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GENERAL MECHANICS

ON TRAVELING WAVES IN LATTICES: THE CASE OF RICCATI LATTICES

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Abstract. The method of simplest equation is applied for analysis of a class of lattices described by differential-difference equations that admit traveling-wave solutions constructed on the basis of the solution of the Riccati equation. We denote such lattices as Riccati lattices. We search for Riccati lattices within two classes of lattices: generalized Lotka–Volterra lattices and generalized Holling lattices. We show that from the class of generalized Lotka–Volterra lattices only the Wadati lattice belongs to the class of Riccati lattices. Opposite to this many lattices from the Holling class are Riccati lattices. We construct exact traveling wave solutions on the basis of the solution of Riccati equation for three members of the class of generalized Holling lattices.
Key words: nonlinear differential-difference equations, method of simplest equation, exact traveling-wave solutions, Lotka–Volterra lattices, Holling lattices, Wadati lattice, Riccati lattices.

1. Introduction

Nonlinear models are used extensively in the research on complex systems [1]–[8]. In many cases, the models consist of nonlinear partial differential equations and it is of great interest to obtain exact analytical solutions of these nonlinear PDEs. Such solutions are useful as initial conditions in the process of obtaining of numerical solutions. In addition, the exact solutions describe important classes of waves and processes in the investigated systems. The researches based on nonlinear PDEs increases steadily and now they are much applied in the theory of solitons [9]–[11], biology [12], theory of dynamical systems, chaos theory and ecology [13]–[16], hydrodynamics and theory of turbulence [17]–[25], in the mathematical social dynamics [26, 27], etc. The inverse scattering transform and the method of Hirota [28]–[31] are famous methods for obtaining exact soliton solutions of various NPDEs. In addition, in the last several years approaches for obtaining exact special solutions of nonlinear PDE have been developed, too [32]–[38]. Exact solutions of many equations have been obtained by means of these approaches such as for an example the Kuramoto-Shivasinsky equation [35], [39]–[41], Sine–Gordon equation [42]–[51], equations, connected to the models of population dynamics [52]–[62], sinh-Gordon or Poisson–Boltzmann equation [63], Lorenz-like systems [64], or water waves [65]–[69].

The discussion below will be devoted to the application of the modified method of simplest equation for obtaining exact and approximate solutions of nonlinear differential–difference equations. The differential–difference equations are much used to describe different processes in complex discrete systems in physics, biology, engineering, etc. We shall discuss below the use of such equations for description of waves in lattices conected to ecological food chains. The method of simplest equation has been established by Kudryashov [41, 61], [70]–[73] on the basis of a procedure analogous to the first step of the test for the Painleve property [74]. The modified method of simplest equation [35, 38, 62] is a simpler for use version of this method where the above-mentioned procedure is substituted by the concept for the balance equation. Modified method of simplest equation is already applied for obtaining exact traveling wave solutions of nonlinear PDEs such as versions of generalized Kuramoto - Sivashinsky equation, reaction–diffusion equation, reaction–telegraph equation [35], [59] generalized Swift–Hohenberg equation and generalized Rayleigh equation [38], generalized Fisher equation, generalized Huxley equation [62], generalized Degasperis–Processi equation and b-equation [75], and to numerous nonlinear PDEs and ODEs [76].

The organization of the paper is as follows. In Section 2 we define the class of the discussed lattices – the Riccati lattices. We shall search for Riccati lattices among the members of two classes of lattices: the class of generalized Lotka-Volterra lattices and the class of the generalized Holling lattices. Section 3 is devoted to a brief description of the modified method of simplest equation. In Section 4 the method is applied to the differential–difference equations describing the generalized Lotka–Volterra lattices. It is shown that from this class of lattices only the generalized Wadati lattice belongs to the class of Riccati lattices. Sect. 5 is devoted to obtaining traveling-wave solutions of the differential - difference equations that describe lattices from the class of generalized Holling lattices. Several concluding remarks are summarized in Section 6.

2. Riccati lattices

2.1. Riccati lattices and Riccati equation

We shall denote as Riccati lattices the class of lattices that admit traveling-wave solutions obtained by the method of simplest equation on the basis of the use of the Riccati equation as simplest equation. The equation of Riccati is:

$$\frac{d\Phi}{d\xi} = b^2 - \Phi^2. \tag{2.1}$$

The solution of (2.1) is

$$\Phi(\xi) = b \tanh[b(\xi + \xi_0)]. \tag{2.2}$$

Here, the following notes are in order:

[1.] Another form of the Riccati equation is:

$$\frac{d\tilde{\Phi}}{d\xi} = \tilde{b}^2 - \tilde{a}^2\tilde{\Phi}^2. \tag{2.3}$$

Eq. (2.3) has the solution:

$$\tilde{\Phi}(\xi) = \frac{\tilde{b}}{\tilde{a}} \tanh[\tilde{a}\ \tilde{b}\ (\xi + \xi_0)], \tag{2.4}$$

where $\tilde{a}^2\tilde{\Phi}(\xi)^2 < \tilde{b}^2$ and ξ_0 is a constant of integration.
The third form of the Riccati equation is:

$$\frac{d\Psi}{d\xi} = a^*[\Psi(\xi)]^2 + b^*\Psi(\xi) + c^*, \tag{2.5}$$

which has as a solution:

$$\Psi(\xi) = -\frac{b^*}{2a^*} - \frac{\theta}{2a^*} \tanh\left[\frac{\theta(\xi + \xi_0)}{2}\right]. \tag{2.6}$$

In Eq. (2.6) $\theta^2 = b^{*2} - 4a^*c^* > 0$. One can easily check that when $\tilde{\Phi}(\xi) = \Psi(\xi) - \frac{b^*}{2a^*}$ and in addition $a^* = -\tilde{a}^2$ as well as $\tilde{b}^2 = \frac{4a^*c^* - b^{*2}}{4a^*}$, then the equation (2.5) is reduced to the Eq. (2.3) and the solution (2.6) is reduced to the solution (2.4).
The Riccati equation (2.3) can be further reduced to Eq. (2.1). Let

$$\tilde{\Phi} = \frac{1}{\tilde{a}^2}\Phi; \qquad b = \tilde{a}\tilde{b}. \tag{2.7}$$

The substitution of Eq. (2.7) in (2.3) leads to Eq. (2.1) and the solution (2.6) is reduced to the solution (2.2). We shall use Eq. (2.1) below and its solution (2.2).

