Journal of Theoretical and Applied Mechanics, Sofia, 2013, vol. 43, No. 4, pp. 29–48

FLUID MECHANICS

ON FARADAY INSTABILITY IN MAGNETIC LIQUIDS: INCE-ERDELYI APPROACH APPLIED TO THE HILL EQUATION DESCRIBING OSCILLATIONS OF A FERROFLUID FREE SURFACE*

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[Received 01 December 2012. Accepted 16 December 2013]

Abstract. When an isothermal ferrofluid is submitted to an oscillating magnetic field, the initially motionless liquid free surface can start to oscillate. This physical phenomenon is similar to the Faraday instability for usual Newtonian liquids subjected to a mechanical oscillation. In the present paper, we consider the magnetic field as a sum of a constant part and a time periodic part. Two different cases for the constant part of the field, being vertical in the first one or horizontal in the second one are studied. Assuming both ferrofluid magnetization and magnetic field to be collinear, we develop the linear stability analysis of the motionless reference state taking into account the Kelvin magnetic forces. The Laplace law describing the free surface deformation reduces to Hill's equation, which is studied using the classical method of Ince and Erdelyi. Inside this framework, we obtain the transition conditions leading to the free surface oscillations.
Key words: Magnetic liquid (ferrofluid), oscillating magnetic field, Faraday instability, Hill equatio.

1. Introduction

An isothermal, nearly non viscous ferrofluid layer is submitted to an *external* magnetic field (abbreviated MF) $\mathbf{H}^{ext}$. It is assumed to be a sum of

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two terms: a *constant* part of amplitude H_0 and a *time oscillating* part at an imposed frequency $\omega = 2\pi f$ in a plane, normal to the unperturbed free surface at $z = 0$, so that the period of the MF is $T_{MF} = 1/f$. The fluid layer is considered to be of infinite lateral extent and has a finite width, much larger than the capillary length.

We study the linear departure from the motionless state of such system. This amounts to the study of the Laplace law describing the upper free surface deformation [1, 2]. The problem can be rewritten as a homogeneous second order differential equation whose zeroth order term is a periodic function. It reduces to Hill's equation that generalizes the Mathieu equation [3, 4, 5, 6]. We will adapt the method of Ince [7] and Erdelyi [8] to find the necessary condition for a *resonant oscillating* solution to exist. It will differ from the well known Cowley-Rosensweig criterion for *static* instability [9, 10]. Moreover, our approach differs in one important aspect from those considered in [11, 12, 13, 14]. Indeed, those authors studied the problem linearizing it with respect to a *small* parameter, the ratio of the amplitude of the oscillating part to the constant part of the MF. They do not change the value of the constant part of the MF. This different choice thus, restricts their linear study to the Mathieu equation [6, 10, 11, 12, 13, 14]. Here, we suppose the constant coefficient multiplying the complete periodic function to be the small parameter, thus, leading to Hill's equation [1, 2, 3, 7, 8]. We compare with the experimental results, the two possible independent resonant conditions [12, 13].

2. Reference motionless state

We start from a motionless ferrofluid layer of width d and of infinite lateral extent submitted to an external periodic magnetic field. Two cases can easily be considered. In the first case, $H_0\mathbf{1}_z$, the constant part of the external magnetic field is normal to the free unperturbed flat surface. In the second case, that part of the external magnetic field $H_0\mathbf{1}_x$ is along the axes x in the reference free surface. In the first case, the unperturbed MF $\mathbf{H}^{ext}$ *outside* the layer is:

$$\mathbf{H}^{ext} = \mathbf{H}_v^{ext} = H_0\left\{[\delta_B + m_{vv}\cos(\omega t)]\mathbf{1}_z + m_{vh}\sin(\omega t)\mathbf{1}_x\right\} \tag{1a}$$

and in the second one:

$$\mathbf{H}^{ext} = \mathbf{H}_h^{ext} = H_0\left\{m_{hv}\sin(\omega t)\mathbf{1}_z + [\delta_B + m_{hh}\cos(\omega t)]\mathbf{1}_x\right\}. \tag{1b}$$

The first subscript a in the matrix m_{ab}, $(a, b) = (v, h)$ refers to the direction of the *unperturbed constant* component of the MF, while the second one b indicates the direction of the *unperturbed oscillating* part. The present paper generalizes thus, our previous studies [1, 2], where we considered only case (1a).

We derive the magnetic field inside the ferrofluid layer, assuming that the magnetization $\mathbf{M} = (\mu_r - 1)\mathbf{H}$ and $\mathbf{H}$ are always collinear, with a constant relative permeability $\mu_r = 1 + \chi$, where χ is the magnetic susceptibility. The period of the imposed magnetic field T is thus much larger than the effective relaxation time of the ferrofluid [15]. The case $\delta_B = 0$, that corresponds to the absence of a constant part in $\mathbf{H}^{ext}$, has been studied by Cebers [10, 11]. The case $\delta_B = 1$ was taken into consideration by Bashtovoi and Rosensweig [14] for a *vertical* constant field and where $m_{vv} \ll 1$ and $m_{vh} = 0$. The case (1b) completes the problem considered by Mekhonoshkin and Lange [16]. Indeed, those authors studied Faraday instability on ferrofluids in a constant *horizontal* magnetic field, but submitted to a vertical vibration.

The unperturbed magnetic fluid $\mathbf{H}^{np} = \mathbf{H}_x^{np} + \mathbf{H}_z^{np}$ *inside* the layer is obtained using the usual boundary conditions $[\mathbf{H} - \mathbf{H^{ext}}] \times \mathbf{n} = \mathbf{0}$ and $[\mu_\mathbf{r}\mathbf{H} - \mathbf{H^{ext}}] \cdot \mathbf{n} = \mathbf{0}$ [9, 10, 17], which are valid also in the perturbed case. We have thus:

A] in the first case (1a), the unperturbed MF *inside* the layer is [1, 2]:

$$\text{(2a)} \qquad \mathbf{H}^{np} \equiv \mathbf{H}_v^{np} = H_0 \left\{ \frac{[\delta_B + m_{vv}\cos(\omega t)]}{\mu_r} \mathbf{1}_z + m_{vh}\sin(\omega t)\mathbf{1}_x \right\};$$

