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ANALYSIS OF MIXED MODE II/III CRACK IN BILAYERED COMPOSITE BEAM

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Abstract. Mixed mode II/III crack investigation in cantilever bilayered unidirectional fiber reinforced composite beam is reported. The crack is situated between the layers. The two crack arms have different widths. Formula for the strain energy release rate is obtained by the linear elastic fracture mechanics methods using the magnitude of the applied forces, geometrical characteristics of the cross-section, and the elastic moduli of the layers. An equivalent shear modulus of the un-cracked beam portion is used. Several diagrams illustrating the results of parametrical analysis of the strain energy release rate are presented. The paper is a part of a research in the field of fracture behaviour of composite beams.
Key words: Mixed mode II/III fracture, cantilever bilayered composite beam, compliance technique.

1. Introduction

One of the main reasons for failure of the constructions made of composite materials is initiation and propagation of cracks. In order to investigate these cracks the fracture mechanics methods has been used. Three main modes of fracture, namely Mode I, Mode II, and Mode III are defined [1, 2]. The difference between them is in the directions of the crack arms mutual displacements, as shown in Fig. 1.

Many studies dealing with the cracks in composite materials are available in the specialized literature [3–12]. Usually, linear elastic fracture mechanics has been applied for theoretical analysis of cracks in composites. Besides,

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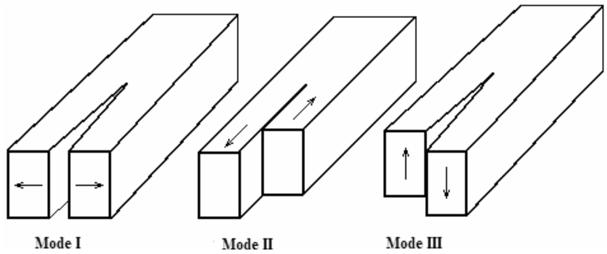

Fig. 1. Fracture modes

composite beam specimens containing an initial crack have been used for experimental characterization of the fracture behaviour.

It should be mentioned that most of the crack investigations have considered the pure modes of fracture. However, in the real practice the crack can propagate in the mixed mode manner. Due to this fact, the present study is concerned with the analysis of mixed mode II/III crack in bilayered composite beam. The two layers are manufactured of unidirectional fiber reinforced composites. The fibers are directed along the beam axis and are uniformly distributed in the cross-sectional plane. The crack of length a is situated between the layers. The load consists of two vertical forces F of opposite directions applied to the two crack arms (Fig. 2). Such loading configuration causes bending moments and shearing forces in the sections behind the crack front and torsion moment in the section ahead of the crack front. However, the object under

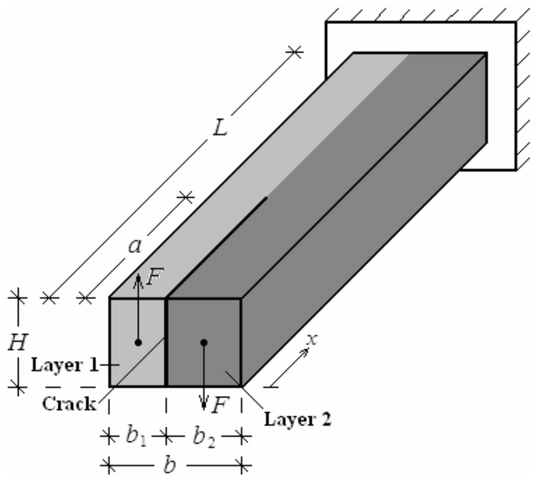

Fig. 2. Bilayered composite beam under consideration

consideration in the study is a long beam where $L \gg H$, and, consequently, the influence of the shearing forces can be neglected. Hence, the beam portion behind the crack front, $0 \leq x \leq a$, is subjected to bending, while the beam portion ahead of the crack front, $a \leq x \leq L$, is subjected to torsion.

The crack investigation will be performed by the compliance technique [13] and the formula for the strain energy release rate,G, will be derived. As has already been mentioned, the beam portions behind and ahead of the crack front are subjected to different loading conditions. Then, the expression for the strain energy release rate will take a form:

$$G = G_B + G_T, \tag{1}$$

where G_B is the component due to the bending, while G_T is the component due to the torsion.

2. Determination of the strain energy release rate

2.1. Determination of G_B

According to the compliance technique, the strain energy release rate can be obtained by the formula:

$$G_B = \frac{F^2}{2H}\frac{dC_B}{da}, \tag{2}$$

where F is the force applied to the beam, H is the length of the crack front, i.e. the height of the cross-section, C_B is the compliance of the beam subjected to bending, a is the crack length.

