

CALCULATING THE RIGHT-EIGENVECTORS OF A SPECIAL VIBRATION CHAIN BY MEANS OF MODIFIED LAGUERRE POLYNOMIALS*

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ABSTRACT. This contribution deals with the identification of the right-eigenvectors of a linear vibration system with arbitrary n degrees of freedom as given in [1]. Applying the special distribution of stiffnesses and masses given in [1] yields a remarkable sequence of matrices for arbitrary n . For computing the (right-)eigenvectors a generalised approach allowing the use of LAGUERRE polynomials is performed.

KEY WORDS: eigenvector; mode shape; eigenvalue; natural value; vibration system; LAGUERRE polynomial.

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1. Introduction

A major task in engineering applications is to determine the eigenfrequencies and eigenvectors of dynamical systems. Extensive scientific research has been performed in order to develop numerical methods for determining or estimating the eigenfrequencies of dynamical systems or at least the upper or lower bounds of these characteristic values, [2]. However, it is inevitable to test these algorithms with respect to their reliability and accuracy. Thus, there is

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a need for reference examples of dynamical structures whose dynamic properties are known exactly. This can only be achieved by analytical methods, [3]. Additionally, analytical methods are much faster than numerical ones.

One example being suitable to verify numerical procedures is a vibration chain according to МИКОТА, see [1]. The eigenfrequencies of this special vibration chain with arbitrary n degrees of freedom were conjectured in [1], two proofs of this conjecture were summarized in [4, 5]. The objective of this paper is the analytical determination of the mode shapes, that is, the eigenvectors, of this vibration chain. Concerning the arising difficulties, which are due to the special structure of the involved matrices characterizing the vibration chain the reader is referred to [6]. An analytical solution of the problem is performed with the approach given here.

A special linear vibration system having arbitrary n degrees of freedom is introduced in [1], an example of which is shown in Fig. 1. Herein, the definition of the single stiffnesses and masses leads to a noteworthy sequence of matrices for arbitrary n . The remarkable property of the given system is its distribution of the eigenfrequencies: the i -th eigenfrequency is given by:

$$(1) \quad \Omega_i = i\Omega, \quad \text{where} \quad i = 1 \dots n, \quad i \in \mathbb{N},$$

and Ω being the first eigenfrequency. Herein, $\Omega = \sqrt{k/m}$ with k being the stiffness of the first spring and m being the mass of the first mass according to Fig. 1.

For the masses:

$$(2) \quad m_i = \frac{1}{i}m, \quad i = 1 \dots n, \quad i \in \mathbb{N},$$

holds, whereas the stiffnesses of the respective springs are calculated from

$$(3) \quad k_i = (n - i + 1)k, \quad i = 1 \dots n, \quad i \in \mathbb{N}.$$

The equation of motion of the system dealt with here thus leads to:

$$(4) \quad \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0}, \quad \text{where:}$$

$$(5a) \quad \mathbf{M} = m \operatorname{diag} \left(\frac{1}{i} \right),$$

$$(5b) \quad \mathbf{K} = \operatorname{diag} \{ (k_1 + k_2, -k_2), \dots, (-k_i, k_i + k_{i+1}, -k_{i+1}), \dots, (-k_n, k_n) \}.$$

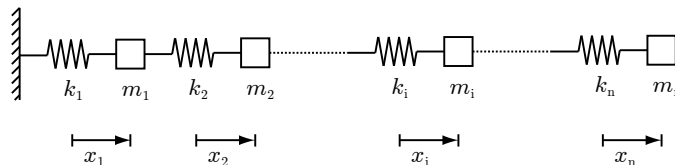


Fig. 1. Sketch of a chain-mass-vibration system with n degrees of freedom

Whereas the author of [1] could calculate the remarkable distribution of eigenfrequencies given in Eq. (1) for a fixed n , he did not succeed in giving a formal proof, [1]. A missing proof is summarised in [4], the idea of a second proof following another approach is presented shortly in [5]. In what follows, the calculation of the (right-)eigenvectors belonging to the respective eigenvalues is dealt with. For this, $k = m = 1$ is set.

A detailed investigation concerning vibration chains can be found in [7, Sect. 3.2].