B] in the second case (1b):

$$\text{(2b)} \qquad \mathbf{H}^{np} \equiv \mathbf{H}_h^{np} = H_0 \left\{ \frac{m_{hv}\sin(\omega t)}{\mu_r} \mathbf{1}_z + [\delta_B + m_{hh}\cos(\omega t)]\mathbf{1}_x \right\}.$$

3. Linearized perturbation problem

3.1. Fourier expansion and the free surface deformation

Supposing a reference state at rest, we develop now the linear stability study. Thus, any perturbed quantity $\delta f(x, y, z, t)$, among which the velocity $\mathbf{v}$, the deformation ξ of the free surface Σ or the magnetic field $\delta\mathbf{h}$, will be given by a Fourier expansion in normal modes, along the horizontal plane $(x,\, y)$, which are orthogonal to one another. Thus, we keep only one single arbitrary mode in the form $\widehat{\delta f}(z, t)\exp[\imath\,(k_x\, x + k_y\, y)]$ where $\imath = \sqrt{-1}$, enabling to algebrize the problem with respect to the variables $x,\, y$. The unperturbed magnetic field $\mathbf{H}^{np}$ being *time dependent*, one cannot separate in $\widehat{\delta f}(z, t)$ the z and t variables anymore [1, 4, 10]. The deformed free surface Σ is $\mathcal{F}(x, y, z, t) \equiv z - \xi(x, y, t) =$

0, and its linearized unit normal is $\mathbf{1}_n = -\dfrac{\partial\xi}{\partial x}\mathbf{1}_x - \dfrac{\partial\xi}{\partial y}\mathbf{1}_y + \mathbf{1}_z$. The Fourier transform [1] of the surface deformation is thus $\xi_\Sigma = \xi_\Sigma(t)$ to which corresponds the Fourier component of the normal $\mathbf{n} = -\imath[k_x\, \xi_\Sigma \mathbf{1}_x \; + k_y\, \xi_\Sigma \mathbf{1}_y] + \mathbf{1}_z$

3.2. Perturbed magnetic field

The perturbed ferrofluid magnetic field is quite simple to obtain [9, 10, 17]. Indeed, since one of the Maxwell equations is $\vec{\nabla} \times \mathbf{H} = \mathbf{0}$, the magnetic field perturbation $\delta\mathbf{h}$ derives from a gradient $\delta\mathbf{h} = \vec{\nabla}\phi$ and the other Maxwell equation $\vec{\nabla}\cdot\mathbf{H}(1+\chi) = 0$ gives $(1+\chi)\vec{\nabla}\cdot\delta\mathbf{h} = 0$. Thus, $\nabla^2\,\phi = 0$ for all three phases: the upper gaseous phase, the underlying solid one and the ferrofluid layer.

The boundary conditions for the magnetic field are valid also when considering the perturbed magnetic field along the deformed free surface. This leads to [10, 14, 17]:

$$\frac{\partial\,\xi}{\partial\,x_l}\left\{\left[\mathbf{H}^{np} - \mathbf{H}^{ext}\right]\cdot\mathbf{1}_z\right\}\Big|_{z=d} + \left[\frac{\partial\phi}{\partial x_l} - \frac{\partial\phi^g}{\partial x_l}\right]\Big|_{z=d} = 0 \quad \text{for } x_l = x, y,$$

(3a)

$$\sum_{l=1}^{2}\frac{\partial\xi}{\partial x_l}\left\{\left[(1+\chi)\mathbf{H}^{np} - \mathbf{H}^{ext}\right]\cdot\mathbf{1}_{x_l}\right\}\Big|_{z=d} = \left[(1+\chi)\frac{\partial\phi}{\partial z} - \frac{\partial\phi^g}{\partial z}\right]\Big|_{z=d}$$

while at the solid-liquid boundary $z = 0$, we have:

(3b)

$$\left[\frac{\partial\phi}{\partial x_l} - \frac{\partial\phi^s}{\partial x_l}\right]\Big|_{z=0} = 0, \quad \text{for } x_l = x, y \quad \text{and} \quad \left[(1+\chi)\frac{\partial\phi}{\partial z} - \frac{\partial\phi^s}{\partial z}\right]\Big|_{z=0} = 0\,,$$

so that all three magnetic potentials should depend on the explicit time function appearing respectively in the above definitions (1a), (1b) of $\mathbf{H}^{ext}$ and (2a), (2b) of $\mathbf{H}^{np}$. Using the boundary conditions (3a, 3b), the Fourier component of the perturbed magnetic potential ϕ, valid for the ferrofluid layer is:

$$\phi = \phi(z,\,t) \;=\; \xi_\Sigma\,\chi\,\Lambda(k\,z)\left[H_z^{np} \;+\; \imath\,\frac{k_x}{k}\,H_x^{np}\right], \tag{4}$$

where:

$$\Lambda(k\,z) = \frac{(2+\chi)\exp(k\,z) + \chi\exp(-k\,z)}{(2+\chi)^2\exp(k\,d) - \chi^2\exp(-k\,d)}.$$

The function $\Lambda(k\,z)$ has appeared initially in the Rayleigh–Benard–Marangoni problem of a ferrofluid submitted to a *vertical and constant* magnetic fluid [18]. The perturbed magnetic problem is linked to the balance of momentum at the interface through the deformation of the surface ξ_Σ [1, 2, 9, 10].

3.3. Momentum balance

Written in Cartesian coordinates $(x_i, i = 1, 2, 3)$, the momentum balance law for a Newtonian fluid in the gravity field takes into account the magnetic field through the Kelvin force term $\frac{\mu_0\,\chi}{2}\vec{\nabla} H^2$ (μ_0 is the magnetic constant) [1, 2, 9, 10, 17, 18]. Since the perturbed magnetic field derives from a gradient $\delta\mathbf{h} = \vec{\nabla}\phi$, its rotational is equal to 0. The Fourier expansion of the momentum balance for an *incompressible* viscous ferrofluid, whose velocity $\mathbf{v} = (v_x,\, v_y,\, v_z)$ obeys to $\vec{\nabla} \cdot \mathbf{v} = 0$, reduces to the classical form [19, 20]:

$$\left[\frac{\partial}{\partial t} - \nu\left(\frac{\partial^2}{\partial z^2} - k^2\right)\right]\left(\frac{\partial^2}{\partial z^2} - k^2\right) W = 0, \tag{5}$$

where $W = W(z, t)$ is a single Fourier mode of the vertical velocity v_z, $\nu = \eta/\rho$ kinematic viscosity, η dynamic viscosity, and ρ density.

