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STRAIN ENERGY RELEASE RATE DETERMINATION IN THE CASE OF MODE II CRACK IN OVERHANGING BILAYERED COMPOSITE BEAM*

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Abstract. Mode II crack in overhanging bilayered composite beam is investigated. The beam has rectangular cross-section and is made by two unidirectional fiber reinforced composites. The formula for strain energy release rate, G, is obtained by linear elastic fracture mechanics compliance technique. The validity of the expression derived is established by comparison with solution for G in which the internal forces in front and behind the crack tip are used. The influence of the two layers moduli of elasticity ratio on the strain energy release rate is investigated. The dependence among the strain energy release rate and the ratio of the lengths of the overhang beam part and the beam span is also analyzed.
Key words: Mode II crack, strain energy release rate, compliance technique, linear elastic fracture mechanics.

1. Introduction

The unidirectional fibre reinforced composites possess higher strength in the longitudinal direction, i.e. in the direction of the fibres, and lower strength transversely to the fibres. Due to this fact, one of the main disadvantages of these materials is the initiation and propagation of cracks in the matrix along the fibres. In the most of the publications concerned with the interlaminar fracture behaviour of the composite materials the objects of investigations are beams [1–4]. The reason is that the beam specimens are suitable for experimental characterization of fracture resistance [5–8]. Solutions of the problem for the strain energy release rate determination for the pure modes as well as

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the mixed modes cracks are derived. The commonest configurations for crack analysis are double cantilever beam (DCB), asymmetric double cantilever beam (ADCB), simply supported beam loaded by a vertical force in the middle (ENF) and simply supported beam loaded by two vertical forces (MMB).

In the present study, the overhanging bilayered unidirectional fibre reinforced composite beam containing a crack between the layers is considered. The reinforcing fibres are situated longitudinally to the beam axis. Besides, the fibres are uniformly distributed in the cross-sectional plane of the beam. The load represents a vertical force F applied at the end of the overhang beam part (Fig. 1).

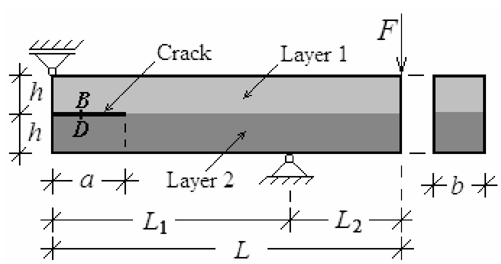

Fig. 1. Beam specimen investigated

It should be noted that, due to the supports position, the force F causes mutual displacements of the two crack arms in the manner of Mode II crack. By definition, in the case of Mode II fracture, the relative displacements of the two crack arms are along the crack, as shown in Fig. 2 [9–13]. Hence, considering the mutual displacements of the crack arms of the beam shown in Fig. 1, it can be concluded that they correspond to the definition of Mode II fracture. Moreover, this fact could be proved by determination of the mutual displacements of two opposite points belonging to the crack arms, for example points B and D (Fig. 1).

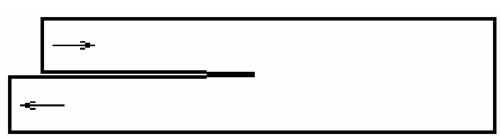

Fig. 2. Mutual displacements of the crack arms in the case of Mode II fracture

2. Determination of the strain energy release rate

The linear elastic fracture mechanics compliance technique is applied in order to obtain the formula for the strain energy release rate, G [14, 15]. According to this technique, the expression for G is:

$$G = \frac{F^2}{2b}\frac{dC}{da}, \tag{1}$$

where F is the external force, b is the length of the crack front (here, b is the width of the beam), C is the beam compliance, a is the crack length.

The beam compliance is obtained on the basis of the formula:

$$C = \frac{w}{F}, \tag{2}$$

where w is the vertical displacement of the beam section where the force F is applied. It should be noted that the influence of the shearing forces on the vertical displacement is taken into account in order to obtain the more complete solution.

