

INTEGRABILITY OF DIFFERENTIAL EQUATIONS WITH FLUID MECHANICS APPLICATION: FROM PAINLEVE PROPERTY TO THE METHOD OF SIMPLEST EQUATION*

ZLATINKA I. DIMITROVA

*“G. Nadjakov” Institute of Solid State Physics, Bulgarian Academy of Sciences,
71, Tzarigradsko Chaussee Blvd, 1784 Sofia, Bulgaria,
e-mail: zdim@phys.bas.bg*

KALOYAN N. VITANOV

*Faculty of Mathematics and Informatics, “St. Kliment Ohridski” University of Sofia,
5, J. Bourchier Blvd, 1164 Sofia, Bulgaria,*

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ABSTRACT. We present a brief overview of integrability of nonlinear ordinary and partial differential equations with a focus on the Painleve property: an ODE of second order possesses the Painleve property if the only movable singularities connected to this equation are single poles. The importance of this property can be seen from the Ablowitz-Ramani-Segur conjecture that states that a nonlinear PDE is solvable by inverse scattering transformation only if each nonlinear ODE obtained by exact reduction of this PDE possesses the Painleve property. The Painleve property motivated much research on obtaining exact solutions on nonlinear PDEs and led in particular to the method of simplest equation. A version of this method called modified method of simplest equation is discussed below.

KEY WORDS: Painleve property, Lorenz system, method of simplest equation.

1. Several words on the integrability of differential equations

The modern natural sciences development led to wide use of natural and social phenomena nonlinear models [1]–[4]. Thus, the problems connected to obtaining exact solutions of the model nonlinear differential equations are very actual [5]–[9]. For many researchers, the question about the integrability of nonlinear differential equations or systems of such equations is not very

*Corresponding author e-mail: zdim@phys.bas.bg

clear. They have probably heard that there exists a complicated method called Inverse Scattering Transformation (IST) [10, 11] that allows the obtaining exact solutions of some nonlinear PDEs. The researchers may have heard that integration by IST can be performed if the corresponding nonlinear differential equation is somehow connected to a property called Painleve property that states something about the singularities of the solution in the complex plane. Many researchers are puzzled by this fact as they have to solve differential equations which contain real variables. We shall discuss below the connection between the methodology of the test for the Painleve property and the research on direct methods for obtaining exact solutions of nonlinear differential equations.

How one can tell if given a differential equation or a system of equations a priori whether or not they are integrable? The existence of a constant first integral for a second order ordinary differential equation (which for an example is the energy for the many problems connected to the physics) leads to a reduction of the order of the equation by 1. Thus, the task for finding a solution is reduced to quadrature. For second order systems especially, if they are not conservative the integration is more difficult. Such systems possess solution but the problem for finding analytical representation of the solution is very difficult to solve. It is extremely useful to find integrals of motion connected to the solved equations. How to find integrals of motion if they do exist? The existence of first integrals may be connected to the analytical structure of the studied differential equation as it is implied by the research of Sofia Kovalevskaya.

2. The research of Kovalevskaya and Painleve

Kovalevskaya won the Bordin prize of the Paris Academy of Sciences in 1888 for the solution of the system of six first-order nonlinear ODEs that describe the motion of a heavy top rotation about a fixed point. Kovalevskaya solved this mechanical problem by the methods of complex variable theory (she obtained conditions under which the only movable singularities of the solutions in the complex plane are single poles). Movable singularity is a singularity which location depends on the initial conditions. Let us consider the simple nonlinear equation in the complex plane [12] (z below denotes a complex number):

$$(1) \quad \frac{df}{dz} + f^2(z) = 0.$$

It has the solution:

$$(2) \quad f(z) = \frac{1}{z + z_0}; \quad z_0 = \frac{1}{f(0)}.$$

In other words, $f(z)$ has a singularity at $z = -z_0$ and this singularity is movable as it depends on the initial condition. Painleve continued this research on the connection between analytic structure and integrability by analysis of the class of second order differential equations:

$$(3) \quad \frac{d^2y}{dz^2} = F\left(\frac{dy}{dz}, y, z\right),$$

where F is analytic function in z and rational in y and $\frac{dy}{dz}$. Painleve obtained 50 types of such equations whose only movable singularities were simple poles. Fourty four of these equations can be integrated in terms of known functions. The remained six equations are called Painleve transcendents and they can not be integrated by quadratures. The simplest two Painleve transcendents are

$$\text{P1: } \frac{d^2y}{dx^2} = 6y^2 + x; \quad \text{P2: } \frac{d^2y}{dx^2} = 2y^3 + xy + \pi; \quad \pi: \text{ free parameter.}$$

