34} \frac{\omega \sin e_\ell}{c_\ell} + i\beta_3 \tau_1 F_\ell^*,$$

$$(\ell = 4, \dots, 6)$$

$$f_\ell = c'_{23} \frac{\omega \sin e_\ell}{c_\ell} F_\ell - c'_{33} \frac{\omega \cos e_\ell}{c_\ell} - c'_{34} \frac{\omega \cos e_\ell}{c_\ell} F_\ell + c'_{34} \frac{\omega \sin e_\ell}{c_\ell} - i\beta'_3 \tau'_1 F_\ell^*,$$

$$(\ell = 7, \dots, 9)$$

$$g_\ell = -c_{24} \frac{\omega \sin e_\ell}{c_\ell} F_\ell + c_{34} \frac{\omega \cos e_\ell}{c_\ell} + c_{44} \frac{\omega \cos e_\ell}{c_\ell} F_\ell - c_{44} \frac{\omega \sin e_\ell}{c_\ell}, \quad (\ell = 1, \dots, 3)$$

$$g_\ell = -c_{24} \frac{\omega \sin e_\ell}{c_\ell} F_\ell - c_{34} \frac{\omega \cos e_\ell}{c_\ell} - c_{44} \frac{\omega \cos e_\ell}{c_\ell} F_\ell - c_{44} \frac{\omega \sin e_\ell}{c_\ell}, \quad (\ell = 4, \dots, 6)$$

$$g_\ell = c'_{24} \frac{\omega \sin e_\ell}{c_\ell} F_\ell - c'_{34} \frac{\omega \cos e_\ell}{c_\ell} - c'_{44} \frac{\omega \cos e_\ell}{c_\ell} F_\ell + c'_{44} \frac{\omega \sin e_\ell}{c_\ell}, \quad (\ell = 7, \dots, 9)$$

$$h_\ell = K_3 F_\ell^* \frac{\omega \cos e_\ell}{c_\ell}, \quad (\ell = 1, \dots, 3), \quad h_\ell = -K_3 F_\ell^* \frac{\omega \cos e_\ell}{c_\ell}, \quad (\ell = 4, \dots, 6)$$

$$h_\ell = -K'_3 F_\ell^* \frac{\omega \cos e_\ell}{c_\ell}, \quad (\ell = 7, \dots, 9)$$

$$\tau'_1 = (1 + i\omega t'_1), \quad \tau_1 = (1 + i\omega t_1)$$

The expressions for $G_0, G_1, G_2, G_3, G_4, G_5$ and G_6 are

$$\begin{aligned} G_0 &= \frac{\bar{K}_3}{C_e \tau^*} (\bar{c}_{34}^2 - \bar{c}_{33}), \quad G_1 = \frac{2\bar{K}_3}{C_e \tau^*} (\bar{c}_{34}\bar{c}_{23} - \bar{c}_{24}\bar{c}_{33}) \\ G_2 &= \left(\bar{c}_{33} - \bar{c}_{34}^2 + \frac{\bar{K}_3}{C_e \tau^*} (1 + \bar{c}_{33}) + \frac{\varepsilon\eta}{c_{44}\tau^*} \bar{\beta}^2 \right) \bar{c}^2 \\ &\quad - \frac{\bar{K}_3}{C_e \tau^*} (\bar{c}_{22}\bar{c}_{33} - 2\bar{c}_{24}\bar{c}_{34} - \bar{c}_{23}^2 - 2\bar{c}_{23}) - \frac{\bar{K}_2}{C_e \tau^*} (\bar{c}_{33} - \bar{c}_{34}^2) \\ G_3 &= 2 \left[\left\{ \bar{c}_{24}\bar{c}_{33} - \bar{c}_{34}\bar{c}_{23} + \frac{\bar{K}_3}{C_e \tau^*} (\bar{c}_{24} + \bar{c}_{34}) + \frac{\varepsilon\eta\bar{\beta}^2}{c_{44}\tau^*} \bar{c}_{24} - \frac{\varepsilon\eta\bar{\beta}}{c_{44}\tau^*} \bar{c}_{34} \right\} \bar{c}^2 \right. \\ &\quad \left. - \frac{\bar{K}_3}{C_e \tau^*} (\bar{c}_{22}\bar{c}_{34} - \bar{c}_{24}\bar{c}_{23}) - \frac{\bar{K}_2}{C_e \tau^*} (\bar{c}_{24}\bar{c}_{33} - \bar{c}_{34}\bar{c}_{23}) \right] \\ G_4 &= - \left\{ (1 + \bar{c}_{33}) + \frac{\bar{K}_3}{C_e \tau^*} + \frac{\varepsilon\eta}{c_{44}\tau^*} \bar{\beta}^2 \right\} \bar{c}^4 + \left\{ \bar{c}_{22}\bar{c}_{33} + 1 + 2\bar{c}_{24}\bar{c}_{34} - (1 + \bar{c}_{23})^2 \right. \\ &\quad \left. + \frac{\bar{K}_3}{C_e \tau^*} (\bar{c}_{22} + 1) + \frac{\bar{K}_2}{C_e \tau^*} (1 + \bar{c}_{33}) + \frac{\varepsilon\eta\bar{\beta}^2}{c_{44}\tau^*} \bar{c}_{22} - 2\frac{\varepsilon\eta\bar{\beta}}{c_{44}\tau^*} (1 + \bar{c}_{23}) + \frac{\varepsilon\eta}{c_{44}\tau^*} \bar{c}_{33} \right\} \bar{c}^2 \\ &\quad + \frac{\bar{K}_3}{C_e \tau^*} (\bar{c}_{24}^2 - \bar{c}_{22}) + \frac{\bar{K}_2}{C_e \tau^*} \left((1 + \bar{c}_{23})^2 - \bar{c}_{22}\bar{c}_{33} - 1 + 2\bar{c}_{24}\bar{c}_{34} \right) \\ G_5 &= 2 \left[-(\bar{c}_{24} + \bar{c}_{34})\bar{c}^4 + \left\{ \bar{c}_{22}\bar{c}_{34} - \bar{c}_{24}\bar{c}_{23} + \frac{\bar{K}_2}{C_e \tau^*} (\bar{c}_{24} + \bar{c}_{34}) + \frac{\varepsilon\eta}{c_{44}\tau^*} (\bar{c}_{34} - \bar{c}_{24}\bar{\beta}) \right\} \bar{c}^2 \right. \\ &\quad \left. + \frac{\bar{K}_2}{C_e \tau^*} (\bar{c}_{24}\bar{c}_{23} - \bar{c}_{22}\bar{c}_{34}) \right] \\ G_6 &= \bar{c}^6 - \left\{ (1 + c_{22}) + \frac{\bar{K}_2}{C_e \tau^*} + \frac{\varepsilon\eta}{c_{44}\tau^*} \right\} \bar{c}^4 + \left\{ (\bar{c}_{22} - \bar{c}_{24}^2) + \frac{\bar{K}_2}{C_e \tau^*} (\bar{c}_{22} + 1) + \frac{\varepsilon\eta}{c_{44}\tau^*} \right\} \bar{c}^2 \\ &\quad + \frac{\bar{K}_2}{C_e \tau^*} (\bar{c}_{24}^2 - \bar{c}_{22}) \end{aligned}$$