3.4. Stress balance on the ferrofluid-gas surface

We assume gas and liquid to be immiscible, so that there is an obvious relation between the deformation of the free surface and the normal component of the velocity along that surface:

$$\frac{\partial\,\xi}{\partial\,t} = \mathbf{v}\Big|_\Sigma \cdot \mathbf{1}_n = v_z\Big|_\Sigma \quad \text{thus} \quad \frac{d\,\xi_\Sigma}{d\,t} = W\Big|_\Sigma . \tag{6a}$$

Call $\mathcal{L}_i \equiv \left[T_{ij} - T^g_{ij}\right]\Big|_\Sigma n_j$, the projection on the normal $\mathbf{1}_n = (n_x,\, n_y,\, n_z)$ at the interface Σ, the difference between the stress tensor T_{ij} in the liquid phase and the stress tensor T^g_{ij} in the inviscid magnetically inert gaseous phase. Then along Σ, one has the linearized Laplace-Marangoni condition [1, 2, 17, 18]:

$$\mathcal{L}_i = 2\mathcal{K}\,\sigma\,\delta_{i3} + (1 - \delta_{i3})\frac{\partial\sigma}{\partial x_i} \quad \text{where} \quad i = 1, 2, 3, \tag{6b}$$

where δ_{ij} is the Kronecker delta, σ the surface tension and $2\mathcal{K} = \frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2}$ the mean curvature. The tensor components T_{ij} and T^g_{ij} are given by (p is the pressure), respectively:

$$T_{ij} = -\left\{p + \frac{\mu_0}{2} H^2\right\} \delta_{ij} + \mu_0(1+\chi)\, H_i\, H_j + \eta \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right],$$

(6c)

$$T^g_{ij} = -\left\{p^g(T) + \frac{\mu_0}{2} H^2\right\} \delta_{ij} + \mu_0\, H_i\, H_j.$$

In the present isothermal problem, we consider that $\frac{\partial \sigma}{\partial x_i} = 0$, simplifying thus (6b). The magnetic field **H** is calculated locally, in each phase along the boundary surface. After a lengthy and straightforward calculation [1, 2, 17], the Fourier decomposition applied to the Laplace equation (6b), gives an expression for a single normal mode. It is given by (g is the gravity acceleration):

$$\left[\nu\left(\frac{\partial^2}{\partial z^2} - 3\,k^2\right) - \frac{\partial}{\partial t}\right] \frac{\partial W}{\partial z}\bigg|_{z=d} = \xi_\Sigma \left[k^2 \left(\frac{\sigma}{\rho}\, k^2 + g\right)\right]$$

(7)

$$-\xi_\Sigma \frac{\mu_0\, \chi^2\, k^3}{\rho} \left[\frac{\mu_r\, \tanh(k\, d) + 1}{(\mu_r^2 + 1)\tanh(k\, d) + 2\,\mu_r}\right] \left[\mu_r\, (H^{np}_z)^2 - \left(\frac{k_x}{k}\right)^2 (H^{np}_x)^2\right].$$

4. Laplace equation (7) in the nearly inviscid case

Last equation (7) can be simplified, while keeping its most interesting features since the main study focuses on the time dependency of the magnetic field. The width of the ferrofluid layer is not very important at this stage, so that we will study the limiting case of a ferrofluid layer of very large depth. Indeed, whenever $d \geq 3\, d_{cap}$, where $d_{cap} = \sqrt{\frac{\sigma}{\rho\, g}}$ is the capillary length, one assumes a quite thick layer [19, 21] and thus, rewrites the RHS of equation (7) as:

$$\xi_\Sigma \left\{k^2 \left[\frac{\sigma}{\rho}\, k^2 + g\right] - \frac{\mu_0\, \chi^2\, k^3}{\rho\,(1 + \mu_r)} \left[\mu_r\, (H^{np}_z)^2 - \frac{k_x^2}{k^2} (H^{np}_x)^2\right]\right\}. \tag{8}$$

Also, we apply to the LHS of (7) an approach [1, 2, 19] originally developed by Landau and Lifshitz [20], to obtain the capillary waves equation for low viscosity fluids which is independent on the external forces provided these last derive from a potential gradient (as is the case here). It becomes

$$\left[\nu\left(\frac{\partial^2}{\partial z^2} - 3\,k^2\right) - \frac{\partial}{\partial t}\right] \frac{\partial W}{\partial z}\bigg|_{z=d} \approx -\,k \left[\frac{\partial^2}{\partial t^2} + 4\,k^2\,\nu\,\frac{\partial}{\partial t}\right] \xi_\Sigma. \tag{9}$$

This transforms the Laplace equation (7) into a homogeneous second order differential equation acting on $\xi_\Sigma = \xi_\Sigma(t)$, the normal mode describing the amplitude of the surface deformation [10, 19, 21]:

$$\left[\frac{d^2}{d\,t^2} + 4\,k^2\,\nu\,\frac{d}{d\,t} + \mathcal{A}_j + \mathcal{B}_j\,\cos(\omega\,t) + \mathcal{C}_j\,\cos(2\,\omega\,t)\right]\xi_\Sigma = 0, \tag{10}$$

where calling $\alpha = \left(\dfrac{k_x}{k}\right)^2$, one has for the vertical case:

$$\mathcal{A}_v = k\left(\frac{\sigma\,k^2}{\rho} + g\right) - \left\{\frac{\mu_0\,[(\mu_r - 1)H_0\,k]^2}{\rho\,(\mu_r + 1)\,\mu_r}\left(\delta_B + \frac{m_{vv}^2 - \alpha\,\mu_r\,m_{vh}^2}{2}\right)\right\},$$

$$\mathcal{B}_v = -\frac{\mu_0\,[(\mu_r - 1)H_0\,k]^2}{\rho\,(\mu_r + 1)\,\mu_r}\,2\,\delta_B\,m_{vv}, \tag{11a}$$