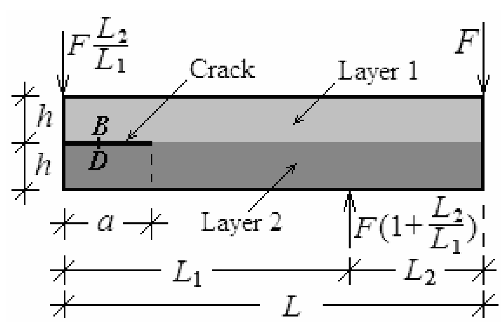

Fig. 3. Support reactions of the beam investigated

First, the support reactions are determined and shown in Fig. 3. Then, since the interaction between the layers in the cracked beam portion the left support reaction is resolved into components for each crack arm by the help of [16]. Thus, the components F_1 and F_2 of the force $F\frac{L_2}{L_1}$ are obtained, as follows (Fig. 4):

$$F_1 = F\frac{L_2}{L_1}\frac{E_1}{E_1+E_2}, \quad F_2 = F\frac{L_2}{L_1}\frac{E_2}{E_1+E_2}, \tag{3}$$

where E_1 and E_2 are the moduli of elasticity of the two layers, respectively.

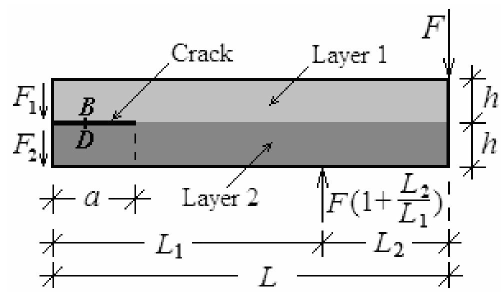

Fig. 4. Resolution of the left support reaction into components

Next step of the solution is to express the un-cracked beam portion modulus of elasticity, E, as well as the shear modulus, G', as functions of the two layers moduli. The modulus of elasticity is derived on the basis of [17], where the stiffness of the bilayered beam subjected to bending is given by the equation:

$$EI = E_1 I_1 + E_2 I_2 + \frac{E_1 A_1 E_2 A_2 c^2}{E_1 A_1 + E_2 A_2}. \tag{4}$$

In (4), $I = \frac{2bh^3}{3}$ is the cross-sectional moment of inertia of the bilayered beam subjected to bending, $I_1 = I_2 = \frac{bh^3}{12}$ are the cross-sectional moments of inertia of the two layers, $A_1 = A_2 = bh$ are the cross-sectional areas of the layers, $c = h$ is the distance between the cross-sectional centers of gravity of the layers. Thus, the formula for the beam un-cracked portion modulus of elasticity takes a form:

$$E = \frac{E_1^2 + E_2^2 + 14E_1E_2}{8(E_1 + E_2)}. \tag{5}$$

The un-cracked beam portion shear modulus is expressed by following equation taken from [18]:

$$G' = \frac{2G_1' G_2'}{G_1' + G_2'}, \tag{6}$$

where G_1' and G_2' are the shear moduli of the two layers, respectively.

Further, the vertical displacement w of the beam section where the force F is applied is determined by Mechanics of materials methods [19] adding the fact that the shearing forces influence is taken into account. Introducing the x-axis of origin the left section of the beam, the vertical displacement at distance $x = L$ is derived as:

$$
\begin{aligned}
w(L) = {} & \frac{4FL_2^2a^3}{(E_1+E_2)\,bh^3L_1^2} + \frac{4FL_2^2\left(LL_1^2-a^3\right)}{bh^3L_1^2}\left(\frac{E_1+E_2}{E_1^2+E_2^2+14E_1E_2}\right)+ \\
& + \frac{kFL_2^2a}{bhL_1^2\,(E_1+E_2)^2}\left(\frac{E_1^2}{G_1'}+\frac{E_2^2}{G_2'}\right)+\frac{kFL_2^2\,(G_1'+G_2')}{4G_1'G_2'bhL_1^2}\left(L_1-a+\frac{L_1^2}{L_2^2}\right),
\end{aligned} \tag{7}
$$

where k is the shear coefficient.

