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VIBRATION MODAL SOLUTIONS DEVELOPING OF THE ELASTIC CIRCULAR MEMBRANE IN POLAR COORDINATES BASED ON THE FOURIER-BESSEL SERIES

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Abstract. This paper is written to show the development of the vibration modal solutions of elastic circular membranes in polar coordinates using the Fourier-Bessel series. The ordinary differential equation approach is utilized and the Laplacian of wave equation in polar coordinates is used to develop the solution of the membrane vibrations. A Fourier-Bessel solution is developed for the vibration of the elastic circular membrane in specific separation of variables is elaborated and based on the initial and boundary conditions. A numerical example is provided to show the application of such theory.
Key words: Membrane vibration, Fourier-Bessel series, eigenvalues.

1. Introduction

In 1738 Daniel Bernoulli published a number of theories on the oscillations of heavy chains and G. N. Watson in 1922 in his book of Treatise on the Theory of Bessel Functions used this similar approach to investigate use of Bessel functions for development of solutions of the oscillating systems using Euler theories. Later these developments lead into development of modal solutions of the oscillating rectangular membranes by various authors such as Nakhle H. Asmar in 2005 and others earlier, using boundary value differential equations theories.

The application of circular membranes in systems such as pizeoelectric pressure sensors, drum heads and biological systems such as eardrums, has developed the need to investigate the modal solutions of such circular membranes in more detail. Also, due to circular geometrical properties of such membranes, the study of the oscillating behavior of such membranes in Polar coordinate becomes more useful than the study in Cartesian coordinate system. Thus, this paper attempts to investigate modal behavior of the circular membranes in polar coordinates in detail using Bessel-Fourier approach.

According to Newton's second law, the sum of the forces on a small portion of a body is equal to the mass of that small portion times the acceleration $\left(\frac{\partial^2 u}{\partial t^2}\right)$ of the body in motion. Also, the sum of forces on a vibrating membrane is equal to the tension on the membrane times the stretch of the membrane during oscillations [1]. This force balance equality gives the equation that governs the behaviour of the vibrating membrane which mathematically is in the Laplacian form. This concept can be derived from elementary geometrical mechanics of vibrating bodies that leads to an equation known as the wave equation describing the motion of a vibrating membrane. The following wave equation according to Kreyszig [1] for circular plates can be utilized to develop the vibration modal solutions of the circular membranes in polar coordinates:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{r \partial r^2} + \frac{\partial^2 u}{r^2 \partial \theta^2} \right) \tag{1}$$

where, "c^2" is the membrane tension divided by the membrane density, "r" is the radius of the membrane and "θ" is the angle around the axis of the membrane.

For this wave equation the following solution is valid,

$$u = F(r, \theta) G(t). \tag{2}$$

Substituting (2) and it's derivative into (1), one would have:

$$\ddot{G} F = c^2 \left(\frac{\partial^2 F}{\partial r^2} G(t) + \frac{\partial F}{r \partial r} G(t) + \frac{\partial^2 F}{r^2 \partial \theta^2} G(t) \right). \tag{3}$$

By separation,

$$\frac{\ddot{G}}{c^2 G} = \frac{1}{F} \left(\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{r \partial r} + \frac{\partial^2 F}{r^2 \partial \theta^2} \right). \tag{4}$$

As such the expressions on both sides must equal a constant to derive a solution. Which must be a negative constant (-k^2) to satisfy the boundary conditions without being zero [1].

$$\frac{\ddot{G}}{c^2 G} = \frac{1}{F} \left(\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{r \partial r} + \frac{\partial^2 F}{r^2 \partial \theta^2} \right) = -k^2. \tag{5}$$

Rewriting the equation (5) in terms of G one would have:

$$\frac{\ddot{G}}{c^2 G} = -k^2. \tag{6}$$

and rewriting equation (5) in terms of "F" one would have:

$$\frac{1}{F}\left(\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{r\partial r} + \frac{\partial^2 F}{r^2 \partial \theta^2}\right) = -k^2. \tag{7}$$