$$\mathcal{C}_v = -\frac{\mu_0\,[(\mu_r - 1)H_0\,k]^2}{\rho\,(\mu_r + 1)\,\mu_r}\,\frac{(m_{vv}^2 + \alpha\,\mu_r\,m_{vh}^2)}{2},$$

and for the horizontal one:

$$\mathcal{A}_h = k\left(\frac{\sigma\,k^2}{\rho} + g\right) - \left\{\frac{\mu_0\,[(\mu_r - 1)H_0\,k]^2}{\rho\,(\mu_r + 1)\,\mu_r}\left[\frac{m_{hv}^2}{2} - \alpha\,\mu_r\left(\frac{m_{hh}^2}{2} + \delta_B\right)\right]\right\},$$

$$\mathcal{B}_h = \alpha\,\frac{\mu_0\,[(\mu_r - 1)H_0\,k]^2}{\rho\,(\mu_r + 1)}\,2\,\delta_B\,m_{hh}, \tag{11b}$$

$$\mathcal{C}_h = \frac{\mu_0\,[(\mu_r - 1)H_0\,k]^2}{\rho\,(\mu_r + 1)\,\mu_r}\,\frac{(m_{hv}^2 + \alpha\,\mu_r\,m_{hh}^2)}{2}.$$

If the MF has a constant component $\delta_B = 1$, Eq. (10) is Hill's equation [7, 8] generalizing the case of a pure oscillating field $\delta_B = 0$ studied by Cebers [10, 11].

5. Inviscid case ($\nu = 0$)

5.1. Cowley-Rosensweig static instability

The static instability for the vertical case at $\omega = 0$, is a well developed theme, since the classical work of Cowley and Rosensweig [9, 10, 15, 17]. Then, Eq. (10) reads:

$$\left(\frac{d^2}{d\,t^2} + \mathcal{D}_v\right)\xi_\Sigma = 0, \quad \text{where:}$$

(12)
$$\mathcal{D}_v = \mathcal{A}_v + \mathcal{B}_v + \mathcal{C}_v = k\left(\frac{\sigma\,k^2}{\rho} + g\right)$$
$$-\frac{\mu_0\;[(\mu_r - 1)H_0\,k]^2}{\rho\,(\mu_0 + 1)\,\mu_r}\,(\delta_B + m_{vv})^2.$$

The marginal steady state ($\omega = 0$) is given by the function $\mathcal{D}_v = \mathcal{D}_v(k) = 0$, which is tangent to the horizontal axis at $k = \sqrt{\rho\,g/\sigma}$, for a critical magnetic field $\mathbf{H}^{ext} = H^{(1)}_{crit}[\delta_B + m_{vv}]\mathbf{1}_z$:

(13)
$$\left[H^{(1)}_{crit}\,(\,\delta_B\,+\,m_{vv})\right]^2 = 8\,\pi\,\sqrt{\rho\,g\,\sigma}\frac{\mu_r\,(\mu_r + 1)}{(\mu_r - 1)^2} \quad \text{for} \quad \mu_0 = \frac{1}{4\,\pi}.$$

Equation (13) is the value of the Cowley and Rosensweig stability criteria [9], expressed in terms of the applied external field, but half the value is reported in Blums et al. [10].

A similar discussion on the constant horizontal field (1b) has no real interest whatsoever. Indeed, when $\omega = 0$, we obtain:

$$\left(\frac{d^2}{d\,t^2} + \mathcal{D}_h\right)\xi_\Sigma = 0, \quad \text{where:}$$

(14)
$$\mathcal{D}_h = \mathcal{A}_h + \mathcal{B}_h + \mathcal{C}_h = k\left(\frac{\sigma\,k^2}{\rho} + g\right)$$
$$+\alpha\frac{\mu_0\;[(\mu_r - 1)H_0\,k]^2}{\rho\,(\mu_0 + 1)\,\mu_r}\,(\delta_B + m_{hh})^2.$$

The function $\mathcal{D}_h(k)$ is an ever increasing positive function of the wavenumber k, starting from $\mathcal{D}_h(0) = 0$. As $\alpha \leq 1$, the horizontal constant magnetic field has no specific action as it increases only the coefficient $\mathcal{D}_h(k)$.

5.2. Oscillating magnetic field

Let us now consider the case of an oscillating magnetic field where $\omega \neq 0$. Furthermore, and solely for didactic reasons, we will restrict here the discussion to the case where $\alpha = 0$. The ordinary differential equation:

$$\left(\frac{d^2}{d\,z^2} + a^2\right) g(z) = 0,$$

where a^2 is a positive integer m^2 has a periodic solution $\cos(m\,z)$ or $\sin(m\,z)$. Therefore, the periodic solution of the Hill equation [7, 8]:

$$\left[\frac{d^2}{d\,z^2} + a^2 + \epsilon f(z)\right] g(z) = 0,$$

where $a^2 \sim O(m^2)$, $\epsilon \ll a^2$ and $f(z) = 1 + \sum_{l=1}^{n} f_l \cos(2\,l\,\pi\,z)$, will be based on $\cos(m\,z)$ completed by a power series in a small parameter ϵ [3, 5, 6, 7, 8].

The equivalent parameters appearing in Eq.(10) of a^2 and ϵ are defined in terms of dimensional quantities as:

$$a^2_{Dim} = k\left(\frac{\sigma\,k^2}{\rho} + g\right) \quad \text{and} \quad \epsilon_{Dim} = \frac{\mu_0\,\left[(\mu_r - 1)H_0\,k\right]^2}{\rho\,(\mu_r + 1)\,\mu_r}.$$

Both quantities have the dimension of t^{-2}. Thus, the necessary condition for possible periodic solution requires $\mathcal{A}_v \geq 0$, applying the previous argument to (10). The marginal oscillation appears at $\mathcal{A}(k) = 0$ for $k = \sqrt{\rho\,g/\sigma}$, and for the critical field $H_0 = H^{(2)}_{crit}$:

$$\left(H^{(2)}_{crit}\right)^2 \left(\delta_B + \frac{m_v^2}{2}\right) = 2\sqrt{\rho\,g\,\sigma}\,\frac{\mu_r\,(\mu_r + 1)}{\mu_0(\mu_r - 1)^2}. \tag{15}$$

In the case studied by Cebers and Maiorov [11], the applied magnetic field is only oscillating, so that $\delta_B = 0$. Since $\mu_0 = \dfrac{1}{4\,\pi}$, Eq. (15) gives:

$$\left(H^{(2)}_{crit} m_v\right)^2 = 16\,\pi\,\sqrt{\rho\,g\,\sigma}\,\frac{\mu_r\,(\mu_r + 1)}{(\mu_r - 1)^2}, \tag{16}$$

which is exactly the value reported in Blums et al. [10]. It rests on the necessary validity of (15), not on the static stability criteria (12). Equation (15) substantiates the experimental results obtained by Mahr and Rehberg [12] and Pi et al. [13]. As their results satisfy it very nearly, the slight discrepancy being related to $\nu = 0$. The criteria (13) and (16) are thus very similar, but they apply to highly different cases.