The expression for compliance C_B has a form:

$$C_B = \frac{w}{F}. \tag{3}$$

Here, w is the vertical displacement of the beam section where the force F is applied.

The methods of Mechanics of materials [14] are used in order to determine w and the result is:

$$w = \frac{4Fa^3}{H^3}\left(\frac{1}{E_1 b_1} + \frac{1}{E_2 b_2}\right), \tag{4}$$

where E_1 and E_2 are the moduli of elasticity of the two layers, respectively.

Then, the equation for the compliance is:

$$C_B = \frac{4a^3}{H^3}\left(\frac{1}{E_1 b_1} + \frac{1}{E_2 b_2}\right). \tag{5}$$

Furthermore, the first order derivative of C_B with respect to the crack length is obtained:

$$\frac{dC_B}{da} = \frac{12a^2}{H^3}\left(\frac{1}{E_1 b_1} + \frac{1}{E_2 b_2}\right). \tag{6}$$

Finally, the formula for G_B takes the following form:

$$G_B = \frac{F^2}{2H}\frac{dC_B}{da} = \frac{6F^2a^2}{H^4}\left(\frac{1}{E_1 b_1} + \frac{1}{E_2 b_2}\right). \tag{7}$$

2.2. Determination of G_T

The compliance technique is applied again. The formula for the strain energy release rate for a beam subjected to torsion is:

$$G_T = \frac{T^2}{2H}\frac{dC_T}{da}, \tag{8}$$

where T is the torsion moment acting on the beam, while C_T is the compliance of the beam subjected to torsion.

The torsion moment in the beam portion ahead of the crack front caused by the two external forces has a magnitude shown below:

$$T = \frac{Fb}{2} = \frac{F(b_1 + b_2)}{2}. \tag{9}$$

The compliance of the beam subjected to torsion is given by the expression:

$$C_T = \frac{\varphi}{T}. \tag{10}$$

In (10), φ is the angle of twist of the beam section where the torsion moment acts. Here, torsion moment is applied at the section directly ahead of the crack front, i.e. the beam section of coordinate $x = a$.

The solution from Theory of elasticity [15] will be used for determination of φ due to the fact that the beam under consideration has rectangular cross-section. The angle of relative twist, θ, according to this solution is:

$$\theta = \frac{d\varphi}{dx} = -T\frac{\pi^6}{256}\frac{\left(b^2 + H^2\right)}{G' b^3 H^3}, \tag{11}$$

where G' is the shear modulus of bilayered beam, i.e. shear modulus of the un-cracked beam portion.

Solving (11), the expression for φ results in:

$$\varphi = 3.755T\frac{\left(b^2+H^2\right)}{G'b^3H^3}\left(L-a\right). \tag{12}$$

Further, the compliance of the beam subjected to torsion is:

$$C_T = \frac{3.755\left(b^2+H^2\right)}{G'b^3H^3}\left(L-a\right). \tag{13}$$

Then, the first order derivative of C_T with respect to a is obtained:

$$\frac{dC_T}{da} = -\frac{3.755\left(b^2+H^2\right)}{G'b^3H^3}. \tag{14}$$

Finally, substituting (9) and (14) in (8), the formula for G_T takes the form:

$$G_T = \frac{T^2}{2H}\frac{dC_T}{da} = -\frac{0.47F^2\left(b^2+H^2\right)}{G'bH^4}. \tag{15}$$

2.3. Determination of the shear modulus of the un-cracked beam portion, G'

Next step of the solution is to derive the shear modulus of the un-cracked beam portion. It will be expressed as a function of the two layers shear moduli, G'_1 and G'_2. In order to perform that, the solution of Muskhelishvili [16] for the stiffness of rectangular bilayered beam subjected to torsion, labelled by D, is used, namely:

$$\begin{aligned} D = G'I = {} & \frac{8}{3}(G'_1b_1+G'_2b_2)\left(\frac{H}{2}\right)^3 + \\ & + \left(\frac{4}{\pi}\right)^5\left(\frac{H}{2}\right)^4\sum_{n=0}^{\infty}\frac{G'^2_1 chmb_2 + G'^2_2 chmb_1 - \left(G'^2_1+G'^2_2\right)chmb_1 chmb_2}{(2n+1)^5(G'_1 chmb_2 shmb_1 + G'_2 chmb_1 shmb_2)} - \\ & - \left(\frac{4}{\pi}\right)^5\left(\frac{H}{2}\right)^4 G'_1G'_2\sum_{n=0}^{\infty}\frac{chmb_1 + chmb_2 - chm(b_1-b_2) - 1}{(2n+1)^5(G'_1 chmb_2 shmb_1 + G'_2 chmb_1 shmb_2)}, \end{aligned} \tag{16}$$