[2.] The Riccati equation is one of the many possible simplest equations (for other possibilities see for an example [76]). Tanh-function and the Riccati equation are already applied to many lattices described by differential–difference equations. Tanh-function was used for an example for obtaining exact traveling wave solutions of a nonlinear lattice Klein–Gordon model [77]. Tanh-function was used also for obtaining of exact traveling-wave solution of differential–difference equation in [78]. Exact traveling wave solution are obtained [78] for the Ablowitz–Ladik lattice, several variants of the non-relativistic and relativistic Toda lattice for Volterra lattice, for discretized mKdV lattice and for the hybrid (Wadati) lattice. Solution of the Riccati equation was used for investigation of the Wadati lattice equation by Xie and Wang [79]. Recently Aslan [80, 81] used the (G'/G)–method for obtaining exact traveling waves of differential - difference equations. Kudryashov [82] has shown that the (G'/G)–method is equavalent to the method of simplest equation for the case when the equation of Riccati is used as simplest equation.

[3.] The new knowledge this paper adds to the significant amount of research of differential–difference equations is as follows: First, we show that from the class of generalized Lotka–Volterra lattices discussed below only the Wadati lattice belongs also to the class of Riccati lattices. In addition, we discuss exact traveling wave solutions of the class of Holling lattices and show that many of these lattices are Riccati lattices.

2.2. Generalized Lotka–Volterra lattices. Holling lattices

Let us consider a chain of species. The number of each kind of species is $M_1, M_2, \dots$. Let us assume that the number of n-th species M_n increases by collision with $n+1$-th species and decreases by collision with $n-1$-th species. Then we can write:

$$\frac{dM_n}{dt} = M_n(M_{n+1} - M_{n-1}). \tag{2.8}$$

Eq. (2.8) is a simple example of a differential–difference equation that models a Lotka–Volterra lattice. Wadati [83] has discussed the class of lattices:

$$\frac{dM_n}{dt} = (\alpha + \beta M_n + \gamma M_n^2)(M_{n+1} - M_{n-1}), \tag{2.9}$$

which generalizes the Lotka - Volterra latttices of kind Eq. (2.8).

We shall discuss below the following two generalizations of the Wadati and Lotka - Volterra lattices:
(1.) Generalized Lotka–Volterra lattices:

$$\frac{dM_n}{dt} = F(M_n)(M_{n+1} - M_{n-1}), \tag{2.10}$$

where $F(M_n)$ is a polynomial of M_n and
(2.) Generalized Holling Lattices:

$$\frac{dM_n}{dt} = \frac{F(M_n)}{G(M_n)}(M_{n+1} - M_{n-1}) \tag{2.11}$$

where $F(M_n)$ and $G(M_n)$ are polynomials of M_n.

As we can see, Eq. (2.10) is a straightforward generalization of the Wadati lattice equation (2.9). Eq. (2.11) reflects the possibility of Holling functional responce in population dynamics [84]. We shall denote because of this the lattices modeled by this equation as generalized Holling lattices.

We shall study below the conditions which ensure that the lattices described by Eqs. (2.10) and (2.11) belong to the class of Riccati lattices defined above. The basis of our investigation will be the modified method of simplest equation for obtaining exact and approximate solutions of nonlinear PDEs.

3. The modified method of simplest equation

Let us have a partial differential equation and let by means of an appropriate ansatz this equation be reduced to the nonlinear ODE:

$$P\left(F(\xi), \frac{dF}{d\xi}, \frac{d^2F}{d\xi^2}, \dots\right) = 0. \tag{3.1}$$

For large class of equations from the kind (3.1) exact solution can be constructed as finite series:

$$F(\xi) = \sum_{\mu=-\nu}^{\nu_1} a_\mu [\Phi(\xi)]^\mu, \tag{3.2}$$

where $\nu > 0$, $\mu > 0$, p_μ are parameters and $\Phi(\xi)$ is a solution of some ordinary differential equation referred to as the simplest equation. The simplest equation is of lesser order than (3.1) and we know the general solution of the simplest equation or we know at least exact analytical particular solution(s) of the simplest equation [70, 71].

The modified method of simplest equation can be applied to nonlinear partial differential equations of the kind:

$$E\left(\frac{\partial^{\omega_1}F}{\partial x^{\omega_1}}, \frac{\partial^{\omega_2}F}{\partial t^{\omega_2}}, \frac{\partial^{\omega_3}F}{\partial x^{\omega_4}\partial t^{\omega_5}}\right) = G(F), \tag{3.3}$$

where $\omega_3 = \omega_4 + \omega_5$ and

1. $\frac{\partial^{\omega_1}F}{\partial x^{\omega_1}}$ denotes the set of derivatives:
$$\frac{\partial^{\omega_1}F}{\partial x^{\omega_1}} = \left(\frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial x^2}, \frac{\partial F^3}{\partial x^3}, \dots\right).$$
2. $\frac{\partial^{\omega_2}F}{\partial t^{\omega_2}}$ denotes the set of derivatives:
$$\frac{\partial^{\omega_2}F}{\partial t^{\omega_2}} = \left(\frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial t^2}, \frac{\partial f^3}{\partial t^3}, \dots\right).$$
3. $\frac{\partial^{\omega_3}F}{\partial x^{\omega_4}\partial t^{\omega_5}}$ denotes the set of derivatives:
$$\frac{\partial^{\omega_3}F}{\partial x^{\omega_4}\partial t^{\omega_5}} = \left(\frac{\partial^2 F}{\partial x \partial t}, \frac{\partial^3 F}{\partial x^2 \partial t}, \frac{\partial F^3}{\partial x \partial t^2}, \dots\right).$$
4. $G(F)$ can be:
   (a) polynomial of F or;
   (b) function of F which can be reduced to polynomial of F by means of Taylor series for small values of F.
5. The function E can be an arbitrary sum of products of arbitrary number of its arguments. Each argument in each product can have arbitrary power. Each of the products can be multiplied by a function of F which can be:
   (a) polynomial of F or;
   (b) function of F which can be reduced to polynomial of F by means of Taylor series for small values of F.

