

## EXACT PERIODIC SEISMIC WAVES IN THE NIKOLAEVSKII MODEL

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**ABSTRACT.** In the present paper three different in their structure families, of exact periodic solutions of the nonlinear evolution equation of Nikolaevskii, have been obtained. The common dynamic structure of these families of periodic solutions has been shown as well as the spatial displacements, typical of the non-integrable evolution equations, for each separate harmonics. These exact solutions are published for the first time. **KEY-WORDS:** Hirota bilinear operators, Hirota-Matsuno bilinear transformation method, Weierstrass elliptic functions, Jacobi elliptic functions, Jacobi theta functions.

### 1. Introduction

Experimental studies show that the earth's crust is elastically nonlinear and contains the sources of accumulated elastic energy. This dissipative medium generates seismic emissions of low and high frequency range. Beresnev and Nikolaevskii [1], [2] proposed the following evolutionary non-integrable equation to describe seismic waves in a viscoelastic medium:

$$(1) \quad u_t + [\beta u - (u + 2uu_{xx} + u_{xxx})]_{xx} + uu_x = 0,$$

where  $u(x, t) \in C^6(\Omega)$  characterizes the elevation of the earth's crust, and  $\beta$  is a small positive parameter controlling the distance from a certain origin. The model does not allow  $\beta$  to be reduced to zero. By  $\Omega$  we have denoted the semi-infinite two-dimensional stripe  $\Omega = \{(x, t) \in \mathbb{R}^2, 0 \leq x \leq L, t > 0\}$ . Equation (1) could be also interpreted as a generalization of Burgers equation [3]. Let us mention the fact that it is invariant with respect to the Galilean transformation, i.e. if  $U(x, t)$  is a solution of (1), then the function  $u(t, x) = u_0 + U(t, x - u_0 t)$ , where  $u_0 = \text{const}$ , is also a solution of equation (1).

The numerical analysis of the evolution equation (1) performed in [1], [2] shows the evolution of quasi-sinusoidal, stationary groups of waves. The authors of [3] have also established numerically that equation (1) has a threshold of short-wave instability under the dynamics of slow long waves. So, contrary to the conventional scenarios, turbulence arises out of a spatially homogeneous condition as a result of only one supercritical bifurcation.

Authors mentioned Hao-wen Xi, R. Toral, J. Gunton, M. Tribelsky [3] use the evolution equation (1) like the simplest possible model of an extensive spatio-temporal chaos. Spatio-temporal chaos is observed in a wide range of models of dissipative systems and can be interpreted as a macroscopic dynamic analog of a phase transition of second kind, usually called weak turbulence. Equation (1) has been studied mostly numerically [3], [4], [5], [6] and Kudryashov [7] has found an exact elliptic solution.

The focus of the present paper is to obtain exact periodic solutions of the evolution equation (1) in a closed analytic form. For convenience in further analysis, the equation will be represented in the form:

$$(2) \quad u_t + uu_x = \lambda u_{xx} + 2u_{xxxx} + u_{xxxxx},$$

where  $\lambda = 1 - \beta$ . In its classic form, the bilinear transformation method of Hirota-Matsuno [8] is not applicable because equation (2) is non-integrable. Hence, we have used a “spatial” modification of this classic method that allows overcoming the serious mathematical problems ensuing from the Nikolaevskii equation [8], [9], [10]. There are also other mathematical models [15], [16], [17] applied successfully in modelling of seismic and hydrodynamic processes in dispersive medium.

## 2. Periodic solution

We will look for a periodic solution of equation (2) in the two-dimensional space  $\Omega$  assuming that it is equipped with a “strong topology”, i.e. if  $u(x, t) \in C(\Omega)$  is a localized solution of equation (2), then  $u_t, u_x, u_{xx}, u_{xxxx}, u_{xxxxx}$  are also continuously differentiable functions on the two-dimensional stripe  $\Omega$ . Let us represent the solution of (2), applying the Hirota-Satsuma transformation [11]:

$$(3) \quad u(x, t) = a + \mu(\ln \zeta(x, t))_{xx},$$

where  $a, \mu$  are unknown parameters for the time being (possibly complex as well), while  $\zeta(x, t)$  is an unknown periodic function of the class  $C^8(\Omega)$ . If we

substitute  $u(x, t)$  from (3) into equation (2) and employ the expressions for the logarithmic derivatives through the Hirota bilinear operator [12] (See Appendix A), by a single integration over  $x$ , we will obtain the following bilinear form of equation (2):