The solutions even for simple ODEs in complex plane are not known but the nature of their movable singularities can be determined by studying the local properties of the solutions. Let as above the function F is analytic in z and rational in other arguments. The behaviour of the movable singularities of the n -th order ODE:

$$(4) \quad \frac{d^ny}{dz^n} = F\left(\frac{d^{n-1}y}{dz^{n-1}}, \dots, \frac{dy}{dz}, y, z\right),$$

can be determined by leading order analysis, i.e., by the ansatz:

$$(5) \quad y(z) = a(z - z_0)^\alpha.$$

Let us discuss an equation which contains as particular case the first Painleve transcendent (A below is a parameter) [12]:

$$(6) \quad \frac{d^2y}{dz^2} = 6y^2 + Az.$$

We substitute Eq. (5) in Eq. (6) and obtain as a result a relationship that contains the powers $\alpha - 2$; 2α ; and α . In order to obtain a non-zero expression of kind (5) we have to balance at least two of the powers. Taking into account that the highest order singularity arises from the lowest (negative) value of α the appropriate balance is: $\alpha - 2 = 2\alpha \rightarrow \alpha = -2$. Then, at z close to z_0 the leading order of the solution is $\frac{1}{(z - z_0)^2}$ i.e. second order (movable simple) pole. In order to characterize the behavior better in the neighborhood of the singularity a local expansion of the solution has to be considered. The

expansion is a Laurent series if the singularity is a movable pole. In our case (the second-order pole above), the candidate for Laurent series is:

$$(7) \quad y(z) = \sum_{i=0}^{\infty} a_i (z - z_0)^{i-2}.$$

One obtains a system of recursion equations ($a_0 = 1$) after the substitution of Eq. (7) in the equation (6): $a_i(i+1)(i-6) = 6 \sum_{l=1}^{i-1} a_{i-l}a_l + Aa_{i-2}$. The solution of the system of equations is: $a_1 = 0$; $a_2 = -\frac{A}{12}$; $a_3 = 0$; $a_4 = -\frac{A^2}{24}$; $a_5 = 0$; $0 \cdot a_6 = 12a_2a_4 + Aa_4 = 0$. Thus, the series (7) have two free parameters a_6 and z_0 (the arbitrariness of the pole position). As the investigated ODE (6) is of second order, its general solution contains two free parameters. These free parameters are contained in the local series expansion, i.e., (7) are Laurent series of the general solution, indeed. In addition, in the neighborhood of the movable singularity z_0 the solution of Eq.(6) behaves as a second order pole, i.e., the equation has the Painleve property (which is a strong hint for its integrability).

The powers of $(z - z_0)$ at which the arbitrary coefficient appears are called resonances. At the resonances, one obtains a compatibility condition which must be satisfied in order to ensure the arbitrariness of the corresponding coefficient. In our case the resonance occurred at $i = 6$ and the compatibility condition is: $12a_2a_4 + Aa_4 = 0$.

The leading order of the solution behaviour near the pole as well as the value of the resonance can be integer number (and then the pole is simple and the equation has the Painleve property) but these two numbers can be also non-integer (for an example irrational or even complex). In this case, the singularity is not a simple pole and the equation does not have the Painleve property (which is a hint of non-integrability from the point of view of the research of Kovalevskaya and Painleve).

In some cases, the Painleve property can be established only for selected values of the parameters of the studied nonlinear ODEs. In such case, one can attempt the Ψ -series technique, that will not be discussed here.