The modified method of simplest equation for this class of equations allows us in principle to search for:

1. Exact traveling-wave solutions of (3.3) if $G(F)$ and the multiplication functions form item 5. above are polynomials;

2. Approximate traveling-wave solutions for small F in all other cases.

The application of the modified method of simplest equation is based on the following steps:

- The solved class of NPDE of kind (3.3) is reduced to a class of nonlinear ODEs of the kind (3.1) by means of an appropriate ansatz (for an example the traveling-wave ansatz);
- The finite-series solution (3.2) is substituted in (3.1) and as a result a polynomial of $\Phi(\xi)$ is obtained. Eq. (3.2) is a solution of (3.1) if all coefficients of the obtained polynomial of $\Phi(\xi)$ are equal to 0;
- One ensures by means of a balance equation that there are at least two terms in the coefficient of the highest power of $\Phi(\xi)$. The balance equation gives a relationship between the parameters of the solved class of equations and the parameters of the solution;
- The application of the balance equation and the equalizing the coefficients of the polynomial of $\Phi(\xi)$ to 0 leads to a system of nonlinear relationships among the parameters of the solution and the parameters of the solved class of equation;
- Each solution of the obtained system of nonlinear algebraic equations leads to a solution of a nonlinear PDE from the investigated class of nonlinear PDEs.

4. The uniqueness of the Wadati lattice

Let us apply the modified method of simplest equation to the lattice equation (2.10). We are interested in traveling waves and introduce the traveling - wave coordinate $\xi_n = c\,t + d\,n + \xi_0$, where c, d and ξ_0 are parameters. After the substitution of the traveling - wave coordinate in Eq. (2.10), we obtain the lattice equation:

$$c\frac{dM_n}{d\xi_n} - F(M_n)\ [M_{n+1} - M_{n-1}] = 0. \tag{4.1}$$

As we are interested in the Riccati lattices we search the traveling - wave solution of Eq. (4.1) as a sum of powers of the solution of the Riccati equation:

$$M_n(\xi_n) = \sum_{k=0}^{K} a_k[\Phi(\xi_n)]^k; \qquad \frac{d\Phi}{d\xi_n} = b^2 - [\Phi(\xi_n)]^2. \tag{4.2}$$

For the polynomial $F(M_n)$ we assume:

$$F(M_n) = \sum_{l=0}^{L} c_l M_n^l = \sum_{l=0}^{L}\left[\sum_{k=0}^{K} a_k \Phi^k\right]^l. \tag{4.3}$$

The substitution of Eqs. (4.2), (4.3) in Eq. (4.1) leads to the following equation:

$$c[b^2 - \sigma^2\Phi^2]^K \sum_{k=0}^{K} (kb^2 a_k \Phi^{k-1} - k a_k \Phi^{k+1}) - \sum_{l=0}^{L} \left[\sum_{k=0}^{K} a_k \Phi^k\right]^l \times \left\{ \sum_{k=0}^{K} a_k b^k [(b+\sigma\Phi)^{K-k}(b-\sigma\Phi)^K (b\sigma+\Phi)^k - (b+\sigma\Phi)^K (b-\sigma\phi)^{K-k}(\Phi - b\sigma)^k] \right\} = 0, \tag{4.4}$$

where $\sigma = \tanh(b\ d)$.

Let us now derive the balance equation for Eq. (4.4). The maximum powers connected to the different groups of terms in Eq. (4.4) are as follows:

$$\textbf{Term} \;\; c[b^2 - \sigma^2\Phi^2]^K \sum_{k=0}^{K} (kb^2 a_k \Phi^{k-1} - k a_k \Phi^{k+1}) \to 3K+1,$$

$$\textbf{Term} \;\; \sum_{l=0}^{L} \left[\sum_{k=0}^{K} a_k \Phi^k\right]^l \left\{ \sum_{k=0}^{K} a_k b^k (b+\sigma\Phi)^{K-k}(b-\sigma\Phi)^K (b\sigma+\Phi)^k \right\} \to KL+2K,$$

$$\textbf{Term} \;\; \sum_{l=0}^{L} \left[\sum_{k=0}^{K} a_k \Phi^k\right]^l \left\{ \sum_{k=0}^{K} a_k b^k (b+\sigma\Phi)^{K}(b-\sigma\phi)^{K-k} (\Phi - b\sigma)^k \right\} \to KL+2K. \tag{4.5}$$

Thus we have two possibilities for balance equation:

- Balance between the first and the second term in Eq. (4.5):

$$K+1 = KL. \tag{4.6}$$

- Balance between the second and the third term in Eq. (4.5):

$$KL+2K = KL+2K. \tag{4.7}$$

As K and L must be integers and from Eq. (4.6) $L = 1 + \frac{1}{K}$ the balance (4.6) is valid only for the case $K = 1$, $L = 2$. In all other cases the balance has to be (4.7).

It is easily to see that the balance equation (4.7) is not acceptable. The application of this balance equation to Eq. (4.4) leads to terms of the kind

$$\sigma^{2K-k}\Phi^{2K}[(-1)^K - (-1)^{K-k}], \qquad k = 0, 1, \dots, K$$

Except for the case $K = 0$ in all other cases terms arise for which $[(-1)^K - (-1)^{K-k}] \neq 0$. This fact requires $\sigma = 0$ which leads to $d = 0$ which is not acceptable for the discussed problem. We shall discuss because of this below the balance equation (4.6).

As we have mentioned above Eq. (4.6) leads to $K = 1$ and $L = 2$ which is exactly the case of the generalized Wadati lattice. The application of the modified method of simplest equation to Eq. (4.4) with $K = 1$ and $L = 2$ leads to the following system of 5 nonlinear algebraic relationships among the parameters of the equation and the parameters of the solution:

$$\begin{aligned}
\sigma a_1[c\sigma + 2c_2 a_1^2 b] &= 0,\\
2\sigma a_1^2 b[c_1 + 2c_2 a_0] &= 0,\\
a_1 b[-bc(1+\sigma^2) + 2\sigma(c_0 + c_1 a_0 + c_2 a_0^2 - 2c_2 a_1^2 b^2)] &= 0,\\
2\sigma a_1^2 b^3[c_1 + 2c_2 a_0] &= 0,\\
a_1 b^3[bc - 2\sigma(c_0 + c_1 a_0 + c_2 a_0^2)] &= 0.
\end{aligned} \tag{4.8}$$

One solution of this system is:

$$c = \frac{\sigma(4c_0c_2 - c_1^2)}{2bc_2}; \quad a_0 = -\frac{c_1}{2c_2}; \quad a_1 = \frac{\sigma\sqrt{c_1^2 - 4c_0c_2}}{2bc_2}, \tag{4.9}$$

and the corresponding solution of Eq. (4.4) is:

$$M_n(\xi_n) = -\frac{c_1}{2c_2} + \frac{\tanh(bd)\sqrt{c_1^2 - 4c_0c_2}}{2c_2}\tanh\left[b\left(-\frac{(c_1^2 - 4c_0c_2)\tanh(b\ d)\ t}{2bc_2} + d\ n + \xi_0\right)\right]. \tag{4.10}$$

The solution (4.10) has been obtained by different authors. The interesting point is that the discussion of the possible balance equations above have shown the unique position of the Wadati lattice as the only Riccati lattice of the kind (4.2) from the class of the generalized Lotka–Volterra lattices discussed here.