where I is the moment of inertia of the beam of rectangular cross-section subjected to torsion, $m = \dfrac{(2n+1)\pi}{H}$, n is an integer. The formula for D represents the Fourrier's series and can be easily expanded. Thus, in the case of narrow rectangle, i.e. $b_1 \geq 3H$ and $b_2 \geq 3H$, (16) can be simplified, as follows [16]:

$$D^* = \frac{H^3}{3}(G'_1 b_1 + G'_2 b_2) - 0,21H^4 \frac{(G'^2_1 + G'^2_2)}{G'_1 + G'_2}. \tag{17}$$

Furthermore, the moment of inertia I of rectangular beam subjected to torsion is obtained by the Saint-Venant's solution [17] modified by Love [18]:

$$I = \frac{(b_1 + b_2)H^3}{3} - \left(\frac{4}{\pi}\right)^5 \left(\frac{H}{2}\right)^4 \sum_{n=0}^{\infty} \frac{thm(b_1 + b_2)}{(2n+1)^5}. \tag{18}$$

It should be mentioned, if the cross-section is narrow rectangle, $b_1 \geq 3H$ and $b_2 \geq 3H$, then, (18) results in [17, 18]:

$$I^* = \frac{H^3}{3}(b_1 + b_2) - 0.21H^4. \tag{19}$$

Finally, the formula for G' takes the form:

$$\begin{aligned} G' = \frac{D}{I} = \Bigg[& \frac{8}{3}(G'_1 b_1 + G'_2 b_2) + \\ & + \frac{H}{2}\left(\frac{4}{\pi}\right)^5 \sum_{n=0}^{\infty} \frac{G'^2_1 chmb_2 + G'^2_2 chmb_1 - (G'^2_1 + G'^2_2) chmb_1 chmb_2}{(2n+1)^5 (G'_1 chmb_2 shmb_1 + G'_2 chmb_1 shmb_2)} - \\ & - \frac{H}{2}\left(\frac{4}{\pi}\right)^5 G'_1 G'_2 \sum_{n=0}^{\infty} \frac{chmb_1 + chmb_2 - chmb\,(b_1 - b_2) - 1}{(2n+1)^5 (G'_1 chmb_2 shmb_1 + G'_2 chmb_1 shmb_2)} \Bigg] \times \\ & \times \left[\frac{8}{3}(b_1 + b_2) - \frac{H}{2}\left(\frac{4}{\pi}\right)^5 \sum_{n=0}^{\infty} \frac{thm(b_1 + b_2)}{(2n+1)^5} \right]^{-1}. \end{aligned} \tag{20}$$

Again, in the case of narrow rectangle of $b_1 \geq 3H$ and $b_2 \geq 3H$ the expression for the equivalent shear modulus can be simplified, as follows:

$$G'^* = \frac{D^*}{I^*} = \frac{(G'_1 + G'_2)(G'_1 b_1 + G'_2 b_2) - 0.63H(G'^2_1 + G'^2_2)}{(G'_1 + G'_2)(b_1 + b_2 - 0.63H)}. \tag{21}$$

It should be specified, that for a homogeneous beam, i.e. $G'_1 = G'_2 = G'$, formula (21) yields $G'^* = G'$.

2.4. Expression derivation for the strain energy release rate, *G*

Equations (7) and (15) are substituted in (1). The result is:

$$G = \frac{6F^2a^2}{H^4}\left(\frac{1}{E_1b_1}+\frac{1}{E_2b_2}\right) - \frac{0,47F^2}{G'}\left(\frac{b^2+H^2}{bH^4}\right) = \tag{22}$$

$$= \frac{F^2}{H^4}\left[6a^2\left(\frac{1}{E_1b_1}+\frac{1}{E_2b_2}\right) - \frac{0,47}{G'}\left(\frac{(b_1+b_2)^2+H^2}{b_1+b_2}\right)\right].$$

3. Comparison of formula (22) with the result obtained in [11]

In order to verify (22) the following formula for G is used [11]:

$$G = \frac{F^2}{2H}\left(\frac{2a^2}{EI_{y_1}} + \frac{H^2}{4G'I_{y_1}}\right). \tag{23}$$

It should be specified, that the object of investigation in [11] is a unidirectional fiber reinforced composite beam containing a crack of equal arms.