$$\begin{aligned}
& \frac{1}{2\zeta^2}(D_t D_x + 2aD_x^2 - 8B)\zeta \cdot \zeta + \frac{1}{2} \left( \frac{D_x^2 \zeta \cdot \zeta}{2\zeta^2} \right) \left[ \mu \left( \frac{D_x^2 \zeta \cdot \zeta}{2\zeta^2} \right) - 2a \right] \\
(4) \quad & = \frac{\partial}{\partial x} \left\{ \frac{1}{2\zeta^2} [(a + \lambda)D_x^2 + 2D_x^4 + D_x^6 - 8C] \zeta \cdot \zeta \right. \\
& \left. + \left( \frac{D_x^2 \zeta \cdot \zeta}{2\zeta^2} \right) \left[ 120 \left( \frac{D_x^2 \zeta \cdot \zeta}{2\zeta^2} \right)^2 - 30 \left( \frac{D_x^4 \zeta \cdot \zeta}{2\zeta^2} \right) - 12 \left( \frac{D_x^2 \zeta \cdot \zeta}{2\zeta^2} \right) - a \right] \right\},
\end{aligned}$$

where, for convenience, we have denoted by  $B$ :

$$(5) \quad B = \frac{1}{8\mu} \left( B_0 - \frac{a^2}{2} \right),$$

and  $B_0$  is an integration constant, while  $C$  is a constant. By  $D_x^n$ ,  $n = 1, 2, \dots$  we have denoted the Hirota bi-differential operator [12], defined by the equality:

$$\begin{aligned}
& D_t^m D_x^n \varphi(x, t) \cdot \psi(x', t') \\
& = (\partial/\partial t - \partial/\partial t')^m (\partial/\partial x - \partial/\partial x')^n \varphi(x, t) \psi(x', t') \Big|_{\substack{x=x' \\ t=t'}} , \quad m, n \in \mathbb{N}.
\end{aligned}$$

The bilinear equation (4) can be represented as a conjunction of the following four equations:

$$(6) \quad (D_t D_x + 2aD_x^2 - 8B)\zeta \cdot \zeta = 0,$$

$$(7) \quad \mu D_x^2 \zeta \cdot \zeta - 4a\zeta^2 = 0,$$

$$(8) \quad [(a + \lambda)D_x^2 + 2D_x^4 + D_x^6 - 8C]\zeta \cdot \zeta = 0,$$

$$(9) \quad 30(D_x^2 \zeta \cdot \zeta)^2 - 15\zeta^2(D_x^4 \zeta \cdot \zeta) - 6\zeta^2(D_x^2 \zeta \cdot \zeta) = a\zeta^4.$$

Let us clarify that the equations generated by the bilinear analogue of the initial equation are usually called residual. Their great number in this particular case is explained by the high degree of singularity ( $\sigma = 5$ ) of equation (2), but

the problem here follows from their structural inhomogeneity. The structure of equations (7) and (9) is quite different with its nonlinear bilinearities, while the left sides of equations (6) and (8) have the structure  $F(D_t, D_x)$ , where  $F$  is a polynomial of two variables. As will be seen further, this circumstance is causing the major mathematical difficulties, however they can be overcome.

A sufficiently smooth function  $\zeta(x, t)$  would be a localized solution of equation (2) provided it satisfies all residual solutions for certain values of the parameters  $a, \mu, B, C$  having  $\mu \neq 0$ , for reasons of non-triviality of the solution  $u(x, t)$ . Let us assume that  $\zeta(x, t)$  is expressed by the fourth<sup>1</sup>  $\theta$ -function of Jacobi [13]:

$$(10) \quad \zeta(x, t) = \theta_4(\xi, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2in\xi},$$

where  $\xi = kx + \omega t + \delta$  is the phase variable,  $k, \omega, \delta, k \neq 0, \omega \neq 0$  are unknown at this stage parameters (possibly complex as well). The function  $\theta_4(\xi, q)$  is well-defined if the perturbation parameter  $q = e^{i\pi\tau}$  ( $\text{Im } \tau > 0$ ) is such that:

$$(11) \quad 0 < |q| < 1.$$

All  $\theta$ -functions are biperiodic, and  $\theta_4(\xi, q)$  in particular has a real period  $\pi$  (by  $\xi$ ) and an imaginary period  $2$  (by  $q$ ). If we substitute  $\zeta(x, t)$  from (10) in the first residual equation (6), we will obtain the following infinite chain of algebraic equations:

$$(12) \quad \sum_{m=-\infty}^{\infty} F(m)(-1)^m e^{2im\xi} = 0,$$

where  $F(m) = \sum_{n=-\infty}^{\infty} [-(4k\omega + 2ak^2)(2n - m)^2 - 8B]q^{n^2+(n-1)^2}$ .