2. Theoretical background

In order to calculate these eigenvectors the time-dependency:

$$(6) \quad \mathbf{x} = \mathbf{x}_A \exp \{ \iota \Omega t \},$$

is chosen. Then, Eq. (4) takes the form:

$$(7) \quad (-\Omega^2 \mathbf{M} + \mathbf{K}) \mathbf{x}_A \exp \{ \iota \Omega t \} = \mathbf{0}, \quad (\mathbf{A} - \Omega^2 \mathbf{I}) \mathbf{x}_A \exp \{ \iota \Omega t \} = \mathbf{0}.$$

Herein, \mathbf{I} is the identity matrix and $\mathbf{A} := \mathbf{M}^{-1} \mathbf{K}$ is set for the sake of brevity. Additionally, the substitution $\lambda = \Omega^2$ is performed. Due to the facts that both matrices \mathbf{M} , \mathbf{K} are symmetric and \mathbf{M} obviously is positive-definite the eigenvalues $\lambda_1, \dots, \lambda_n$ are positive, see e. g. [8]. Furthermore, it can be shown that \mathbf{K} is positive-definite as well. Now, the adjoint* $\mathbf{B}(\lambda)$ of the characteristic matrix $(\mathbf{A} - \lambda \mathbf{I})$ given with Eq. (7) is calculated by means of [9]:

$$(8) \quad \mathbf{B}(\lambda) = \mathbf{I} \lambda^{n-1} + \mathbf{B}_1 \lambda^{n-2} + \mathbf{B}_2 \lambda^{n-3} + \dots + \mathbf{B}_{n-1}.$$

The matrix coefficients \mathbf{B}_j with $j = 0, 1, \dots, n-1$ are constant matrices. Columns of $\mathbf{B}(\lambda)$ not being null vectors represent right-eigenvectors of the

*that is, not the classical adjoint

non-symmetric matrix \mathbf{A} , see [9]. Obviously, the coefficients of the characteristic polynomial $\det(\lambda\mathbf{I} - \mathbf{A})$ are needed for calculating the matrix coefficients \mathbf{B}_j in a recursive manner. Of course, these coefficients of the characteristic polynomial are not known for arbitrary $n \in \mathbb{N}$. Thus, the calculation of the sought eigenvectors by means of Eq. (8) is not appropriate. The question occurs, whether all coordinates of the eigenvector are polynomials in λ of degree $(n - 1)$. Therefor, without loss of generality an example is chosen. For this, $n = 5$ is set. For the right-eigenvector \mathbf{u} the approach:

$$(9) \quad \mathbf{u} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ \mathbf{0} & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix}, \quad \alpha_1, \dots, \alpha_5 \in \mathbb{R}$$

is made. The well-known equation of eigenvalues:

$$(10) \quad (\mathbf{A} - \lambda\mathbf{I}) \mathbf{u} = \mathbf{0},$$

then yields:

$$(11) \quad \begin{aligned} & \begin{bmatrix} (1 \cdot 9 - \lambda) & -4 & 0 & 0 & 0 \\ -8 & (2 \cdot 7 - \lambda) & -6 & 0 & 0 \\ 0 & -9 & (3 \cdot 5 - \lambda) & -6 & 0 \\ 0 & 0 & -8 & (4 \cdot 3 - \lambda) & -4 \\ 0 & 0 & 0 & -5 & (5 \cdot 1 - \lambda) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ \mathbf{0} & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} \\ &= \begin{bmatrix} (9 - \lambda) & -4 & 0 & 0 & 0 \\ -8 & (14 - \lambda) & -6 & 0 & 0 \\ 0 & -9 & (15 - \lambda) & -6 & 0 \\ 0 & 0 & -8 & (12 - \lambda) & -4 \\ 0 & 0 & 0 & -5 & (5 - \lambda) \end{bmatrix} \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \\ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \\ \alpha_3 + \alpha_4 + \alpha_5 \\ \alpha_4 + \alpha_5 \\ \alpha_5 \end{bmatrix} \\ &= \begin{bmatrix} (9 - \lambda) & (5 - \lambda) & (5 - \lambda) & (5 - \lambda) & (5 - \lambda) \\ -8 & (6 - \lambda) & -\lambda & -\lambda & -\lambda \\ 0 & -9 & (6 - \lambda) & -\lambda & -\lambda \\ 0 & 0 & -8 & (4 - \lambda) & -\lambda \\ 0 & 0 & 0 & -5 & -\lambda \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \mathbf{0}. \end{aligned}$$