Here, to perform the comparison, expressions (22) and (23) are rearranged, as follow:

$$\frac{G}{F^2} = \frac{1}{H^4}\left[6a^2\left(\frac{1}{E_1b_1}+\frac{1}{E_2b_2}\right) - \frac{0.47}{G'}\left(\frac{(b_1+b_2)^2+H^2}{b_1+b_2}\right)\right], \tag{24}$$

$$\frac{G}{F^2} = \frac{1}{2H}\left(\frac{2a^2}{EI_{y_1}} + \frac{H^2}{4G'I_{y_1}}\right). \tag{25}$$

Then, the following data from [11] are substituted in (24) and (25):

$$a = 0.127 \text{ m}; \quad b_1 = b_2 = 0.0102 \text{ m}; \quad H = 0.0127\text{m};$$

$$E = E_1 = E_2 = 43.5 \times 10^9 \text{ Pa}; \qquad G' = G'_1 = G'_2 = 4.14 \times 10^9 \text{ Pa}.$$

The result is:

– by formula (24): $\dfrac{G}{F^2} = 1.664 \times 10^{-2}\ (\text{Nm})^{-1}$;

– by formula (25): $\dfrac{G}{F^2} = 1.699 \times 10^{-2}\ (\text{Nm})^{-1}$.

The difference between the results obtained is very small. This means that formula (22) can be applied for the strain energy release rate determination in the case of mixed mode II/III crack.

4. Investigation on the influence of some parameters on the strain energy release rate

In order to perform this investigation, (22) is rearranged for the case of narrow rectangle with $b_1 = 3H$, i.e. the shear modulus of the un-cracked beam portion is expressed by (21). The result is:

$$(26)\quad G^* = \frac{F^2}{H^4}\left\{\left[6a^2\left(\frac{1}{E_1 b_1}+\frac{1}{E_2 b_2}\right)\right] - \right.$$

$$\left. -\frac{0,47(G'_1+G'_2)(b_1+b_2-0.63H)}{(G'_1+G'_2)(G'_1 b_1+G'_2 b_2)-0.63H(G'^2_1+G'^2_2)}\left[\frac{(b_1+b_2)^2+H^2}{b_1+b_2}\right]\right\}.$$

Then, (26) is rewritten in a dimensionless form:

$$(27)\quad \frac{G^*}{G'_1 H} = \frac{F^2}{G'_1 H^5}\left\{\left[6a^2\left(\frac{1}{E_1 b_1}+\frac{1}{E_2 b_2}\right)\right] - \right.$$

$$\left. -\frac{0.47(G'_1+G'_2)(b_1+b_2-0.63H)}{(G'_1+G'_2)(G'_1 b_1+G'_2 b_2)-0.63H(G'^2_1+G'^2_2)}\left[\frac{(b_1+b_2)^2+H^2}{b_1+b_2}\right]\right\}.$$

Furthermore, the coefficients $\theta = \frac{E_1}{E_2}$, $\eta = \frac{G'_1}{G'_2}$ and $\delta = \frac{b_1}{b_2}$ are introduced in (27). The result is:

$$(28)\quad \frac{G^*}{G'_1 H} = \frac{6F^2 a^2}{E_1 G'_1 b_1 H^5}(1+\delta\theta) -$$

$$-\frac{0.47F^2 b_1}{G'^2 H^5}\left[\frac{\eta(1+\eta)(1+0.79\delta)(1+2\delta+1.11\delta^2)}{\delta(1+\delta)(1+0.79\delta\eta^2+\delta\eta+\eta-0.21\delta)}\right].$$

4.1. Influence of the two layers moduli of elasticity ratio on the strain energy release rate

In order to investigate this influence, θ takes values 1, 2, 3, 4, and 5, the other coefficients are assumed $\eta = 2$ and $\delta = 2$, while the expressions out of the brackets are supposed to be equal to one.

The influence of the two layers moduli of elasticity ratio on the strain energy release rate is shown in Fig. 3. It is obvious, that G^* increases when θ increases. This is due to the fact that the increasing θ leads to the increase of the difference between the two crack arms bending stiffness and G^*, as well.