5. Holling lattices

To the best of our knowledge, the class of generalized Holling lattice equations was not discussed up to now. Thus, the obtained below traveling-wave solutions are new.

Let us now discuss Eq. (2.11) where $F(M_n)$ be the same as in (4.3) ,M_n be given by Eq. (4.2). In addition let $G(M_n)$ be:

$$G(M_n) = \sum_{p=0}^{P} d_p M_n^p = \sum_{p=0}^{P} d_p \left[\sum_{k=0}^{K} a_k \Phi^k \right]^p . \tag{5.1}$$

The substitution of Eqs. (4.2), (4.3) and (5.1) in Eq. (2.11) and the switching to the traveling-wave coordinate leads to the equation:

$$\begin{aligned} c[b^2 - \sigma^2 \Phi^2]^K \left[\sum_{p=0}^{P} d_p \left(\sum_{k=0}^{K} a_k \Phi^k \right)^p \right] \sum_{k=0}^{K} (k b^2 a_k \Phi^{k-1} - k a_k \Phi^{k+1}) \\ - \sum_{l=0}^{L} \left[\sum_{k=0}^{K} a_k \Phi^k \right]^l \times \Bigg\{ \sum_{k=0}^{K} a_k b^k [(b+\sigma\Phi)^{K-k} (b - \sigma \Phi)^K (b\sigma + \Phi)^k \\ - (b+\sigma\Phi)^K (b - \sigma\phi)^{K-k} (\Phi - b\sigma)^k] \Bigg\} = 0. \end{aligned} \tag{5.2}$$

Here, we again have two possibilities for balancing equations:

$$KP + K + 1 = KL, \tag{5.3}$$

and

$$KL + 2K = KL + 2K. \tag{5.4}$$

The balance equation (5.4)as in previous section leads to $d = 0$ which is unacceptable for the discussed problem. We shall work on the basis of the balance equation (5.3) because of this below. We note that when $P = 0$ Eq. (5.3) reduces to the balance equation (4.6) from the previous section. In addition, from Eq. (5.3) we obtain $L = P + 1 + \frac{1}{K}$. As L, P and K must be integer then we must set $K = 1$ in Eq. (5.3). We shall discuss below the simplest cases $P = 1$, $P = 2$ and $P = 3$.

5.1. Case $P = 1$, $L = 3$

For this case Eq. (2.11) becomes:

$$\frac{dM_n}{dt} = \frac{c_0 + c_1 M_n + c_2 M_n^2 + c_3 M_n^3}{d_0 + d_1 M_n}(M_{n+1} - M_{n-1}). \tag{5.5}$$

The application of the modified method of simplest equation reduces Eq. (5.5) to the following system of nonlinear algebraic relationships:

$$\begin{aligned}
\sigma a_1^2[c\sigma d_1 + 2c_3 a_1^2 b] &= 0,\\
\sigma a_1[c\sigma(d_0 + d_1 a_0) + 2a_1^2 b(c_2 + 3c_3 a_0)] &= 0,\\
a_1^2 b[2\sigma(2c_2 a_0 + c_1 + 3c_3 a_0^2) - b(2c_3 a_1^2 b\sigma - cd_1 - c\sigma^2 d_1)] &= 0,\\
a_1 b\{\sigma[2(c_0 + c_1 a_0 + c_3 a_0^3 + c_2 a_0^2) - 2b^2(c_2 a_1^2 + 3c_3 a_0 a_1^2)] - &\\
bc[(d_0 + d_1 a_0) - \sigma^2(d_0 + d_1 a_0)]\} &= 0,\\
a_1^2 b^3[bcd_1 - 2\sigma(2c_2 a_0 + c_1 + 3c_3 a_0^2)] &= 0,\\
a_1 b^3[bc(d_0 + d_1 a_0) - 2\sigma(c_0 + c_1 a_0 + c_3 a_0^3 + c_2 a_0^2)] &= 0.
\end{aligned} \tag{5.6}$$

One solution of this system is:

$$\begin{aligned}
c &= -\frac{\sigma(2c_2 c_3 d_0 d_1 + c_2^2 d_1^2 - 4c_1 c_3 d_1^2 - 3c_3^2 d_0^2)}{2bc_3 d_1^3},\\
a_1 &= \frac{\sigma\sqrt{2c_2 c_3 d_0 d_1 + c_2^2 d_1^2 - 4c_1 c_3 d_1^2 - 3c_3^2 d_0^2}}{2bc_3 d_1},\\
a_0 &= -\frac{c_2 d_1 - c_3 d_0}{2c_3 d_1}; \quad c_0 = \frac{d_0(c_1 d_1^2 + c_3 d_0^2 - c_2 d_0 d_1)}{d_1^3},
\end{aligned} \tag{5.7}$$

and the corresponding traveling-wave is:

$$\begin{aligned}
M_n(\xi_n) = -\frac{1}{2c_3 d_1}\Bigg\{ & c_2 d_1 - c_3 d_0 \\
& + \frac{\tanh(b\ d)}{d}\sqrt{2c_2 c_3 d_0 d_1 + c_2^2 d_1^2 - 4c_1 c_3 d_1^2 - 3c_3^2 d_0^2} \times \\
\tanh\Bigg[& -\frac{\tanh(b\ d)(2c_2 c_3 d_0 d_1 + c_2^2 d_1^2 - 4c_1 c_3 d_1^2 - 3c_3^2 d_0^2)}{2c_3 d_1^3} t + d\ n + \xi_0 \Bigg]\Bigg\}.
\end{aligned} \tag{5.8}$$

5.2. Case $P = 2$, $L = 4$

For this case Eq. (2.11) becomes:

$$\frac{dM_n}{dt} = \frac{c_0 + c_1 M_n + c_2 M_n^2 + c_3 M_n^3 + c_4 M_n^4}{d_0 + d_1 M_n + d_2 M_n^2}(M_{n+1} - M_{n-1}). \tag{5.9}$$