Concerning the backward calculation, choosing $\alpha_5 = 1$ is not adjuvant as this choice leads to rational numbers which hinder in finding the appropriate rules for determining the missing coefficients. This disadvantage is eliminated when

taking the product of the lower diagonal entries of the matrix in Eq. (11) instead:

$$(12) \quad \alpha_5 = -8 \cdot (-9) \cdot (-8) \cdot (-5) = 2880.$$

Inserting this into the homogeneous equation system (11) gives

$$\begin{array}{ll} \text{row 5: } \alpha_5 = 2880 & \alpha_4 = -576\lambda, \\ \text{row 4: } -8\alpha_3 + (4-\lambda)\alpha_4 - \lambda\alpha_5 = 0 & \alpha_3 = 72\lambda^2 - 648\lambda, \\ \text{row 3: } -9\alpha_2 + (6-\lambda)\alpha_3 - \lambda\alpha_4 - \lambda\alpha_5 = 0 & \alpha_2 = -8\lambda^3 + 184\lambda^2 - 752\lambda, \\ \text{row 2: } -8\alpha_1 + (6-\lambda)\alpha_2 - \lambda\alpha_3 - \lambda\alpha_4 - \lambda\alpha_5 = 0 & \alpha_1 = \lambda^4 - 29\lambda^3 + 295\lambda^2 - 843\lambda. \end{array}$$

The first row of the homogeneous system (11) clearly is the characteristic equation. With equation (9) the eigenvector \mathbf{u} may be expressed by means of the polynomials $p_4(\lambda), \dots, p_0(\lambda)$, thus leading to:

$$(13) \quad \mathbf{u} = \begin{bmatrix} p_4(\lambda) \\ p_3(\lambda) \\ p_2(\lambda) \\ p_1(\lambda) \\ p_0(\lambda) \end{bmatrix} = \begin{bmatrix} \lambda^4 - 37\lambda^3 + 551\lambda^2 - 2819\lambda + 2880 \\ -8\lambda^3 + 256\lambda^2 - 1976\lambda + 2880 \\ 72\lambda^2 - 1224\lambda + 2880 \\ -576\lambda + 2880 \\ 2880 \end{bmatrix}.$$

These polynomials are not divisible without remainder, that is, $p_{k+1}(\lambda) \neq p_k(\lambda)(c\lambda + f)$ with $c, f \in \mathbb{R}$ and $k = 0, 1, 2, \dots$. Already this example reveals that the degree of the polynomials decreases monotonically from $5 - 1 = 4$ to 0 with increasing index of the coordinate. Hereby, the sign of the respective highest power alternates. This monotonicity is a specific property of the so-called matrix of MIKOTA-type, one reason of which is its tridiagonality. With respect to the backward calculation given above, independently from $n = 5$ the polynomial obtained lastly is multiplied by a linear factor in λ , thus consequently giving the characteristic polynomial. Hence, for an increasing index of coordinates the degree of the polynomial always decreases by one.

Additionally, the sequence of matrices $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$ of MIKOTA-type has a constant column sum s being independently from n and leading to the eigenvalue $\lambda_1 = s = 1$ with the associated (left-)eigenvector $\mathbf{e}_n = (1, 1, \dots, 1)^T$, see [5]. As is generally known, this (left-)eigenvector \mathbf{e}_n is orthogonal to all (right-)eigenvectors \mathbf{u} with $\lambda \neq 1$. Hence, the scalar product $\mathbf{e}_n^T \mathbf{u}$ equals the characteristic polynomial $p_n(\lambda)$ divided by the linear factor $(\lambda - 1)$:

$$(14) \quad \mathbf{e}_n^T \mathbf{u} = \frac{p_n(\lambda)}{\lambda - 1} = 0,$$

see Eq. (13) for the example with $n = 5$. As the eigenvalues are used within this equation, $\frac{p_n(\lambda)}{\lambda-1} = 0$ is the characteristic equation for the eigenvalues $\lambda_2, \lambda_3, \dots, \lambda_n$.