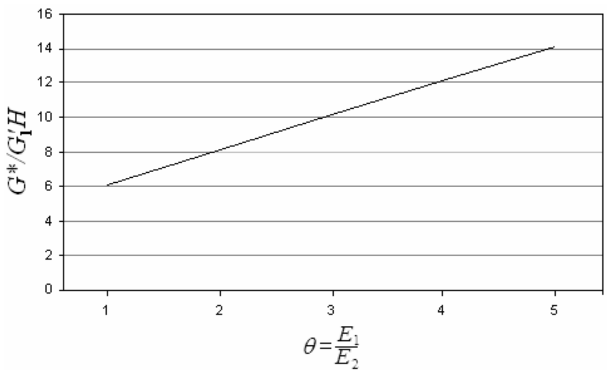

Fig. 3. Diagram of the influence of the two layers moduli of elasticity ratio on the strain energy release rate

4.2. Influence of the two layers shear moduli ratio on the strain energy release rate

In (28), η is assumed to take values 1, 2, 3, 4, and 5, the other two parameters are assumed $\theta = 2$ and $\delta = 2$, while the expressions out of the brackets are supposed to be equal to one.

The dependence between the strain energy release rate and the two layers shear moduli ratio is depicted in Fig. 4. It is evident that the increase of η causes increase of G^*. The explanation is that the increase of η results in the increase of the difference between the torsion stiffness of the layers and increasing G^*, as well.

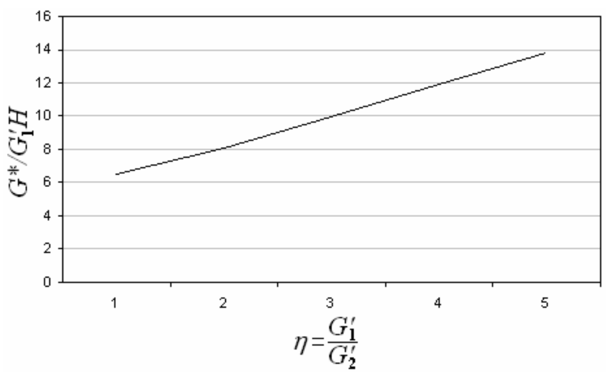

Fig. 4. Diagram of the influence of the two layers shear moduli ratio on the strain energy release rate

4.3. Influence of the two crack arms widths ratio on the strain energy release rate

Now, the values 1, 2, 3, 4, and 5 for δ are assumed, the other coefficients are taken $\theta = 2$ and $\eta = 2$, while the expressions out of the brackets are supposed to be equal to one.

The diagram of the influence of δ on the strain energy release rate is illustrated in Fig. 5. It is clear, that G^* increases when δ increases. This fact could be explained by the increase of δ leading to the increase of the two crack arms mutual displacements, which influences on the strain energy release rate value.

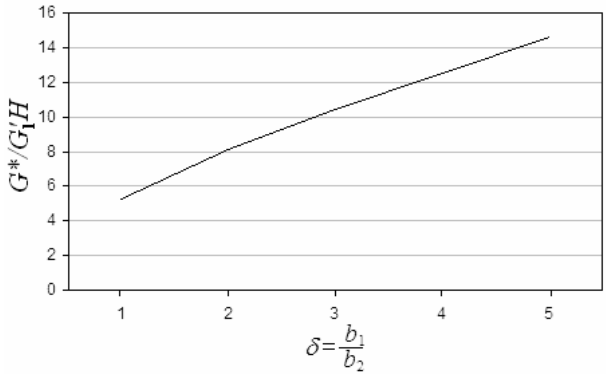

Fig. 5. Diagram of the influence of the two crack arms widths ratio on the strain energy release rate

5. Conclusions

The object of investigation in the present study is the mixed mode II/III crack in cantilever bilayered beam of rectangular cross-section. The crack arms are made of two unidirectional fiber reinforced composites and have different widths. The loading condition causes bending in the beam sections behind the crack front and torsion in the beam section ahead of the crack front. The shear modulus of the un-cracked portion of the beam is derived on the basis of the solutions of Saint-Venant, Love and Muskhelishvili for the rectangular beam subjected to torsion.

The formula for the strain energy release rate is obtained by the compliance technique of linear elastic fracture mechanics. The validity of the expression for G is established by comparison with a known solution.

The dependences between the strain energy release rate and the two layers moduli of elasticity ratio, the two layers shear moduli ratio as well as the crack arms widths ratio are investigated. It is observed, that the strain energy release rate increases when these ratios increase.

It should be noted that formula (22) might be successfully applied for the strain energy release rate critical value determination on the basis of the data from the experimental investigation of mixed mode II/III crack in bilayered composite beams. In order to perform that, the critical value F_c of the external force (the value obtained during the experiment which corresponds to the beginning of the crack propagation) has to be substituted in (22).

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