Then, the expression of the beam compliance results in:

$$
\begin{aligned}
C = \frac{w(L)}{F} = {} & \frac{4L_2^2a^3}{(E_1+E_2)\,bh^3L_1^2} + \frac{4L_2^2\left(LL_1^2-a^3\right)}{bh^3L_1^2}\left(\frac{E_1+E_2}{E_1^2+E_2^2+14E_1E_2}\right)+ \\
& + \frac{kL_2^2a}{bhL_1^2\,(E_1+E_2)^2}\left(\frac{E_1^2}{G_1'}+\frac{E_2^2}{G_2'}\right)+\frac{kL_2^2\,(G_1'+G_2')}{4G_1'G_2'bhL_1^2}\left(L_1-a+\frac{L_1^2}{L_2^2}\right).
\end{aligned} \tag{8}
$$

After that, the first order derivative of the compliance C with respect to the crack length a is determined. The result is:

$$
\begin{aligned}
\frac{dC}{da} = {} & \frac{12L_2^2a^2}{(E_1+E_2)\,bh^3L_1^2} - \frac{12L_2^2a^2}{bh^3L_1^2}\left(\frac{E_1+E_2}{E_1^2+E_2^2+14E_1E_2}\right)+ \\
& + \frac{kL_2^2}{bhL_1^2}\left[\frac{E_1^2}{G_1'\,(E_1+E_2)^2}+\frac{E_2^2}{G_2'\,(E_1+E_2)^2}-\frac{G_1'+G_2'}{4G_1'G_2'}\right].
\end{aligned} \tag{9}
$$

Substituting (9) in (1) the strain energy release rate formula takes the form:

$$
\begin{aligned}
G = \frac{F^2}{2b}\frac{dC}{da} = {} & \frac{6F^2L_2^2a^2}{(E_1+E_2)\,b^2h^3L_1^2} - \frac{6F^2L_2^2a^2}{b^2h^3L_1^2}\left(\frac{E_1+E_2}{E_1^2+E_2^2+14E_1E_2}\right)+ \\
& + \frac{kF^2L_2^2}{2bhL_1^2}\left[\frac{E_1^2}{G_1'\,(E_1+E_2)^2}+\frac{E_2^2}{G_2'\,(E_1+E_2)^2}-\frac{G_1'+G_2'}{4G_1'G_2'}\right].
\end{aligned} \tag{10}
$$

Finally, simplifying (10), for G is obtained:

(11)
$$G = \frac{72F^2L_2^2a^2}{b^2h^3L_1^2}\left[\frac{E_1E_2}{(E_1+E_2)(E_1^2+E_2^2+14E_1E_2)}\right]+ \\ +\frac{kF^2L_2^2}{2bhL_1^2}\left[\frac{1}{(E_1+E_2)^2}\left(\frac{E_1^2}{G_1'}+\frac{E_2^2}{G_2'}\right)-\frac{G_1'+G_2'}{4G_1'G_2'}\right].$$

3. Check of the solution validity

In order to validate (11), the strain energy release rate is also determined by the following formula [20]:

(12)
$$G = \frac{1}{2b}\left(\frac{N_1^2}{E_1A_1}+\frac{N_2^2}{E_2A_2}-\frac{N^2}{EA}\right)+\frac{1}{2b}\left(\frac{M_1^2}{E_1I_1}+\frac{M_2^2}{E_2I_2}-\frac{M^2}{EI}\right)+ \\ +\frac{k}{2b}\left(\frac{V_1^2}{G_1'A_1}+\frac{V_2^2}{G_2'A_2}-\frac{V^2}{G'A}\right),$$

where N_1, V_1, M_1 are the internal forces in the upper crack arm of the beam section behind the crack tip, N_2, V_2, M_2 are the internal forces in the lower crack arm of the beam section behind the crack tip, N, V, M are the internal forces of the beam section in front of the crack tip.

The internal forces of the beam sections in front and behind the crack tip are obtained, as follow:

(13)
$$N_1 = 0, \quad V_1 = -F\frac{L_2}{L_1}\frac{E_1}{E_1+E_2}, \quad M_1 = -Fa\frac{L_2}{L_1}\frac{E_1}{E_1+E_2};$$
$$N_2 = 0, \quad V_2 = -F\frac{L_2}{L_1}\frac{E_2}{E_1+E_2}, \quad M_2 = -Fa\frac{L_2}{L_1}\frac{E_2}{E_1+E_2};$$
$$N = 0, \quad V = -F\frac{L_2}{L_1}, \quad M = -Fa\frac{L_2}{L_1}.$$

Substituting (13) in (12) the expression for G results in:

$$G = \frac{72F^2L_2^2a^2}{b^2h^3L_1^2}\left[\frac{E_1E_2}{(E_1+E_2)(E_1^2+E_2^2+14E_1E_2)}\right]+ \tag{14}$$
$$+\frac{kF^2L_2^2}{2bhL_1^2}\left[\frac{1}{(E_1+E_2)^2}\left(\frac{E_1^2}{G_1'}+\frac{E_2^2}{G_2'}\right)-\frac{G_1'+G_2'}{4G_1'G_2'}\right].$$

It is obvious that (11) and (14) completely match, meaning that the formula derived can be successfully applied for Mode II crack investigations in overhanging bilayered composite beams.