Equations (6) and (7) yield the two differential equations following:

$$\ddot{G} + \lambda^2 G = 0, \tag{8}$$

where $\lambda = ck$ and,

$$\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{r\partial r} + \frac{\partial^2 F}{r^2 \partial \theta^2} + k^2 F = 0. \tag{9}$$

Substitute the following expression into equation (9):

$$F = W(r)Q(\theta). \tag{10}$$

One would have the expression:

$$QW'' + \frac{1}{r}QW' + \frac{1}{r^2}WQ'' + k^2 WQ = 0 \tag{11}$$

which can be simplified as,

$$Q'' + Q\frac{r^2 W'' + rW' + r^2 k^2 W}{W} = 0. \tag{12}$$

Letting,

$$-\frac{Q''}{Q} = \frac{r^2 W'' + rW' + r^2 k^2 W}{W} = n^2 \tag{13}$$

the following expressions are derived;

$$Q'' + n^2 Q = 0 \tag{14}$$

$$r^2 W'' + rW' + (k^2 r^2 - n^2)W = 0. \tag{15}$$

Letting $s = kr$, then $(k/s) = (1/r)$ and $ds/dr = k$, Now,

$$\frac{dW}{dr} = \frac{dW}{ds}k \tag{16}$$

$$\frac{d^2W}{dr^2} = \frac{d^2W}{ds^2}k^2. \tag{17}$$

Substitute, expression (16) and (17) into equation (15) and the following expression can be derived:

$$\frac{d^2W}{ds^2}k^2 + \frac{1}{r}\frac{dW}{ds}k + \left(k^2 - \frac{n^2}{r^2}\right) W = 0. \tag{18}$$

Substitute $r = s/k$ and one would have:

$$\frac{d^2W}{ds^2} + \frac{1}{s}\frac{dW}{ds} + \left(1 - \frac{n^2}{k^2r^2}\right) W = 0. \tag{19}$$

This is a Bessel's equation as described in many mathematical literature [1]. Which has the Bessel equation solution of,

$$W_n(r) = J_n(s) = J_n(kr). \tag{20}$$

Farther more expression (14) has the solution [1]:

$$Q_n = \cos(n\theta) \qquad \text{for} \quad n = 0, 1, 2, \ldots \tag{21}$$

Thus, following the expression (10) one would have:

$$F = J_n(kr)\cos(n\theta) \tag{22}$$

and by expression (2):

$$u(r, t, \theta) = F(r, \theta)G(t) = J_n(kr)\cos(n\theta)G(t) \tag{23}$$

where, the Eigen-function $G(t)$ can be expressed by the Fourier series as following [2],

$$G_{mn}(t) = a_{mn}\cos(ck_{mn}t) + b_{mn}(ck_{mn}t) \quad \text{for} \quad m = 0, 1, 2, \ldots \tag{24}$$

Thus, expression (23) can be rewritten as,

$$u_{mn}(r, t, \theta) = [a_{mn}\cos(ck_{mn}t) + b_{mn}\sin(ck_{mn}t)]J_n(k_{mn}r)\cos(n\theta) \tag{25}$$

(for R being the radius of the circular membrane, $k_{mn} = \alpha_{mn}/R$), where the Fourier series coefficients are:

$$a_{mn} = \frac{2}{R^2 J_1^2(\alpha_{mn})} \int_0^R r f(r) J_0\left(\frac{\alpha_{mn}}{R} r\right) dr \tag{26}$$

$$b_{mn} = \frac{2}{c\alpha_{mn} R J_1^2(\alpha_{mn})} \int_0^R r g(r) J_0\left(\frac{\alpha_{mn}}{R} r\right) dr \tag{27}$$

and initial displacement and initial velocity are as follows, respectively:

$$f(r) = u(r,0) \quad \text{and} \quad g(r) = \left.\frac{\partial u}{\partial t}\right|_{t=0}. \tag{28}$$