3. Weiss-Tabor-Carnevale research on Painleve property for nonlinear PDEs

Weiss, Tabor, and Carnevale [13] continued the research of Painleve in the direction of finding hints for obtaining exact solutions of nonlinear partial

differential equations. Their work was motivated by the conjecture of Ablowitz, Ramani and Segur [14], that a nonlinear PDE is solvable by inverse scattering transform (IST) only if each nonlinear ODE obtained by exact reduction of this PDE possesses the Painleve property. In other words, we have to test all reductions of the PDE to ODEs for Painleve property and if all ODEs have the Painleve property then the corresponding nonlinear PDE has to be integrable by IST. But what are all these reductions? We don't know them. Then, one has to follow another way and one possible way the generalized Laurent expansion approach of Weiss, Tabor and Carnevale. The key point in this approach is that as difference to the case of complex function of one complex variable for the case of functions of several complex variables, the singularities can't be isolated and these singularities are located in singular manifolds. The singularities of f are located in analytic manifolds (called singular manifolds) of dimension $2n - 2$, if $f(z_1, \dots, z_n)$ is a meromorphic function of the complex variables z_1, \dots, z_n , which are determined by conditions like:

$$(8) \quad \psi(z_1, \dots, z_n) = 0,$$

where ψ is analytic function in the neighborhood of the manifold defined by Eq.(8). Weiss, Tabor and Carnevale have proposed generalized Laurent expansion of analytic function $f(z_1, \dots, z_n)$ as follows:

$$(9) \quad f(z_1, \dots, z_n) = \frac{1}{\phi^\alpha} \sum_{i=0}^{\infty} u_i \phi^i,$$

where $\phi(z_1, \dots, z_n)$ and $u_i(z_1, \dots, z_n)$ are analytic functions of z_1, \dots, z_n in the neighborhood of the manifold given by Eq.(8). The parameter α is an integer. The difference to the classic Laurent series is that instead of $z - z_0$ we have the function ϕ and u_i , which are functions and not constant coefficients as in the classic case. Additional requirement to the function ϕ is that its gradients do not vanish on the singular manifold.

The test for the generalized Painleve property happens as in the case of ordinary differential equations. First, Eq.(9) is substituted in the studied nonlinear PDE and the possible values of α and the recursion relations for the function u_i are determined. The recursion relations are coupled PDEs containing the functions ϕ and u_i . As in the case of the discussed above ODEs there will be powers of ϕ for which the corresponding u_i are arbitrary (resonances). Some kind of generalized Ψ -series can be introduced to ensure their arbitrariness, if some u_i are not arbitrary at the resonance.

4. Application to systems connected to fluid mechanics

We shall discuss the application the methodology for detecting Painleve property to the Lorenz system and one of its generalizations, namely:

$$(10) \quad \begin{aligned} \frac{dx}{dt} &= -\sigma x + \sigma y - \omega yz, \\ \frac{dy}{dt} &= rx - y - mxz, \quad \frac{dz}{dt} = -bz + xy. \end{aligned}$$

where $\sigma, \omega, r, m,$ and b are parameters. When $\omega = 0; m = 1$ the system (10) is reduced to the classical Lorenz system well known from meteorology, nonlinear dynamics and chaos theory. The classic Lorenz system was tested for presence of Painleve property in [15]. Let us transform the equations of the system (10) by setting $x = \frac{X}{\epsilon}; y = \frac{Y}{\sigma\epsilon^2}; z = \frac{Z}{\sigma\epsilon^2}; t = \epsilon T; \epsilon = \frac{1}{\sqrt{\sigma r}}$. The system (10) becomes:

$$(11) \quad \begin{aligned} \frac{dX}{dT} &= -\sigma\epsilon X + Y - \frac{\omega}{\sigma^2\epsilon} YZ, \\ \frac{dY}{dT} &= X - \epsilon Y - mXZ, \quad \frac{dZ}{dT} = -b\epsilon Z + XY. \end{aligned}$$

We shall investigate the system of equations (11) for presence of Painleve property.

First of all, we have to perform the leading order analysis by setting:

$$(12) \quad X = \frac{a}{T^\alpha}; Y = \frac{b}{T^\beta}; Z = \frac{c}{T^\gamma},$$

where a, b, c and α, β, γ are parameters. The substitution of Eqs. (12) in the system (11) leads to the following relationships, after balancing the largest powers in the denominators of the resulting three equations: $\alpha = \beta = \gamma = 1$. This result is different from the case of classical Lorenz system where $m = 1$ and $\omega = 0$ (the last relationships leads to significant changes of the right-hand side of the first equation from (11) – the nonlinearity there is removed). For this case, the substitution of Eqs. (12) in the system (11) leads to the result $\alpha = 1; \beta = \gamma = 2$. For the case of generalized Lorenz system, we obtain the following system of equation for the parameters a, b, c (after equating the numerators of the two largest powers from the denominators from the equations for the leading order analysis):

$$(13) \quad -a + \frac{\omega bc}{\sigma^2\epsilon} = 0, \quad -b + mac = 0, \quad -c - ab = 0.$$

The non-trivial solution of the system of equations (13) is $a = \pm \frac{i}{\sqrt{m}}$; $b = \pm i\sqrt{\frac{\epsilon}{\omega}}$; $c = \sqrt{\frac{\epsilon}{m\omega}}$.