3. Numerical calculation of the (right-)eigenvectors

In this section the calculation of the coordinates of the (right-)eigenvector according to equation (10) is carried out in a recursive manner. Herein, use of the results of the precedent sections is made. The k -th row of the characteristic matrix $(\mathbf{A} - \lambda\mathbf{I})$ having the dimension $n \times n$ can be written as:

$$(15) \quad \begin{aligned} k = 1 & : \{(2n - 1) - \lambda, -(n - 1), 0, 0, \dots, 0\}, \\ k = 2, 3, \dots, n - 1 & : \{0, \dots, 0, -k(n - k + 1), \\ & \quad k[2n - (2k - 1)] - \lambda, -k(n - k), 0, \dots, 0\}, \\ k = n & : \{0, \dots, 0, -n, n - \lambda\}. \end{aligned}$$

On the other hand, the k -th coordinate of the (right-)eigenvectors:

$$(16) \quad \mathbf{u} = \begin{bmatrix} p_{n-1}(\lambda) \\ p_{n-2}(\lambda) \\ p_{n-3}(\lambda) \\ \vdots \\ p_1(\lambda) \\ p_0(\lambda) \end{bmatrix},$$

is $p_{n-k}(\lambda)$. Inserting Eqs. (15, 16) into the equation of the eigenvalues (10) and applying the scalar products yields:

$$(17) \quad \begin{aligned} k = 1 & : \{(2n - 1) - \lambda\}p_{n-1}(\lambda) - (n - 1)p_{n-2}(\lambda) = 0, \\ k = 2, 3, \dots, n - 1 & : -k(n - k + 1)p_{n-k+1}(\lambda) + \{k[2n - (2k - 1)] - \\ & \quad - \lambda\}p_{n-k}(\lambda) \cdots - k(n - k)p_{n-k-1}(\lambda) = 0, \\ k = n & : -np_1(\lambda) + (n - \lambda)p_0(\lambda). \end{aligned}$$

In an analogy to Eq. (12) the product of the lower diagonal entries is used again to obtain:

$$(18) \quad p_0(\lambda) = (-1)^n n!(n - 1)!.$$

Although this scaling leads to a clear arrangement as the coefficients of all polynomials are integers, a drawback of this scaling surely are the absolute

values of the coefficients, which increase enormously. The recursion formulae now are:

$$\begin{aligned}
 (19) \quad & k = n : p_0 \quad \text{according to Eq. (18),} \\
 & k = n - 1 : (n - \lambda) \frac{p_0(\lambda)}{n}, \\
 & k = n - 1, n - 2, \dots, 2 : \{k[2n - (2k - 1)] - \lambda\} \frac{p_{n-k}(\lambda)}{k(n - k + 1)} - \dots \\
 & \quad \dots - k(n - k) \frac{p_{n-k-1}(\lambda)}{k(n - k + 1)}.
 \end{aligned}$$

By this, the polynomials occurring within the coordinates of the (right-)eigenvector \mathbf{u} according to Eq. (16) can be calculated recursively.

4. Connection between the coordinates of the eigenvectors and the LAGUERRE polynomials

The LAGUERRE differential equation is given by:

$$(20) \quad xy'' + (\alpha + 1 - x)y' + ny = 0, \quad \text{with } n = 0, 1, 2, \dots \quad \text{and } \alpha \in \mathbb{R},$$

see among others [10]. It can be shown that the degree of freedom n of the linear vibration system corresponds to the parameter n in the LAGUERRE differential Eq. (20). Hence, $n \geq 1$ holds. The solutions of the LAGUERRE differential equation, that is, the associated LAGUERRE polynomials, are:

$$(21) \quad L_n^{(\alpha)} = \sum_{j=0}^n \binom{n + \alpha}{n - j} \frac{(-x)^j}{j!},$$

according to [11, 12]. Beneath an orthogonality relation for $\alpha > -1$ the following recursion formula holds for $n \geq 1$:

$$(n + 1) L_{n+1}^{(\alpha)}(x) = (-x + 2n + \alpha + 1) L_n^{(\alpha)}(x) - (n + \alpha) L_{n-1}^{(\alpha)}(x),$$

and

$$(22) \quad (n + 1) L_{n+1}^{(\alpha)}(x) - (-x + 2n + \alpha + 1) L_n^{(\alpha)}(x) + (n + \alpha) L_{n-1}^{(\alpha)}(x) = 0,$$

respectively. A comparison of this recursion formula with the one given in Eq. (19) reveals certain similarities, see also [6]. This may tempt the reader to calculate the polynomials of the (right-)eigenvectors of the system shown

in Fig. 1 by means of Eq. (21). However, this intent fails due to the terms $-xL_n^{(\alpha)}(x)$ in Eq. (22) and $-\lambda p_{n-k}(\lambda)$ in Eq. (19), respectively. This problem was already discovered in [6] but not pursued. Hence, the recursion formulae (22) have to be expanded by an addend absorbing the aforementioned term. On the other hand, this addend itself can only be created by means of a particular solution $l_n^{(\alpha)}(x)$ of a differential equation generalising equation (20):

$$(23) \quad xy'' + (\alpha + 1 - x)y' + ny = g(x), \quad \text{with } n = 0, 1, 2, \dots \quad \text{and } \alpha \in \mathbb{R}.$$

In what follows, this inhomogeneous differential Eq. (23) having a disturbance function $g(x)$ which is a polynomial itself, is called generalised LAGUERRE differential equation.