The application of the modified method of simplest equation reduces Eq. (5.9) to a system of 7 nonlinear algebraic relationships among the parameters of the equation and the parameters of the solution. One solution of this nonlinear algebraic system is:

$$
\begin{aligned}
a_0 &= \frac{c_4 d_1 - c_3 d_2}{2c_4 d_2}, \\
a_1 &= \frac{\sigma}{2bc_4 d_2}\sqrt{\frac{4c_2 d_2^2 c_4^2 d_1 - 4c_2 d_2^3 c_4 c_3 - 3d_2 c_3 c_4^2 d_1^2 + d_2^3 c_3^3 + 4c_1 c_4^2 d_2^3 + 2c_4^3 d_1^3}{c_3 d_2 - 2c_4 d_1}}, \\
c &= -\frac{\sigma}{2bc_4 d_2^3(c_3 d_2 - 2c_4 d_1)}\Big[4c_2 d_2^2 c_4^2 d_1 - 4c_2 d_2^3 c_4 c_3 - 3d_2 c_3 c_4^2 d_1^2 + d_2^3 c_3^3 \\
&\quad +4c_1 c_4^2 d_2^3 + 2c_4^3 d_1^3\Big] \\
c_0 &= -\frac{1}{d_2^4(2c_4 d_1 - c_3 d_2)^2}\Big[d_1^3 d_2^3 c_3^3 + c_3^2 c_1 d_1 d_2^5 - 2c_3^2 c_2 d_1^2 d_2^4 - 3c_3^2 c_4 d_1^4 d_2^2 \\
&\quad -c_3 c_1 c_2 d_2^6 + c_3 d_2^5 c_2^2 d_1 - c_3 c_4 d_1^2 c_1 d_2^4 + 4c_3 c_2 c_4 d_1^3 d_2^3 + 3c_3 c_4^2 d_1^5 d_2 + c_1^2 c_4 d_2^6 \\
&\quad -d_2^4 c_2^2 c_4 d_1^2 - 2c_2 c_4^2 d_1^4 d_2^2 - c_4^3 d_1^6\Big] \\
d_0 &= \frac{c_3 d_1^2 d_2 - c_2 d_1 d_2^2 + c_1 d_2^3 - c_4 d_1^3}{d_2(c_3 d_2 - 2c_4 d_1)}
\end{aligned}
$$

(5.10)

The corresponding traveling-wave is:

$$
\begin{aligned}
(5.11)\quad M_n(\xi_n) &= \frac{1}{2c_4 d_2}\Bigg\{c_4 d_1 - c_3 d_2 \\
&+ \tanh(b\ d)\sqrt{\frac{4c_2 d_2^2 c_4^2 d_1 - 4c_2 d_2^3 c_4 c_3 - 3d_2 c_3 c_4^2 d_1^2 + d_2^3 c_3^3 + 4c_1 c_4^2 d_2^3 + 2c_4^3 d_1^3}{c_3 d_2 - 2c_4 d_1}} \\
&\times \tanh\Bigg[-\frac{b\tanh(b\ d)}{2bc_4 d_2^3(c_3 d_2 - 2c_4 d_1)}(4c_2 d_2^2 c_4^2 d_1 - 4c_2 d_2^3 c_4 c_3 - 3d_2 c_3 c_4^2 d_1^2 + d_2^3 c_3^3 \\
&\quad + 4c_1 c_4^2 d_2^3 + 2c_4^3 d_1^3)\ t + d\ n + \xi_0\Bigg]\Bigg\}.
\end{aligned}
$$

5.3. Case $P = 3$, $L = 5$

For this case Eq. (2.11) becomes:

$$
(5.12)\quad \frac{dM_n}{dt} = \frac{c_0 + c_1 M_n + c_2 M_n^2 + c_3 M_n^3 + c_4 M_n^4 + c_5 M_n^5}{d_0 + d_1 M_n + d_2 M_n^2 + d_3 M_n^3}(M_{n+1} - M_{n-1}).
$$

The application of the modified method of simplest equation reduces Eq. (5.12) to a system of 8 nonlinear algebraic relationships among the parameters of the equation and the parameters of the solution. One solution of this nonlinear algebraic system is:

$$
\begin{aligned}
c &= \frac{\sigma(3c_5^2d_2^2 - 2c_5d_2d_3c_4 - 4d_1d_3c_5^2 - d_3^2c_4^2 + 4c_3c_5d_3^2)}{2bc_5d_3^3},\\
a_0 &= \frac{c_5d_2 - c_4d_3}{2c_5d_3},\\
a_1 &= \frac{\sigma\sqrt{-3c_5^2d_2^2 + 2c_5d_2d_3c_4 + 4d_1d_3c_5^2 + d_3^2c_4^2 - 4c_3c_5d_3^2}}{2bc_5d_3},\\
d_0 &= \frac{c_0d_3^3}{c_3d_3^2 - d_3c_4d_2 - d_3d_1c_5 + d_2^2c_5},\\
c_1 &= \frac{C_1}{d_3^3(c_3d_3^2 - d_3c_4d_2 - d_3d_1c_5 + d_2^2c_5)},\\
C_1 &= -2d_1d_3^3c_3c_4d_2 - 2c_5^2d_2^2d_1^2d_3 + c_5^2d_2^4d_1 - 2c_5d_2^3d_3c_4d_1\\
&\quad +2c_5d_2^2d_1d_3^2c_3 + d_2^2d_3^2c_4^2d_1 + 2c_5d_2d_1^2d_3^2c_4 + d_1^3d_3^2c_5^2 + d_1d_3^4c_3^2\\
&\quad -2d_1^2d_3^3c_5c_3 - c_5d_2c_0d_3^4 + d_3^5c_4c_0,\\
c_2 &= \frac{C_2}{d_3^3(c_3d_3^2 - d_3c_4d_2 - d_3d_1c_5 + d_2^2c_5)}\\
C_2 &= c_5^2d_2^5 + c_0c_5d_3^5 + 4c_5d_2^2d_1d_3^2c_4 - d_2d_3^3c_4^2d_1 - 2c_3d_3^3d_2^2c_4\\
&\quad -3d_1d_3^3c_5c_3d_2 + d_1d_3^4c_3c_4 - 2c_5d_2^4d_3c_4 + d_2^3d_3^2c_4^2 - 3c_5^2d_2^3d_1d_3\\
&\quad +2c_3c_5d_3^2d_2^3 + 2d_1^2d_3^2c_5^2d_2 - d_1^2d_3^3c_5c_4c_3^2d_3^4d_2.
\end{aligned} \tag{5.13}
$$