What follows, are the power series expansions around the singularities. On the basis of the above results, we shall consider the following expansions for the three unknown functions around the singularity of the solution which we assume to be at (X_0, Y_0, Z_0) :

$$\begin{aligned} X &= \frac{i}{\sqrt{m}(T-T_0)} \sum_{j=0}^{\infty} a_j (T-T_0)^j, \quad Y = \frac{i\sqrt{\epsilon}}{\sqrt{\omega}(T-T_0)} \sum_{j=0}^{\infty} b_j (T-T_0)^j, \\ Z &= \frac{\sqrt{\epsilon}}{\sqrt{m\omega}(T-T_0)} \sum_{j=0}^{\infty} c_j (T-T_0)^j. \end{aligned} \quad (14)$$

These series are different from the series for the classical Lorenz system. For the classical Lorenz system, the system of equations for the parameters a, b, c is:

$$(15) \quad a + b = 0, \quad ac - 2b = 0, \quad 2c + ab = 0,$$

which solution is $a = \pm 2i$; $b = \mp 2i$; $c = -2$, and the expansions for X, Y, Z around the singularity become:

$$\begin{aligned} X &= \frac{2i}{(T-T_0)} \sum_{j=0}^{\infty} a_j (T-T_0)^j, \quad Y = \frac{-2i}{(T-T_0)^2} \sum_{j=0}^{\infty} b_j (T-T_0)^j, \\ (16) \quad Z &= \frac{-2}{(T-T_0)^2} \sum_{j=0}^{\infty} c_j (T-T_0)^j. \end{aligned}$$

The next step is the recurrence equations obtaining. We have to substitute the expansion (14) in the generalized Lorenz system (10) and by equalizing to 0 the coefficients of the powers $(T-T_0)^{-n}$, $n = -1, 0, 1, \dots$ we shall obtain a system of nonlinear recurrent equations for the coefficients a_j, b_j and c_j . We obtain that $a_0 = b_0 = c_0 = 1$. by setting to 0 the coefficients of $(T-T_0)^{-2}$ in the resulting equations setting to 0 the coefficients of $(T-T_0)^{-1}$ in the resulting equations, we obtain the following system for the coefficients a_1, b_1, c_1 :

$$\begin{aligned} \sigma\sqrt{\epsilon m}(\sqrt{\epsilon} - \sqrt{\omega}) + c_1 + b_1 &= 0, \quad \sigma\epsilon + c_1 + a_1\sigma - \sqrt{\omega}m\epsilon = 0, \\ (17) \quad a_1 + b_1 + \epsilon b &= 0. \end{aligned}$$

The solution of this system is:

$$\begin{aligned}
 a_1 &= \frac{1}{2} \left[-\epsilon + \sigma \sqrt{\frac{\omega}{\epsilon m}} - \sigma \sqrt{\frac{m}{\epsilon \omega}} - \sigma \epsilon - \epsilon b \right], \\
 b_1 &= - \left[\sigma \epsilon - \sigma \sqrt{\frac{\epsilon m}{\omega}} - \frac{1}{2} \left(\epsilon - \frac{1}{\sigma} \sqrt{\omega \epsilon m} - \sigma \sqrt{\frac{m \epsilon}{\omega}} + \sigma \epsilon - b \epsilon \right) \right], \\
 (18) \quad c_1 &= \frac{1}{2} \left[-\epsilon + \sigma \sqrt{\frac{\omega}{\epsilon m}} + \sigma \sqrt{\frac{m \epsilon}{\omega}} - \sigma \epsilon + \epsilon b \right].
 \end{aligned}$$