Comparing the coefficients in Eq. (22) both:

$$(24) \quad \begin{aligned} (n+1) : \quad & y(x, n+1) = L_{n+1}^{(\alpha)}(x) + l_{n+1}^{(\alpha)}(x), \\ -(-x + 2n + \alpha + 1) : \quad & y(x, n) = L_n^{(\alpha)}(x) + l_n^{(\alpha)}(x), \\ (n+\alpha) : \quad & y(x, n-1) = L_{n-1}^{(\alpha)}(x) + l_{n-1}^{(\alpha)}(x), \end{aligned}$$

and

$$(25) \quad (n+1)l_{n+1}^{(\alpha)}(x) - (-x + 2n + \alpha + 1)l_n^{(\alpha)}(x) + (n+\alpha)l_{n-1}^{(\alpha)}(x) \stackrel{!}{=} -xL_n^{(\alpha)}(x),$$

has to hold. Equation (25) leads to the cognition that the particular solution $l_n^{(\alpha)}(x)$ itself is a polynomial of degree n . Due to non-constant coefficients, $-xL_n^{(\alpha)}(x)$ is not a linear combination of fundamental solutions.

Following the structure of the homogeneous solutions according to Eq. (21), the following approach for the particular solutions:

$$(26) \quad l_n^{(\alpha)}(x) = \sum_{j=0}^n c_j x^j, \quad c_j = c_j(\alpha),$$

is made and inserted in Eq. (25):

$$\begin{aligned} (n+1) \sum_{j=0}^{n+1} c_j x^j - (-x + 2n + \alpha + 1) \sum_{j=0}^n c_j x^j + \dots \\ \dots + (n+\alpha) \sum_{j=0}^{n-1} c_j x^j = \sum_{j=0}^n \binom{n+\alpha}{n-j} \frac{(-x)^{j+1}}{j!}, \end{aligned}$$

and

$$(27) \quad \begin{aligned} (n+1) \sum_{j=0}^{n+1} c_j x^j + \sum_{j=0}^n c_j x^{j+1} - (2n+\alpha+1) \sum_{j=0}^n c_j x^j + \dots \\ \dots + (n+\alpha) \sum_{j=0}^{n-1} c_j x^j = \sum_{j=0}^n \binom{n+\alpha}{n-j} \frac{(-1)^{j+1} x^{j+1}}{j!}, \end{aligned}$$

respectively. By this, a functional equation for the particular solution $l_n^{(\alpha)}(x)$ for a given homogeneous solution $L_n^{(\alpha)}(x)$ is at hand. The coefficients $c_0, c_1, \dots, c_n, c_{n+1}$ in Eq. (27) are identified by equating coefficients in the order x^{n+1}, x^n, \dots, x^1

$$(28) \quad \begin{aligned} x^{n+1} : (n+1)c_{n+1} + c_n &= \binom{n+\alpha}{0} \frac{(-1)^{n+1}}{n!}, \\ x^n : (n+1)c_n + c_{n-1} - (2n+\alpha+1)c_n &= \binom{n+\alpha}{1} \frac{(-1)^n}{(n-1)!}, \\ x^{n-1} : (n+1)c_{n-1} + c_{n-2} - \dots & \\ \dots - (2n+\alpha+1)c_{n-1} + (n+\alpha)c_{n-1} &= \binom{n+\alpha}{2} \frac{(-1)^{n-1}}{(n-2)!}, \\ \vdots & \vdots \\ x^1 : (n+1)c_1 + c_0 - (2n+\alpha+1)c_1 + (n+\alpha)c_1 &= \binom{n+\alpha}{n} \frac{(-1)^1}{0!} \\ &= -\binom{n+\alpha}{\alpha}. \end{aligned}$$

Herein, the approach (26) directly leads to $c_{n+1} = 0$.