Also, the "α_{mn}" is the mth positive zeros of $J_n(s)$ that have the following values for the first three modes: (The positive zeros can be determined by plotting the zero-th order Bessel function for a range of "s"; assuming "s" from 0 to 10 or higher):

$$\alpha_{10} = 2.4048, \quad \alpha_{20} = 5.5201, \quad \alpha_{30} = 8.6537. \tag{29}$$

2. Boundary conditions

The following boundary conditions are the trivial boundary conditions that are applicable to the solution of vibration modes for the circular membrane in polar coordinates. For a circular membrane fixed at the outer boundary, the deflection of the membrane at radius $r = R$ is null thus boundary equation 29 following can be derived.

$$u(R, t, \theta) = 0. \tag{30}$$

To develop a numerical solution one can also assume that the initial displacement of the membrane, $f(r)$ is zero and there exists an initial velocity of the vibrating membrane, $g(r)$.

$$f(r) = 0 \quad \text{and} \quad g(r) \neq 0. \tag{31}$$

3. Numerical example

A numerical example for the first three vibrational modes is illustrated here. The MathCAD software is utilized to carry out the numerical example and to illustrate the determination of the constants of the functions. The following k_{mn} factors used in equation (25) for a circular membrane with radius R of 1 ft (0.3048 m) and c of 2ft/sec (.111 Nm^2/kg), are determined for $m = 3$ and $n = 0$:

$$\begin{aligned} k_{1,0} &:= \frac{alp_{1,0}}{R}; \qquad & k_{1,0} &= 2.405; \\ k_{2,0} &:= \frac{alp_{2,0}}{R}; \qquad & k_{2,0} &= 5.52; \\ k_{3,0} &:= \frac{alp_{3,0}}{R}; \qquad & k_{3,0} &= 8.654. \end{aligned} \tag{32}$$

The following Fourier series for an initial velocity of 8.333E-3 ft/sec (0.002539898 m/sec) and initial deflection of zero coefficients are determined: (note: a_{mn} coefficients are all zero since the initial deflection is zero):

$$b_{1,0} := \left[\frac{2}{(c \cdot alp_{1,0} \cdot R)(J_1(alp_{1,0}))^2}\right] \cdot \int_0^R r\left(\frac{1}{12}\right) \cdot J_0\left(alp_{1,0}\frac{r}{R}\right) dr;$$

$$b_{1,0} = 2.776 \times 10^{-3};$$

$$b_{2,0} := \left[\frac{2}{(c \cdot alp_{2,0} \cdot R)(J_1(alp_{2,0}))^2}\right] \cdot \int_0^R r\left(\frac{1}{12}\right) \cdot J_0\left(alp_{2,0}\frac{r}{R}\right) dr;$$

$$b_{2,0} = -8.037 \times 10^{-4};$$

$$b_{3,0} := \left[\frac{2}{(c \cdot alp_{3,0} \cdot R)(J_1(alp_{3,0}))^2}\right] \cdot \int_0^R r\left(\frac{1}{12}\right) \cdot J_0\left(alp_{3,0}\frac{r}{R}\right) dr;$$

$$b_{3,0} = 4.099 \times 10^{-4}.$$

By substituting these values into equation (25), one would have the following vibrational deflection modes, for modes 1, 2 and 3, respectively. The x-axis is the radial distance of the membrane. The y-axis is the deflection of the