6. Laplace law (10) as Hill equation

6.1. Case $\epsilon_{dim} \ll a^2_{dim}$

We will reduce still the mathematical difficulty of (10) and use the approximation $\epsilon_{dim} \ll a^2_{dim}$ for both horizontal and vertical cases. The dimensionless variables have been defined in [1, 2]. Introducing the dimensionless time $\sqrt[4]{\sigma/\rho g^3}$ and dimensionless length $\sqrt{\sigma/\rho g}$, we obtain the following form of the Laplace law (10):

$$\left[\frac{\omega^2}{k^2}\frac{d^2}{d\,y^2} + \frac{8\,\omega}{\sqrt[4]{Fi}}\frac{d}{d\,y} + 4\left(k + \frac{1}{k}\right)\right]\xi_\Sigma = \tag{17}$$

$$8\,H_0^2\,[\mathcal{C}_0 + \mathcal{C}_1\,\cos(y) + \mathcal{C}_2\,cos(2\,y)]\xi_\Sigma,$$

where for the vertical constant part of the field:

$$\mathcal{C}_0^v = \delta_B + \frac{m^2_{vv}}{2} - \frac{\alpha}{2}(1+\chi)m^2_{vh},$$

$$\mathcal{C}_1^v = 2\,\delta_B\,m_{vv}, \tag{18a}$$

$$\mathcal{C}_2^v = \frac{m^2_{vv}}{2} + \frac{\alpha}{2}(1+\chi)m^2_{vh},$$

and for the horizontal constant part of the field:

$$\mathcal{C}_0^h = \frac{m^2_{hv}}{2} - \alpha(1+\chi)\left(\delta_B + \frac{m^2_{hh}}{2}\right),$$

$$\mathcal{C}_1^h = -\,2\,\delta_B\,m_{hh}\,\alpha(1+\chi), \tag{18b}$$

$$\mathcal{C}_2^h = -\left[\frac{m^2_{hv}}{2} + \frac{\alpha}{2}(1+\chi)m^2_{hh}\right].$$

Here, $y = \omega\,t$ and the dimensionless viscosity is expressed by the Kapitza number $Fi = \dfrac{\sigma^3}{\rho\,g\,\nu^4}$ [2]. In both cases, the coefficient $\mathcal{C}_1$ differs from zero only if $\delta_B = 1$, so that there is a non zero constant component of the magnetic field. We recover the Mathieu equation, if $\mathcal{C}_1 = 0$, studied by Cebers [10, 11]. On the contrary, the coefficient $\mathcal{C}_2$ is independent from the value of δ_B and it reflects only the oscillatory part of the MF.

To obtain the standard form of Hill's equation [3], we use still another dimensionless time $x = \dfrac{y}{2} = \dfrac{\omega\, t}{2}$ in Eq. (17). It contains still a first order derivative of $\xi_\Sigma(x)$, which we eliminate through the well known transformation $\psi(x)$ [8, 10], defined by:

$$\xi_\Sigma = \psi(x)\,\exp\left(-\frac{4\,k^2\,x}{\omega\,Fi^{1/4}}\right), \tag{19}$$

and get the final dimensionless equation [3, 5, 7, 8]:

$$\begin{gathered}\left\{\frac{d^2}{d\,x^2} + \theta_0 + 2\,\epsilon\,[\theta_1\,\cos(2x) + \,\theta_2\,\cos(4\,x)]\right\}\psi(x) = 0,\\ \theta_0 = 4\,\frac{k^3+k}{\omega^2} - 16\frac{k^4}{\omega^2\,\sqrt{Fi}} - 2\,\epsilon\,\mathcal{C}_0 \quad \text{and} \quad \theta_1 = -\mathcal{C}_1, \quad \theta_2 = -\mathcal{C}_2.\end{gathered} \tag{20}$$

Here, $\epsilon = 4\left(\dfrac{H_0\,k}{\omega}\right)^2$ is the corresponding version of ϵ_{Dim}. Equation (20) is a model Hill equation[1], to which the classical Floquet theory applies [3, 4, 6, 7, 8]. Its solution is $\psi(x) = \exp(\lambda\,x)\,\Phi(x)$, where $\Phi(x)$ is a periodic function of x satisfying:

$$\left\{\frac{d^2}{d\,x^2} + 2\,\lambda\,\frac{d}{d\,x} + \theta_0 + \lambda^2 + 2\,\epsilon\,[\theta_1\,\cos(2x) + \,\theta_2\,\cos(4\,x)]\right\}\Phi(x) = 0.$$

We assume that $\epsilon \ll 1$ [10]. Neglecting everywhere terms of $O(\epsilon^2)$ like $\lambda^2 \approx O(\epsilon^2)$ [7, 8], we get the following equation

$$\left\{\frac{d^2}{d\,x^2} + 2\,\lambda\,\frac{d}{d\,x} + \theta_0 + 2\,\epsilon\,[\theta_1\,\cos(2x) + \,\theta_2\,\cos(4\,x)]\right\}\Phi(x) = 0. \tag{21}$$

However, the function which should remain periodic is:

$$\xi_\Sigma = \exp\left[\left(\lambda - \frac{4\,k^2}{\omega\,Fi^{1/4}}\right)\,x\right]\,\Phi(x), \qquad \text{so that} \quad \lambda = \frac{4\,k^2}{\omega\,Fi^{1/4}}. \tag{22}$$

Since Eq. (21) contains the trigonometric functions $\cos(2\,x)$ and $\cos(4\,x)$, we know from [5, 7, 8] that one must consider at least two cases which will give two independent resonance conditions. The first one corresponds to $\theta_0 \simeq O(1)$ and

[1] This is the equation studied in 1934 by Erdelyi [8] for electric circuits.

the second one to $\theta_0 \simeq O(4)$. Equation (21) is linear in the parameter ϵ but not in the various $m_{a\,b}$ that appears also as square in the various coefficients θ_i, $i =$ 0, 1, 2. Thus, where our approach differs fundamentally from the few experimental ones which we know [12, 13, 14], since there the various $m_{a\,b}$, being small parameters, appear only *linearly*.