The situation with the classic Lorenz system ($\omega = 0$, $m = 1$) is different again. In this case, we have to substitute the series (16) in (10) (with $\omega =$ and $m = 1$). We obtain the result $a_0 = b_0 = c_0 = 1$ by setting to 0 the coefficient of $(T - T_0)^{-2}$ in the first resulting equation and the coefficients of $(T - T_0)^{-3}$ in the second and third resulting equation. Next, we have to set to 0 the coefficient of $(T - T_0)^{-1}$ in the first resulting equation and the coefficients of $(T - T_0)^{-1}$ in the second and third resulting equation. We obtain the following system of equations for a_1 , b_1 and c_1 :

$$2\sigma\epsilon i + 2ib_1 = 0, \quad b_1 - \epsilon - 2c_1 - 2a_1 = 0, \quad c_1 - 2b_1 - 2a_1 - \epsilon b = 0.$$

The solution of this system is:

$$(19) \quad a_1 = \frac{(3\sigma - 2b - 1)\epsilon}{6}, \quad b_1 = -\epsilon\sigma, \quad c_1 = \frac{b - 1 - 3\sigma}{3}.$$

For $j = 2, 3, \dots$ for the generalized Lorenz system we obtain the system of recurrence equations:

$$\begin{aligned}
 \frac{1}{\sqrt{m}}(j-1)a_j + \frac{\sigma\epsilon}{\sqrt{m}}a_{j-1} - \frac{\sqrt{\epsilon}}{\sqrt{\omega}}b_{j-1} + \frac{1}{m\sigma^2} \sum_{l=0}^j b_l c_{j-l} &= 0, \\
 \frac{\sqrt{\epsilon}}{\sqrt{\omega}}(j-1)b_j - \frac{1}{\sqrt{m}}a_{j-1} + \frac{\epsilon\sqrt{\epsilon}}{\sqrt{\omega}}b_{j-1} + \frac{\sqrt{\epsilon}}{\sqrt{m}\sqrt{\omega}} \sum_{l=0}^j a_l c_{j-l} &= 0, \\
 (20) \quad \frac{1}{m}(j-1)c_j + \frac{b\epsilon}{m}c_{j-1} + \frac{1}{\sqrt{m}} \sum_{l=0}^j a_l b_{j-l} &= 0.
 \end{aligned}$$

Let us obtain the first relationship from the system (20) in more detail. The substitution of the series (14) in the system of generalized Lorenz equations (11), leads to the relationship:

$$\sum_{j=0}^{\infty} (T - T_0)^{j-2} \left[\frac{1}{\sqrt{m}}(j-1)a_j + \frac{\sigma\epsilon}{\sqrt{m}}a_{j-1} - \right.$$

$$(21) \quad \left. \frac{\sqrt{\epsilon}}{\sqrt{\omega}} b_{j-1} \right] + \frac{1}{m\sigma^2} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} b_l c_k (T - T_0)^{l+k-2} = 0.$$

From Eq.(21), we want to select the terms of the same power of $(T - T_0)$. This means that we must have $j - 2 = l + k - 2$, i.e., $j = l + k$. In other words, from the double sum in (21) we have to select the terms for which $k = j - l$ and in addition we have to take into account that $k \geq 0$ which leads to cutting the sum with respect to l from $l = 0$ to $l = j$. Then the terms from the double sum that participate in the recurrence relationship are given by ($\delta_{i,j}$ is the delta-symbol of Kronecker):

$$(22) \quad \sum_{l=0}^j \sum_{k=0}^{\infty} b_l c_k \delta_{k,j-l} (T - T_0)^{l+k-2} \rightarrow \sum_{l=0}^j b_l c_{j-l} (T - T_0)^{j-2}.$$

We note that in the last relationship j was fixed. We obtain the first relationship from (20) putting all together for fixed value of j in Eq.(21). The remaining relationships from (20) are obtained in the same way.

For comparison the situation in the classic Lorenz system is simpler. We obtain for $j = 2, 3, \dots$ the system of recurrence equations:

$$(23) \quad \begin{aligned} (j-1)a_j + b_j + \sigma \epsilon a_{j-1} &= 0, \\ 2a_j + (j-2)b_j + 2c_j + 2 \sum_{k=1}^{j-1} a_{j-k} c_k + a_{j-2} + \epsilon b_{j-1} &= 0, \\ 2a_j + 2b_j + (j-2)c_j + 2 \sum_{k=1}^{j-1} a_{j-k} b_k + \epsilon b c_{j-1} &= 0. \end{aligned}$$