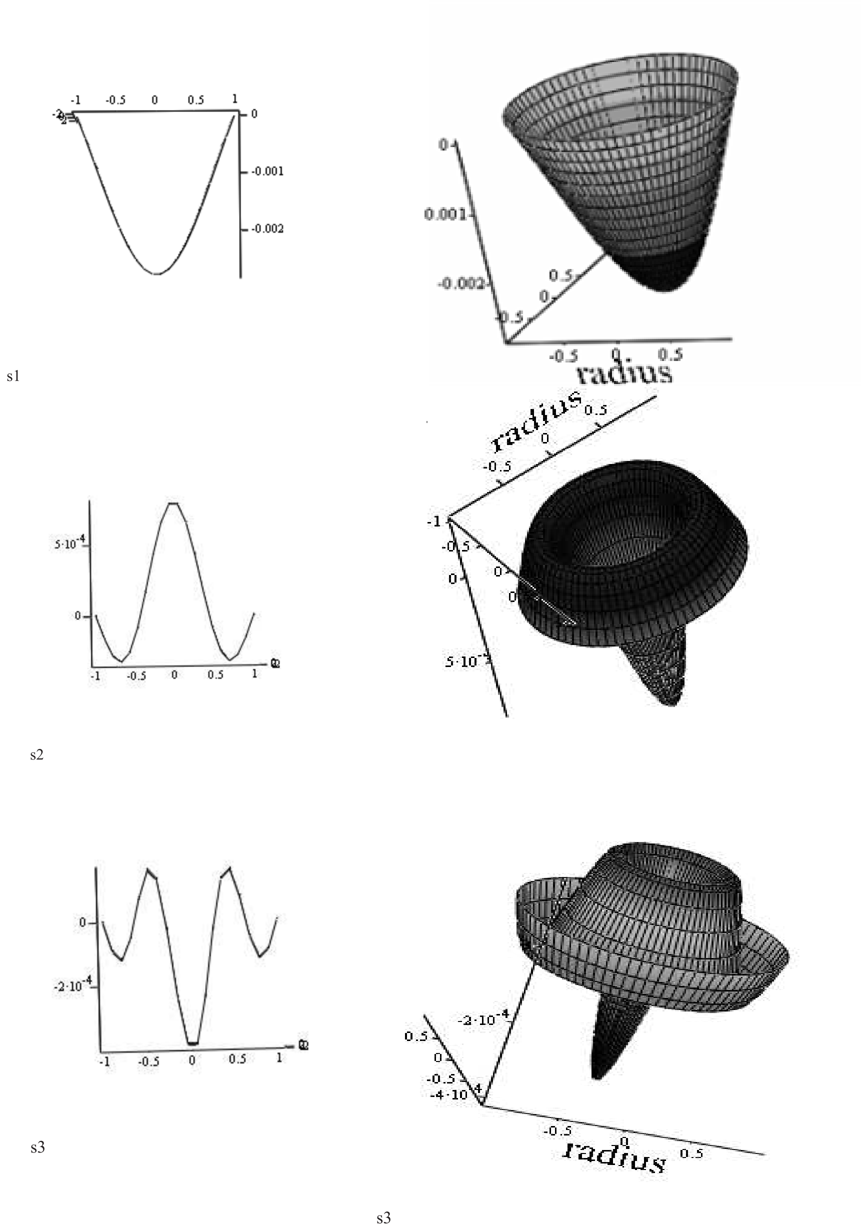

Fig. 1. The first three vibrational modes of the circular membrane

membrane. The S1 is mode 1, S2 is mode 2 and S3 is mode 3 for the circular membrane. Whereas, the figures in the left are the deflection magnitudes and the figures in the right are the 3D depictions of the mode shapes of the circular membrane.

4. Conclusion

The Fourier-Bessel solution of the circular membrane vibration modes was developed utilizing the wave equation in polar coordinates. The developed vibration modes are based on Bessel functions with solution derivatives from the Fourier series. The solutions are a purely mathematical approach to vibrational normal modes development in Polar coordinates that are an extension of the solution development in Cartesian coordinate systems.

REFERENCES

[1] Kreyszig, E. Advanced Engineering Mathematics, New York, John Wiley and Sons, 1999.

[2] Churchill, R. V., J. W. Brown. Fourier Series and Boundary Value Problems, New York, McGrawhill, 1987.

[3] Jackson, D. Fourier series and orthogonal polynomials, Mathematical Assoc. of America, D.C., Washington, 1941.

[4] Krantz, S. G. A Panorama of Harmonic Analysis, Mathematical Assoc. of America, D.C., Washington, 1999.

[5] Walker, J. S. Fourier Analysis, Oxford, Oxford Univ. Press, 1988.

[6] Zygmund, A. Trigonometric Series, Cambridge, Cambridge Univ. Press, 1968.