Thus, $\Phi(x)$ should be a series of functions whose zeroth order term should be proportional to $\cos(n\,x-\gamma)$ (or equivalently to $\sin(n\,x-\gamma)$), where $n = 0, 1, 2, \ldots$ and γ is a phase angle that will be related to the viscosity. We will study three cases $n = 0$, $n = 1$, $n = 2$ and keep all results up to $O(\epsilon^2)$. Calling $\widehat{\theta}_j = \epsilon\,\theta_j$, Ince [7] considers that $\theta_0(\widehat{\theta}_1, \widehat{\theta}_2)$, where $\widehat{\theta}_1$, $\widehat{\theta}_2$ are independent from one another and introduces the following developments in (21):

$$\begin{aligned} \theta_0 &= t_0 + \textstyle\sum_{j=1}^{2} t_j\,\widehat{\theta}_j + \cdots\,, \\ \lambda &= \lambda_0 + \textstyle\sum_{j=1}^{2} \lambda_j\,\widehat{\theta}_j + \cdots\,, \\ \Phi(x) &= \Phi_0(x) + \textstyle\sum_{j=1}^{2} f_j(x)\,\widehat{\theta}_j + \cdots\,. \end{aligned} \tag{23}$$

We obtain thus a hierarchy of equations where each is multiplied by some power of $\widehat{\theta}_j$, $j = 1, 2$ and which are to be solved independently from one another. At each step, the condition to be obeyed is that no secular term can appear, giving relations between the unknowns coefficients. We must take into account that the development concerning θ_0 and λ have to be simultaneously satisfied and that their explicit relation with k is given by (20) and (22).

This being said, we have still a certain freedom in our choice. It will be guided by the fact that $n = 0$ or $n \neq 0$. We obtain then:

$$\begin{aligned} &\left[\frac{d^2}{d\,x^2} + 2\,\lambda_0 \frac{d}{d\,x} + \left(t_0 + \lambda_0^2\right)\right] \Phi_0 = 0, \quad \text{and for} \quad j = 1, 2, \\ &\left[\frac{d^2}{d\,x^2} + 2\,\lambda_0 \frac{d}{d\,x} + \left(t_0 + \lambda_0^2\right)\right] f_j \\ &+ 2\,\lambda_j \frac{d\,\Phi_0}{d\,x} + (t_j + 2\,\lambda_0\,\lambda_j)\,\Phi_0 + 2\,\cos(2\,j\,x)\,\Phi_0 = 0. \end{aligned} \tag{24}$$

6.2. Transition condition for Eq. (21) at $n = 0$

Let us note that this case has never been considered in the past by experimentalists [10, 12, 13, 14]. The solution of Eqs. (24) at $n = 0$ is quite easy to obtain. Indeed, we have

$$\Phi_0(x) = \Phi_0 = const.$$

(25)
$$f_j(x) = \frac{\Phi_0}{2\,(j^2 + \lambda_0^2)} \left[\cos(2\,j\,x) - \frac{\lambda_0}{j}\,\sin(2\,j\,x)\right].$$

To avoid secular terms, the following conditions have to be satisfied:

(26)
$$t_0 \;=\; -\lambda_0^2, \quad \text{and} \quad 2\,\lambda_0\,\lambda_j \;+\; t_j \;=\; 0 \quad \text{for } j = 1,\, 2.$$

We suppose that both $\lambda_1 = \lambda_2 = 0$, following Ince [7], so that $t_1 = t_2 = 0$. We obtain thus back the explicit boundary condition as (see [7]):

(27)
$$-2\,\epsilon\,\mathcal{C}_0 + 4\,\frac{k^3 + k}{\omega} \;=\; 0.$$

6.3. Transition conditions for Eq. (21) at $n \neq 0$

We will consider explicitly the development for $n = 1,\, 2$. The term which is independent from both $\widehat{\theta}_j$, $(j = 1,\, 2)$ obeys to:

(28)
$$\frac{d^2\,\Phi_0(x)}{d\,x^2} = -n^2\,\Phi_0(x)$$
$$= -\,2\,\lambda_0 \frac{d\,\Phi_0(x)}{d\,x} - (t_0 \,+ \lambda_0^2)\,\Phi_0(x).$$

Thus, whatever integer $n \neq 0$ we consider, $\lambda_0 = 0$ and $t_0 = n^2$ and the function $\Phi_0(x)$, as the zeroth order term of the series, has a dimensionless period $\frac{2\,\pi}{n}$, to which corresponds a dimensional period T_{Resp}. But, by definition $2\,x = \omega\,t$ ($\omega\,t$ being always dimensionless quantity), so that $4\,\pi\,/\,n = \omega\,T_{Resp}$. Since the period of the magnetic field is $T_{MF} = \frac{2\,\pi}{\omega} = \frac{1}{f}$, it means that $T_{Resp} = 2\,T_{MF}\,/\,n$. For $n = 1$, the deformation $\xi_\Sigma(t)$ has thus a period $2/f$ which is the double of the period of the applied magnetic field. This corresponds to the observed period by Mahr and Rehberg [12] or Pi et al.[13]. While for $n = 2$, we have $T_{Resp} = T_{MF}$ and for any other n, we should observe that both periods are linked by a rational fraction $\frac{2}{n}$.