With the descent to the next lower power the addend $\sum_{j=0}^n c_j x^{j+1}$, gives a new coefficient with the relative lowest index. Hence, a hierarchised system of equations for the prefactors c_0, c_1, \dots, c_n is at hand and its solution is unique. In what follows, the system of equations (28) thus is treated as to be solved. Then, the disturbance function $g(x)$ in Eq. (23), which leads to the particular solution $l_n^{(\alpha)}(x)$, can be determined. These particular solutions and its

derivatives are:

$$l_n^{(\alpha)}(x) = \sum_{j=0}^n c_j x^j,$$

$$l_n'^{(\alpha)}(x) = \sum_{j=1}^n c_j j x^{j-1},$$

$$l_n''^{(\alpha)}(x) = \sum_{j=2}^n c_j j(j-1) x^{j-2}.$$

Inserting this into Eq. (23) yields:

$$x \sum_{j=2}^n c_j j(j-1) x^{j-2} + (\alpha+1-x) \sum_{j=1}^n c_j j x^{j-1} + n \sum_{j=0}^n c_j x^j = g(x),$$

$$(29) \quad (\alpha+1)c_1 + nc_0 \sum_{j=1}^{n-1} [c_{j+1}(j+1)(\alpha+j+1) + c_j(n-j)] x^j = g(x).$$

Investigating Eq. (29) reveals $g(x)$ to be a polynomial of degree $(n-1)$.

The (homogeneous) LAGUERRE differential equation stems from mathematical physics. There are strong connections to e. g. SCHRÖDINGER's equation or BESSEL differential equation, [13]. On the other hand, investigating the sequence of matrices given here requires a generalised LAGUERRE differential equation. By means of the (homogeneous) LAGUERRE differential Eq. (20) and a disturbance function $g(x)$ as given with Eq. (29) an inhomogeneous second order differential equation:

$$(30) \quad xy'' + (\alpha+1-x)y' + ny = g(x),$$

with

$n = 0, 1, 2, \dots$ and $\alpha \in \mathbb{R}$, as well as:

$$g(x) = (\alpha+1)c_1 + nc_0 \sum_{j=1}^{n-1} [c_{j+1}(j+1)(\alpha+j+1) + c_j(n-j)] x^j,$$

can be constructed. Its solutions:

$$(31) \quad y(x, n) = y_h + y_p,$$

are composed of both a homogeneous solution:

$$y_h(x, n) = L_n^{(\alpha)}(x) = \sum_{j=0}^n \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!},$$

which are the well-known (generalised or associated) LAGUERRE polynomials, and a particular solution:

$$y_p(x, n) = l_n^{(\alpha)}(x) = \sum_{j=0}^n c_j x^j,$$

respectively. Herein, the coefficients c_j of this particular solution are calculated using the hierarchised inhomogeneous system of Eqs. (28). It is apparent that the particular solution itself is a polynomial and this gives a restriction with respect to the choice of the class of the disturbance function $g(x)$.

The solutions $y(x, n)$ of the generalised LAGUERRE differential equation given with equation (30), where the disturbance function $g(x)$ is a polynomial itself, are called modified LAGUERRE polynomials. Please note, that these are NOT the generalised or associated LAGUERRE polynomials as given with e. g. [12, 13].

5. Conclusions

With this contribution, the finding in [6], that the coordinates of the (right-)eigenvectors of the vibration system introduced with Fig. 1 and Eq. (5) do have a similar structure compared to the recursion formula of the LAGUERRE polynomials could be stated. On the other hand, a disturbance function $g(x)$ for the LAGUERRE differential equation absorbing the term $-xL_n^{(\alpha)}(x)$ in Eq. (22) was found. The solution $y(x, n)$ given in Eq. (31) now allows computing the coordinates of the (right-)eigenvectors of the vibration system shown with Fig. 1.

Future tasks involve the analytical determination of the left-eigenvectors. First investigations have shown that the left-eigenvectors show interesting properties. For example, the left-eigenvector belonging to the first eigenvalue $\lambda_1 = 1$ is given with $\mathbf{e}_n = (1, 1, \dots, 1)^T$, see [5]. Investigating the components $C_i^{(\lambda_2=4)}$ of the left-eigenvector belonging to the second eigenvalue $\lambda_2 = 4$ it seems that:

$$(32) \quad C_i^{(\lambda_2=4)} = K_1 i + K_0, \quad i = 1, 2, \dots, n \quad \text{and} \quad K_0, K_1 \in \mathbb{R},$$

holds and thus it is conjectured that these components increase linearly from $C_1^{(\lambda_2=4)}$ to $C_n^{(\lambda_2=4)}$. However, the proof of this conjecture is yet an open task.

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