4. Investigation of the influence of the two layers moduli of elasticity ratio and the ratio of the lengths of the overhang beam part and the beam span on the strain energy release rate

In order to perform this investigation, (11) is rearranged in the manner:

$$\frac{G}{E_1b} = \frac{72F^2L_2^2a^2}{E_1b^3h^3L_1^2}\left[\frac{E_1E_2}{(E_1+E_2)(E_1^2+E_2^2+14E_1E_2)}\right]+ \tag{15}$$
$$+\frac{kF^2L_2^2}{2E_1b^2hL_1^2}\left[\frac{1}{(E_1+E_2)^2}\left(\frac{E_1^2}{G_1'}+\frac{E_2^2}{G_2'}\right)-\frac{G_1'+G_2'}{4G_1'G_2'}\right].$$

After that, the coefficients $\theta=\dfrac{E_1}{E_2}$, $r=\dfrac{L_2}{L_1}$, $\eta=\dfrac{G_1'}{G_2'}$ are introduced. The result is:

$$\frac{G}{E_1b} = \frac{72F^2a^2}{E_1^2b^3h^3}\left[\frac{\theta^2}{(1+\theta)(\theta^2+14\theta+1)}\right]r^2+ \tag{16}$$
$$+\frac{kF^2}{2E_1G_1'b^2h}\left[\frac{\theta^2+\eta}{(1+\theta)^2}-\frac{1+\eta}{4}\right]r^2.$$

Further, the coefficients θ and r are assumed to take values 1, 2, 3, 4, and 5, while the parameter η and the expressions out of the brackets are supposed to be equal to one. The influence of the parameters θ and r on G is given in Table 1. It is evident, that when the two layers moduli of elasticity ratio has minimum value, i.e. $\theta=1$, the strain energy release rate takes the smallest value, too. This finding could be explained by the fact that in the case of equal moduli of elasticity of the two layers, the stiffnesses of the two crack arms are equal, which reduces the crack arms mutual displacements and G, as well.

Table 1. Influence of the coefficients $\theta = \dfrac{E_1}{E_2}$ and $r = \dfrac{L_2}{L_1}$ on $\dfrac{G}{E_1 b}$.

	θ \ r	1	2	3	4	5
$\dfrac{G}{E_1 b}$	1	0.031	0.125	0.281	0.500	0.781
	2	0.096	0.384	0.864	1.534	2.398
	3	0.168	0.673	1.514	2.692	4.207
	4	0.224	0.895	2.015	3.581	5.596
	5	0.266	1.068	2.403	4.272	6.675

Besides, it is obvious that when the ratio of the lengths of the overhang beam part and the beam span takes the smallest value, i.e. $r = 1$, the strain energy release rate also has the smallest magnitude. The explanation of this circumstance is the short pitch of the applied force when $r = 1$. The magnitudes of the bending moments in the beam sections in front and behind the crack tip is directly related to the applied force and the pitch of the force, as well. Then, the longer pitch results in the bigger bending moments and strain energy release rate.

5. Conclusions

The overhanging bilayered beam made by two unidirectional fibre reinforced composite materials is investigated. The beam contains an interlaminar crack situated between the layers and orientated parallel to the fibres. The expression for the strain energy release rate is obtained on the basis of linear elastic fracture mechanics. The formula derived is verified by comparison with the solution for G in which the internal forces in the beam sections in front and behind the crack tip are used. Complete coincidence is established. The influence of the two layers moduli of elasticity ratio on the strain energy release rate is analyzed. The increase of this ratio leads to the increase of G since the horizontal mutual displacements of the crack arms increase. The increase of the ratio of the lengths of the overhang beam part and the beam span also causes the increase of G. The reason is the increase of the bending moments in the beam sections in front and behind the crack tip.

The overhanging beam considered might be used as a specimen for experimental investigation of Mode II interlaminar cracks in composite materials. Moreover, formula (11) can be directly applied for the strain energy release rate determination on the basis of experimental data. In order to perform that, the magnitude of force F and the corresponding crack length a obtained during the experiment have to be introduced in (11).

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