We end up with the following hierarchy valid for any non zero n where $j = 1,\, 2$:

$$(29)\qquad \left(\frac{d^2}{d\,x^2} + n^2\right) f_j(x) \;=\; 2\,\lambda_j\, n\, \sin(n\,x - \gamma)$$
$$-t_j\, \cos(n\,x - \gamma) - 2\cos(2\,j\,x)\, \cos(n\,x - \gamma).$$

6.3.1. Case $\theta_0 \simeq O(1)$

We define the following expansions, using the approach of Ince [7] and Erdelyi [8], which seemingly increase the number of functions to be found:

$$\Phi(x) = \cos(x-\gamma) + \epsilon\, b_1(x,\,\gamma)\,\theta_1 + \epsilon\, b_2(x\,,\gamma)\,\theta_2 \;+\; O(\epsilon^2),$$
$$(30)\qquad \theta_0 = 1 + \,\epsilon\,[\beta_1(\gamma)\,\theta_1 + \,\beta_2(\gamma)\,\theta_2] \;+\; O(\epsilon^2),$$
$$\lambda = \epsilon\,[q_1(\gamma)\,\theta_1 + q_2(\gamma)\,\theta_2] \;+\; O(\epsilon^2).$$

We introduce the development (30) in Eq. (21), neglecting $O(\epsilon^2)$ terms, to obtain the following system:

$$\left(\frac{d^2}{d\,x^2} + 1\right)\, b_1(x,\,\gamma) = \sin(x-\gamma)\,[2\,q_1(\gamma) + \sin(2\,\gamma)]$$
$$-\cos(x-\gamma)\,[\beta_1(\gamma) + \cos(2\,\gamma)] - \cos(3\,x - \gamma),$$
$$(31)\qquad \left(\frac{d^2}{d\,x^2} + 1\right)\, b_2(x,\,\gamma) = 2\,q_2(\gamma)\sin(x-\gamma)$$
$$-\cos(x-\gamma)\,\beta_2(\gamma) - [\cos(3x+\gamma) + \cos(5\,x-\gamma)]\,.$$

The RHS of both equations (31) contain terms proportional to $\cos(x-\gamma)$ and $\sin(x-\gamma)$, which are solutions of the homogeneous differential equation $\left(\frac{d^2}{d\,x^2} + 1\right) f(x) = 0$. As is well known, the general solutions of Eq.(31) will be a polynom in x multiplied by a trigonometric function. But, this is now where the Floquet theory comes to our help. Indeed, to study the resonant case, $\Phi(x)$ has to be a periodic function. Thus, we must exclude increasing secular terms in the final expression of $b_1(x,\,\gamma)$ and $b_2(x,\,\gamma)$ so that all the coefficients proportional to $\cos(x-\gamma)$ or $\sin(x-\gamma)$ should be equal to 0 in the

RHS of (31) [4, 5, 7, 8]. This gives us the following relations:

$$q_1(\gamma) = -\frac{\sin(2\,\gamma)}{2} \quad \text{and} \quad \beta_1(\gamma) = -\cos(2\,\gamma), \tag{32a}$$
$$q_2(\gamma) \;=\; \beta_2(\gamma) \;=\; 0,$$

so that up to an unknown constant, we have:

$$b_1(x,\,\gamma) = \frac{\cos(3\,x-\gamma)}{8} \quad \text{and} \quad b_2(x,\,\gamma) = \frac{\cos(3\,x+\gamma)}{8} + \frac{\cos(5\,x-\gamma)}{24}.$$

Introducing these results in (30), we obtain finally:

$$\lambda \;=\; \frac{\epsilon\,\mathcal{C}_1\,\sin(2\,\gamma)}{2} \quad \text{and} \quad \theta_0 = 1 + \,\epsilon\,\mathcal{C}_1\cos(2\,\gamma). \tag{32b}$$

As we noticed before, the present development rests on $\delta_B \;=\; 1$, it is thus especially interesting in the present problem as it distinguishes itself from the situation studied by Cebers and Maiorov [11]:

A] When the magnetic field constant part is vertical (see(1a)), the previous equations (32b) gives:

$$\lambda = 4\left(\frac{H_0\,k}{\omega}\right)^2\, m_{vv}\,\sin(2\,\gamma), \tag{33a}$$
$$\theta_0 \;=\; 1\;+\;4\left(\frac{H_0\,k}{\omega}\right)^2\, m_{vv}\,\cos(2\,\gamma).$$

Going back to Eq. (19), we conclude that the marginal oscillating state for the deformation $\xi_\Sigma(x)$ is reached when:

$$\frac{k^2}{\omega\,Fi^{1/4}} = \left(\frac{H_0\,k}{\omega}\right)^2\, m_{vv}\,\sin(2\,\gamma), \quad \text{for} \quad \sin(2\,\gamma)\;>\;0, \tag{33b}$$

and the greatest value of the kinematic viscosity ν corresponds to $\gamma\;=\;\pi/4$, where $\theta_0\;=\;1$. The inviscid case corresponds to $\gamma\;=\;0$, where $\theta_0\;=\;1+\left(2\,\frac{H_0\,k}{\omega}\right)^2\, m_{vv}$.

B] When the magnetic field constant part is horizontal (see(1b)), Eq.(32b) leads to:

$$
\begin{aligned}
&\lambda = -\,4\mu_r\,\alpha\,\left(\frac{H_0\,k}{\omega}\right)^2\, m_{hh}\,\sin(2\,\gamma)\,,\\
\text{(34a)}\qquad &\\
&\theta_0\,=\,1\,-\,4\mu_r\,\alpha\,\left(\frac{H_0\,k}{\omega}\right)^2\, m_{hh}\,\cos(2\,\gamma).
\end{aligned}
$$

In this case, the marginal oscillating state for the deformation $\xi_\Sigma(x)$ is reached when:

$$
\text{(34b)}\qquad \frac{k^2}{\omega\,Fi^{1/4}} = -\,\mu_r\,\alpha\,\left(\frac{H_0\,k}{\omega}\right)^2\, m_{hh}\,\sin(2\,\gamma),\quad \text{for}\quad \sin(2\,\gamma)\,\le\,0.
$$

The greatest value of the kinematic viscosity ν corresponds to $\gamma = 3\,\pi/4$, where $\theta_0\,=\,1$. The inviscid case is at $\gamma\,=\,0$, where $\theta_0 = 1 -\,\mu_r\,\alpha\,\left(2\,\frac{H_0\,k}{\omega}\right)^2\, m_{hh}$. Note that the conditions (33b) and (34b) do not depend on the existence of the constant component of the magnetic fields (1a) and (1b), but they are linear in the diagonal coefficients of the matrix $m_{a\,b}$. Moreover, both equations do not depend on the wave number.