The corresponding traveling wave is:

$$
\begin{aligned}
M_n(\xi_n) = \frac{1}{2c_5d_3}\Big\{ c_5d_2 - c_4d_3 &\\
+ \tanh(b\ d)\sqrt{-3c_5^2d_2^2 + 2c_5d_2d_3c_4 + 4d_1d_3c_5^2 + d_3^2c_4^2 - 4c_3c_5d_3^2} &\\
\times \tanh\Big[\Big(\frac{\tanh(b\ d)(3c_5^2d_2^2 - 2c_5d_2d_3c_4 - 4d_1d_3c_5^2 - d_3^2c_4^2 + 4c_3c_5d_3^2)t}{2c_5d_3^3} &\\
+ d\ n + \xi_0\Big)\Big]\Big\}. &
\end{aligned} \tag{5.14}
$$

The differential–difference equations for the corresponding Holling lattices will be reduced to a nonlinear algebraic systems consisting of $9, 10, \dots$ equations the same procedure can be continued for $P = 4, 5, \dots$. As a result of the application of the modified method of simplest equation. Each solution will lead to a traveling wave constructed on the basis of Riccati equation if we are able to solve these nonlinear systems.

6. Concluding remarks

Lattices have many applications in mathematics and physics. This is one of the reason for the importance of the differential–difference equations that often are used to model wave processes in lattices connected to physical chemical or biological systems. In this paper we have applied the modified method of simplest equation for identification of the Riccati lattices among the classes of the generalized Lotka–Volterra lattices and generalizing Holling lattices. The analysis of the balance equation arising from the application of the method of simplest equation has shown that the Wadati lattice is unique in the class of the generalized Lotka–Volterra lattice as it is the only Riccati lattice of class (4.2) among the lattices of the generalized Lotka–Volterra class. Many more Riccati lattices can be found in the class of generalized Holling lattices. We have obtained exact traveling wave solutions for the simplest three Riccati lattices that are Holling lattices too.

The identification of the Riccati lattices is important task as the connected to these lattices waves of tanh-kind describe a kind of switching between the states in the corresponding lattice. The presence of Riccati lattices among the lattice models used in different scientific areas shows that probably this kind of switching between the states is a frequently arising phenomenon and fundamental property of a large class of natural systems. This paper is a first step from a future research on identifying and studying the properties of the Riccati lattices.

REFERENCES

[1] Murray, J. D. Lectures on Nonlinear Differential Equation Models in Biology, UK, Oxford, Oxford University Press, 1977.

[2] Scott, A. C. Nonlinear Science. Emergence and Dynamics of Coherent Structures, UK, Oxford, Oxford University Press, 1999.

[3] May, R. M. Stability and Complexity in Model Ecosystems, New Jersey, Princeton University Press, 2001.

[4] Vitanov, N. K., F. H. Busse. Bounds on the heat transport in a horizontal fluid layer with stress–free boundaries. *ZAMP*, **48** (1997), 310–324.

[5] Hoffmann, N. P., N. K. Vitanov. Upper Bounds on Energy Dissipation in Couette-Ekman Flow. *Phys. Lett. A*, **255** (1999), 277-286.

[6] Colinet, P., J. C. Legros, M. G. Velarde. Nonlinear Dynamics of Surface–Driven Instabilities, Berlin, Wiley–VCH, 2001.

[7] Kantz, H., D. Holstein, M. Ragwitz, N. K. Vitanov. Markov Chain Model for Turbulent Wind Speed Data. *Physica A*, **342** (2004), 315–321.

[8] Vitanov, N. K., E. D. Yankulova. Multifractal Analysis of the Long-Range Correlations in the Cardiac Dynamics Of Drosophila Melanogaster. *Chaos Solitons & Fractals*, **28** (2006), 768–775.

[9] Ablowitz, M.,P. A. Clarkson. Solitons, Nonlinear Evolution Equations and Inverse Scattering, UK, Cambridge, Cambridge University Press, 1991.

[10] Akhmediev, N. N., A. Ankiewicz. Solitons. Nonlinear pulses and beams, London. Chapman & Hall, 1997

[11] Kivshar, Y. C., G. P. Agraval. Optical Solitons, San Diego, Academic Press, 2003.

[12] Scott, A. C. Neuroscience: A Mathematical Primer, New York, Springer, 2002.

[13] Perko L. Differential Equations and Dynamical Systems, New York, Springer, 2001.

[14] Infeld, E., G. Rowlands. Nonlinear Waves, Solitons and Chaos, UK, Cambridge, Cambridge University Press, 1990.

[15] Robertson, R., A. Combs. (Eds.), Chaos Theory in Psychology and the Life Sciences, New Jersey, Mahwah, Lawrence Erlbaum Associates Inc., 1995.

[16] Vitanov, N. K., I. P. Jordanov, Z. I. Dimitrova. On Nonlinear Population Waves. *Applied Mathematics and Computation*, **215** (2009), 2950–2964.

[17] Temam, R. Navier–Stokes Equations: Theory and Numerical Analysis, R. I., Providence, AMS Chelsea Publishing, 2001.

[18] Vitanov, N. K. Upper Bounds on the Heat Transport in a Porous Layer. *Physica D*, **136** (2000), 322–339.

[19] Holmes, P., J. L. Lumley, G. Berkooz. Turbulence, Coherent Structures, Dynamical Systems and Symmetry, UK, Cambridge, Cambridge University Press, 1996.

[20] Boeck, T., N. K. Vitanov. Low-Dimensional Chaos in Zero-Prandtl-Number Benard-Marangoni Convection. *Phys. Rev. E*, **65** (2002), Article number: 037203.

[21] Vitanov, N. K. Upper Bound on the Heat Transport in a Horizontal Fluid Layer of Infinite Prandtl Number. *Phys. Lett. A*, **248** (1998), 338–346.

[22] Foias, C., O. Manley, R. Rosa, R. Temam. Navier–Stokes Equations and Turbulence, UK, Cambridge, Cambridge University Press, 2001.