For $j = 2$ and $j = 4$, there are resonances that lead to the following relationships among the parameters of the equations of the classic Lorenz system:

$$(24) \quad \begin{aligned} 6\sigma^2 - \sigma b - 2\sigma - b(b-1) &= 0, \\ (b-1)[24 + 57(\sigma-1) - 15(b-1)] - 9\sigma(2\sigma-1) &= 0, \\ 6(1+b-\sigma)a_1^2 c + \epsilon(2\sigma-b-5) \left(2\sigma a_1^2 + 6a_1 c_1 \frac{b(1-\sigma\epsilon^2)}{2} \right) &= 0. \end{aligned}$$

The last relationships fix conditions on the values of ϵ , σ and β , for which one the classical Lorenz system is integrable. As ϵ participates only in the third equations from (24) from the first two relationships there one determines σ and b and then the last relationship leads to the corresponding value of ϵ . One example for such solution is $b = 1$; $\sigma = 1/2$, $\epsilon = \infty$. Then, the solution of the Lorenz system can be constructed on the basis of Jacobi elliptic functions.

We let the interested reader to find the values of the parameters for which the generalized Lorenz system is integrable.

5. Method of simplest equation and its version called modified method of simplest equation

We shall give short remarks on the method of simplest equation [16, 17] and on his version called modified method of simplest equation [18]–[20] from the many approaches for obtaining exact analytical solutions of nonlinear PDFs. The method of simplest equation uses the first step of the test for the Painleve property: the leading order analysis to determine the truncation of the sum from Eq. (26) below. This first step is changed by the equivalent procedure of solving of balance equation in the modified method of simplest equation. The methodology is based on the fact that after application of appropriate ansatz large class of NPDEs can be reduced to ODEs of the kind:

$$(25) \quad \mathcal{P} \left(F(\xi), \frac{dF}{d\xi}, \frac{d^2F}{d\xi^2}, \dots \right) = 0,$$

and for some equations of the kind (25) particular solutions can be obtained which are finite series:

$$(26) \quad F(\xi) = \sum_{i=1}^P a_i [\Phi(\xi)]^i,$$

constructed by solution $\Phi(\xi)$ of more simple equation referred to as simplest equation. The simplest equation can be the equation of Bernoulli, equation of Riccati, etc. The substitution of Eq. (26) in Eq. (25) leads to the polynomial equation:

$$(27) \quad \mathcal{P} = \sigma_0 + \sigma_1 \Phi + \sigma_2 \Phi^2 + \dots + \sigma_r \Phi^r = 0,$$

where the coefficients σ_i , $i = 0, 1, \dots, r$ depend on the parameters of the equation and on the parameters of the solutions. Equating all these coefficients to 0, i.e., by setting:

$$(28) \quad \sigma_i = 0, i = 1, 2, \dots, r,$$

one obtains a system of nonlinear algebraic equations. Each solution of this system leads to a solution of kind (26) of Eq. (25).

We have to ensure that σ_r contains at least two terms in order to ensure non-trivial solution by the cited above method. To do this in the modified simplest equation method we have to balance the highest powers of ϕ that are obtained from the different terms of the solved equation of kind (25). As a result

of this, we obtain an additional equation between some of the parameters of the equation and the solution. This equation is called balance equation [18]–[20].

6. Conclusion

As a conclusion, we stress that there is a unproved conjecture that the Painleve property is a direct test of integrability of a differential equation or for a system of differential equations. There is however a solid evidence that the conjecture is true [12]. The Liouville theorem from the complex analysis states that if $f(z)$ is an entire bounded function of the complex variable z then it can take only one (constant) value $f(z) = c$. Example of such entire bounded functions (if we think the time t to be a complex variable) are the time-independent integrals of the motion connected to a differential equation or to a system of such equations. Let us assume, that the studied system of equations is Hamiltonian and algebraic integrable, i.e., the time-independent integrals are polynomials of the canonical variables p_i and q_i . In the complex plane the functions p_i and q_i can possess various kinds of movable singularities, but . . . , as the constructed by them integral has to be a constant in the complex plane all the singularities must cancel at any singularity position t_0 . This is possible for polynomial in p_i and q_i integrals only if the singularities of p_i and q_i are simple poles or rational branch points. It will be not possible to construct a polynomial function in which the corresponding power of singularities cancel for other kinds of terms in the integrals. The words above hint that if the studied system has Hamiltonian with integrals that are not polynomials (and are for an example irrational or transcendental functions) the Painleve property could not be considered as an appropriate test for integrability of corresponding system of differential equations.

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