6.3.2. Case $\theta_0 \simeq O(4)$

Applying further the approach of Ince [7] and Erdelyi [8], we introduce now the following development in Eq. (21):

$$
\begin{aligned}
&\Phi(x) = \cos(2\,x-\gamma) + \epsilon\,B_1(x,\,\gamma)\,\theta_1 + \epsilon\,B_2(x\,,\gamma)\,\theta_2,\\
\text{(35a)}\qquad &\theta_0 = 4 + \,\epsilon\,[\Delta_1(\gamma)\,\theta_1 +\,\Delta_2(\gamma)\,\theta_2]\,,\\
&\lambda = \epsilon\,[r_1(\gamma)\,\theta_1 + r_2(\gamma)\,\theta_2]\,.
\end{aligned}
$$

For this case, we should stress that the periodicity of $\cos(2x-\gamma)$ is 2π, so that the period of the resonance T_{Resp} is the period of the unperturbed MF whatever the phase shift. Following the same procedure as in the previous subsection 6.3.1, we obtain finally:

$$\Delta_1(\gamma) = r_1(\gamma) = 0,$$

$$B_1(x,\,\gamma) = \frac{\cos\,(4\,x-\gamma)}{12} - \frac{\cos(\gamma)}{4},\quad B_2(x,\,\gamma) = \frac{\cos(6\,x-\gamma)}{32},$$

$$\Delta_2(\gamma) = -\cos(2\,\gamma) \quad \text{and} \quad r_2(\gamma) = -\frac{\sin(2\,\gamma)}{4},$$

so that:

(35b) $$\lambda = \frac{\epsilon\,\mathcal{C}_2\,\sin(2\,\gamma)}{4} \quad \text{and} \quad \theta_0 = 4 + \,\epsilon\,\mathcal{C}_2\,\cos(2\,\gamma).$$

A] For the constant vertical part of the magnetic field (1a), we have:

(36a) $$\begin{aligned} \lambda &= \left(\frac{H_0\,k}{\omega}\right)^2 \frac{\sin(2\,\gamma)}{2}\,\left(m_{vv}^2 + \alpha\,\mu_r\,m_{vh}^2\right), \\ \theta_0 &= 4 + 2\,\cos(2\,\gamma)\left(\frac{H_0\,k}{\omega}\right)^2 \left(m_{vv}^2 + \alpha\,\mu_r\,m_{vh}^2\right). \end{aligned}$$

If $m_{vh} = 0$ and $\cos(2\,\gamma) = 0$, then $\theta_0 = 4$, and we find back the critical condition linking the magnetic field to the viscosity given in Blums et al. [10], namely:

(36b) $$\frac{k^2}{\omega\,Fi^{1/4}} = \frac{1}{8}\left(\frac{H_0\,m_{vv}\,k}{\omega}\right)^2.$$

B] For the constant horizontal part of the magnetic field (1b), we obtain:

(37a) $$\begin{aligned} \lambda &= -\left(\frac{H_0\,k}{\omega}\right)^2 \frac{\sin(2\,\gamma)}{2}\,\left(m_{hv}^2 + \alpha\,\mu_r\,m_{hh}^2\right), \\ \theta_0 &= 4 - 2\,\cos(2\,\gamma)\left(\frac{H_0\,k}{\omega}\right)^2 \left(m_{hv}^2 + \alpha\,\mu_r\,m_{hh}^2\right). \end{aligned}$$

When $\gamma = \pi/4$ or $\gamma = 3\,\pi/4$, $\theta_0 = 4$, and from the definition of λ, the case $\gamma = 3\,\pi/4$ gives the relationship between viscosity and frequency for the MF given by (1b):

(37b) $$\frac{k^2}{\omega\,Fi^{1/4}} = \frac{1}{8}\left(\frac{H_0\,k}{\omega}\right)^2 \left(m_{hv}^2 + \alpha\,\mu_r\,m_{hh}^2\right).$$

In both last cases, the conditions (36b) and (37b) for the marginal oscillating state depend on the squares of the coefficients in the matrix $m_{a\,b}$, but they do not depend upon the wavenumber.

For λ to be positive, the phase shift is basically different for the vertical case since one must have $\sin(2\,\gamma) \geq 0$, while for the horizontal one $\sin(2\,\gamma) \leq 0$ is necessary.

6.3.3. Case $\theta_0 \simeq O(n^2)$ for $n > 2$

Applying the above method, it is very easy to convince oneself that there is no resonant terms anymore for $n > 2$, so that for such θ_0 the Floquet coefficient λ is equal to zero and the deformation ξ_Σ is only an exponentially decreasing but oscillating function of time.

7. Conclusion

We develop here the linear study of the resonance conditions of a magnetic Faraday problem, where the small parameter is linked to the magnetic field amplitude and not as has been done previously, to a very small oscillating component. Thus, for a very simple form of the partly oscillating magnetic field, our development ends up with a Hill equation. This choice differs from previous ones, linked only to the Mathieu equation. We apply a classical method developed during last century to obtain the resonant conditions. For our model of the MF, we introduce a phase shift γ around $\cos(n\,\omega\,t - \gamma)$ for $n = 1, 2$, the deformation being always stable for larger values of n. These results are in agreement with experimental results [11, 12, 13]. The magnetic Faraday problem for more complicated MF offers thus new control possibilities than the Faraday vibration. For a horizontal constant part of the MF, or if there is no constant field component, the boundary cases do depend on the kind of pattern observed at resonance (square, hexagons, triangles...). Furthermore, the solution of the Hill equation shows that the resonance conditions obtained for $\theta_0 \approx 1$ (when the series starts with $\cos(\omega\,t - \gamma)$) are linear in the diagonal coefficients of the matrix $m_{a\,b}$ while for $\theta_0 \approx 4$ (thus from a series around $\cos(2\,\omega\,t - \gamma)$) it introduces $m^2_{a\,b}$ terms.

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