[23] Vitanov, N. K., Upper Bounds on Convective Heat Transport in a Rotating Fluid Layer of Infinite Prandtl Number: Case of Intermediate Taylor Numbers. *Phys. Rev. E*, **62** (2000), 3581–3591.

[24] Kudryashov, N. A., D. I. Sinelschikov. Nonlinear Waves in Bubbly Liquids with Consideration for Viscosity and Heat Transport. *Phys. Lett. A*, **374** (2010), 2011–2016.

[25] Vitanov, N. K. Convective Heat Transport in a Fluid Layer of Infinite Prandtl Number: Upper Bounds for the Case Rigid Lower Boundary and Stress-Free Upper Boundary. *European Physical Journal B*, **15** (2000), 349–355.

[26] Vitanov, N. K., Z. I. Dimitrova, M. Ausloos. Verhulst–Lotka–Volterra (VLV) Model of Ideological Struggle. *Physica A*, **389** (2010), 4970–4980.

[27] Vitanov, N. K., S. Panchev. Generalization of the Model of Conflict Between Two Armed Groups. *Compt. rend. Acad. bulg. Sci.*, **61** (2008), 1121–1126.

[28] Gardner, C. S., J. M.Greene, M. D. Kruskal, R. R. Miura. Method for Solving Korteweg–de Vries Equation. *Phys. Rev. Lett.*, **19** (1967), 1095–1097.

[29] Ablowitz, M. J., D. J. Kaup, A. C. Newell, H. Segur. Inverse Scattering Transform – Fourier Analysis for Nonlinear Problems. *Studies in Applied Mathematics*, **53** (1974), 249–315.

[30] Remoissenet, M. Waves Called Solitons, Berlin, Springer, 1993.

[31] Hirota, R. Exact Solution of Korteweg-de Vries Equation for Multiple Collisions of Solitons. *Phys. Rev. Lett.*, **27** (1971), 1192–1194.

[32] Kudryashov, N. A. Exact Solutions of the Generalized Kuramoto–Sivashinsky Equation. *Phys. Lett. A*, **147** (1990), 287–291.

[33] Kudryashov, N. A. Nonlinear Differential Equations with Exact Solutions Expressed Via the Weierstrass Function. *Z. Naturforschung A*, **59** (2004), 443–454.

[34] Kudryashov, N. A., M. V. Demina. Traveling Wave Solutions of the Generalized Nonlinear Evolution Equations. *Applied Mathematics and Computation*, **210** (2009), 551–557.

[35] Vitanov, N. K., Z. I. Dimitrova, H. Kantz. Modified Method of Simplest Equation and its Application to Nonlinear PDEs. *Applied Mathematics and Computation*, **216** (2010), 2587–2595.

[36] Wu, X.-H., J.-H. He. EXP-Function Method and its Application to Nonlinear Equations. *Chaos, Solitons & Fractals*, **38** (2008), 903–910.

[37] Fan, E. Extended Tanh-Method and its Application to Nonlinear Equations. *Phys. Lett. A*, **277** (2000), 212–218.

[38] Vitanov, N. K. Modified Method of Simplest Equation: Powerful Tool for Obtaining Exact and Approximate Traveling-Wave Solutions of Nonlinear PDEs. *Commun. Nonlinear Sci. Numer. Simulat.*, **16** (2011), 1176–1185.

[39] Kudryashov, N. A Solitary and Periodic Solutions of the Generalized Kuramoto-Sivashinsky Equation. *Regular & Chaotic Dynamics*, **13** (2008), 234–238.

[40] Kudryashov, N. A., M. B. Soukharev. Popular Ansatz Methods and Solitary Wave Solutions of the Kuramoto–Sivashinsky Equation. *Regular & Chaotic Dynamics*, **14** (2009), 407–419.

[41] Kudryashov, N. A. Simplest Equation Method to Look for Exact Solutions of Nonlinear Differential Equations. *Chaos Solitons & Fractals*, **24** (2005), 1217–1231.

[42] Lou, S. Symmetry Analysis and Exact Solutions of the $(2+1)$-Dimensional Sine–Gordon System. *J. Math. Phys.*, **41** (2000), 6509–6524.

[43] Martinov, N., N. Vitanov. Running-Wave Solutions of the Two-Dimensional Sine-Gordon Equation. *J. Phys. A: Math. Gen.*, **25** (1992), 3609–3613.

[44] Martinov, N., N. Vitanov. On Some Solutions of the Two-Dimensional Sine-Gordon Equation. *J. Phys. A: Math. Gen.*, **25** (1992), L419–L426.

[45] Martinov, N., N. Vitanov. New Class of Running-Wave Solutions of the $(2+1)$-Dimensional Sine-Gordon Equation. *J. Phys. A: Math. Gen.*, **27** (1994), 4611–4618.

[46] Vitanov, N. K. On Traveling Waves and Double-Periodic Structures in Two-Dimensional Sine-Gordon Systems. *J. Phys. A: Math. Gen.*, **29** (1996), 5195–5207.

[47] Clarkson, P. A., E. L. Mansfield, A. E. Milne. Symmetries and exact solutions of a $(2+1)$-Dimensional Sine–Gordon System. *Phil. Trans. Roy. Soc. London A*, **354** (1996), 1807–1835.

[48] Vitanov, N. K., N. K. Martinov. On the Solitary Waves in the Sine-Gordon Model of the Two-Dimensional Josephson Junction. *Z. Phys. B*, **100** (1996), 129–135.

[49] Vitanov, N. K. Breather and Soliton Wave Families for the Sine-Gordon Equation. *Proc. Roy. Soc. London A*, **454** (1998), 2409–2423.

[50] Radha, R., M. Lakshamanan. The $(2+1)$-Dimensional Sine-Gordon Equation; Integrability and Localized Solutions. *J. Phys A: Math. Gen.*, **29** (1996), 1551–1562.

[51] Nakamura A. Exact Cylindrical Soliton Solutions of the Sine-Gordon Equation, the Sinh-Gordon Equation and the Periodic Toda Equation. *J. Phys. Society Japan*, **57** (1988), 3309–3322.

[52] Ablowitz, M. J., A. Zeppetela. Explicit Solutions of Fisher Equation for a Specifical Wave Speed. *Bull. Math. Biol.*, **41** (1979), 835–840.

[53] Dimitrova, Z. I., N. K. Vitanov. Influence of Adaptation on the Nonlinear Dynamics of a System of Competing Populations. *Phys. Lett. A*, **272** (2000), 368–380.

[54] Dimitrova, Z. I., N. K. Vitanov. Dynamical Consequences of Adaptation of Growth Rates in a System of Three Competing Populations. *J. Phys. A: Math. Gen.*, **34** (2001), 7459–7473.

[55] Dimitrova, Z. I., N. K. Vitanov. Adaptation and its Impact on the Dynamics of a System of Three Competing Populations. *Physica A*, **300** (2001), 91–115.

[56] Dimitrova, Z. I., N. K. Vitanov. Chaotic pairwise competition. *Theoretical Population Biology*, **66** (2004), 1–12.

[57] Vitanov, N. K., Z. I. Dimitrova, H. Kantz. On the Trap of Extinction and its Elimination. *Phys. Lett. A*, **349** (2006), 350–355.

[58] Vitanov, N. K., I. P. Jordanov, Z. I. Dimitrova. On Nonlinear Dynamics of Interacting Populations: Coupled Kink Waves in a System of Two Populations. *Commun. Nonlinear Sci. Numer. Simulat.*, **14** (2009), 2379–2388.

[59] Vitanov, N. K., Z. I. Dimitrova. Application of the method of Simplest Equation for obtaining exact traveling-wave solutions for Two Classes of Model PDEs from Ecology and Population Dynamics. *Commun. Nonlinear Sci. Numer. Simulat.*, **15** (2010), 2836–2845.

[60] Wang, X. Y. Exact and Explicit Wave Solutions for the Generalized Fisher Equation. *Phys. Lett. A*, **131** (1988), 277–279.

[61] Kudryashov, N. A. Exact Solitary Waves of the Fisher equation. *Phys. Lett. A*, **342** (2005), 99–106.

[62] Vitanov, N. K. Application of Simplest Equations of Bernoulli and Riccati Kind for Obtaining Exact Traveling Wave Solutions for a Class of PDEs with Polynomial Nonlinearity. *Commun. Nonlinear Sci. Numer. Simulat.*, **15** (2010), 2050–2060.

[63] Martinov, N. K., N. K. Vitanov. On the Self-Consistent Thermal Equilibrium Structures in Two-Dimensional Negative Temperature Systems. *Canadian Journal of Physics*, **72** (1994), 618–624.

[64] Panchev, S., T. Spassova, N. K. Vitanov. Analytical and Numerical Investigation of two Families of Lorenz-Like Dynamical Systems. *Chaos Solitons & Fractals*, **33** (2007), 1658–1671.

[65] Thaker, W. C. Some Exact Solutions to the Nonlinear Shallow-Water Equations. *J. Fluid Mech.*, **107** (1981), 499–508.

[66] Malfliet, W., W. Hereman. The Tanh-Method: I. Exact Solutions of Nonlinear Evolution and Water Wave Equations. *Physica Scripta*, **54** (1996), 563–568.

[67] Debnath, L. Nonlinear water waves, New York, Academic Press, 1994.

[68] Ivanov, R. I. Water Waves and Integrability. *Phil. Trans. R. Soc. A*, **365** (2007), 2267–2280.

[69] Johnson, R. S. The Classical Problem of Water Waves: a Reservoir of Integrable and Nearly-Integrable Equations. *J. Nonl. Math. Phys.*, **10** (2003), No. 1, 72–92.

[70] Kudryashov, N. A., N. B. Loguinova. Extended Simplest Equation Method for Nonlinear Differential Equations. *Applied Mathematics and Computation*, **205** (2008), 396–402.

[71] Kudryashov, N. A., M. V. Demina. Polygons of Differential Equations for Finding Exact Solutions. *Chaos Solitons & Fractals*, **33** (2007), 480–496.

[72] Kudryashov, N. A., N. B. Loguinova. Be Careful with the Exp-Function Method. *Commun. Nonlinear Sci. Numer. Simulat.*, **14** (2009), 1881–1890.

[73] Kudryashov, N. A. Seven Common Errors in Finding Exact Solutions of Nonlinear Differential Equations. *Commun. Nonlinear Sci. Numer. Simulat.*, **14** (2009), 3507–3529.

[74] Hone, A. N. W. Painleve Tests, Singularity Structure and Integrability. *Lect. Notes Phys.*, **767** (2009), 245–277.

[75] Vitanov N. K., Z. I. Dimitrova, K. N. Vitanov. On the Class of Nonlinear PDEs that Can Be Treated by the Modified Method of Simplest Equation. Application to Generalized Degasperis–Processi Equation and b-Eequation. *Commun. Nonlinear Sci. Numer. Simulat.*, **16** (2011), 3033–3044.

[76] Vitanov, N. K. On Modified Method of Simplest Equation for Obtaining Exact and Approximate Solutions of Nonlinear PDEs: The Role of the Simplest Equation. *Commun. Nonlinear Sci. Numer. Simulat.*, **16** (2011), 4215–4231.

[77] Comte, J. C., P. Marquie, M. Remoissenet. Dissipative Lattice Model with Exact Traveling Discrete Kink-Soliton Solutions: Discrete Breather Generation and Reaction Diffusion Regime. *Phys. Rev. E*, **60** (1999), 7484–7489.

[78] Baldwin, D., Ü. Göktas, W. Hereman. Symbolic Computations of Hyperbolic Tangent Solutions for Nonlinear Differential–Difference Equations. *Computer Physics Communications*, **162** (2004), 203–217.

[79] Xie, F., J. Wang. A New Method for Solving Nonlinear Differential–Difference Equations. *Chaos Solitons & Fractals*, **27** (2006), 1067–1071.

[80] Aslan, I. Analytic Solutions to Nonliner Differential–Difference Equations by Means of the Extended (G'/G)-Expansion Method. *J. Phys. A: Math. Theor.*, **43** (2010), 395207.

[81] Aslan, I. A Discrete Generalization of the Extended Simplest Equation Method. *Commun. Nonlinear Sci. Numer. Simulat.*, **15** (2010), 1967–1973.

[82] KUDRYASHOV, N. A. A Note on the G'/G -Expansion Method. *Applied Mathematics and Computation*, **217** (2010), 1755–1758.

[83] WADATI, M. Transformation Theories for Nonlinear Discrete Systems. *Progr. Theor. Phys. Suppl.*, **59** (1976), 36–63.

[84] HOLLING, C. S. The Components of Predation as Revealed by a Study of Small-Mammal Predation of the European Pine Sawfly. *Canadian Entomologists*, **91** (1959), 293–320.