At first glance, this infinite system is incompatible as there are only five unknowns, but the polynomial structure of equation (6) allows applying the principle of index parity to the system (12). This means that if for a fixed  $m \in Z$  we perform a finite number of reductions  $n \rightarrow n + 1$ , we will get the chain identities:

$$F(m) = F(m - 2)q^{2(m-1)} = F(m - 4)q^{2(2m-4)} = \dots$$

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<sup>1</sup>It is often accepted that  $\theta_4(\xi, q) = \theta_0(\xi, q)$ .

$$= \begin{cases} F(0)q^{m^2/2}, & \text{if } m \text{ is an even number;} \\ F(1)q^{(m^2-1)/2}, & \text{if } m \text{ is an odd number.} \end{cases}$$

If in (12) we sum separately the even and odd addends, the following compact form will be obtained:

$$F(0)\theta_3(2\xi, q^2) - q^{1/2}F(1)\theta_2(2\xi, q^2) = 0,$$

where  $\theta_2(\xi, q)$ ,  $\theta_3(\xi, q)$  are the second and third  $\theta$ -functions of Jacoby, respectively [13], i.e.:

$$\theta_2(\xi, q) = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2} e^{i(2n-1)\xi}; \quad \theta_3(\xi, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2in\xi}.$$

The two simple equations  $F(0) = 0$ ,  $F(1) = 1$  are result from the last equation (in view of the linear independence of  $\theta_2(2\xi, q^2)$  and  $\theta_3(2\xi, q^2)$ ). By means of the bilinear identities (See Appendix B), the last two equations can be transformed into the following simple linear non-homogeneous system:

$$(13) \quad \begin{aligned} kq\theta'_3\omega + \theta_3B &= -2aqk^2\theta'_3 \\ kq\theta'_2\omega + \theta_2B &= -2aqk^2\theta'_2 \end{aligned}$$

where  $\theta_s = \theta_s(0, q^2)$ ,  $s = 2, 3$ , and the symbol  $'$  denotes the derivative with respect to the parameter  $q$ . The system (13) has a unique solution:

$$(14) \quad \omega = -2ka, \quad B = 0, \quad \text{and considering (5), we will have } B_0 = a^2/2.$$

At this stage it is evident that the phase velocity  $\omega$  and the integration constant  $B_0$  would be known, if  $a$  was known.

Let us consider equation (9). This equation does not possess the nice polynomial bilinearity of equation (6). We will obtain a chain of infinite number of equations (from the coefficients in front of  $e^{2im\xi}$ ,  $m = 0, \pm 1, \pm 2, \dots$ ) with one unknown – the parameter  $a$ , if we substitute  $\zeta(x, t)$  from (10) directly into equation (9). Hence, we will represent  $a$  in the formal series:

$$(15) \quad a = 24 \sum_{m=-\infty}^{\infty} a_m,$$

where  $a_m$  depends possibly on  $q$ . Applying Cauchy's formula for the product of two series:

$$\left( \sum_{s=-\infty}^{\infty} A_s \right) \left( \sum_{\nu=-\infty}^{\infty} B_\nu \right) = \sum_{m,n=-\infty}^{\infty} A_m B_{n-m},$$

for the terms of equation (9), we will get the following compatible algebraic system:

$$(16) \quad \begin{aligned} a_m \sum_{n=-\infty}^{\infty} q^{6n^2-16mn+10m^2} &= 10k^4 \sum_{n=-\infty}^{\infty} [2m^2(2n-3m)^2 - (2n-m)^4] q^{6n^2-4mn} \\ &+ k^2 \sum_{n=-\infty}^{\infty} (2n-m)^2 q^{6n^2-4mn}, \quad m = 0, \pm 1, \pm 2, \dots \end{aligned}$$

By accounting for  $\sum_{n=-\infty}^{\infty} q^{6n^2-16mn+10m^2} = \theta_3(\pi\tau(5m^2-2mn), q^6)$  and introducing for convenience, the denotations:

$$(17) \quad \begin{aligned} a_0(q) &= \sum_{m,n=-\infty}^{\infty} \frac{[2m^2(2n-3m)^2 - (2n-m)^4] e^{-4i\pi mn\tau}}{\theta_3(\pi\tau(5m^2-2mn), q^3)} q^{6n^2}, \\ a_1(q) &= \sum_{m,n=-\infty}^{\infty} \frac{(2n-m)^2 e^{-4i\pi mn\tau}}{\theta_3(\pi\tau(5m^2-2mn), q^3)} q^{6n^2}, \end{aligned}$$

we can represent equality (16) in the compact form (taking into account (15)):

$$(18) \quad a = 24[10k^4 a_0(q) + k^2 a_1(q)].$$

We have to note that under the condition of hypothesis (11) both infinite series  $a_0(q)$  and  $a_1(q)$  defined in (17) are absolutely convergent, i.e. these functions are well-defined.

Now, let us analyze the residual equation (8) where the two unknown parameters are  $a$  and  $C$ . Taking into account that this equation has a polynomial bilinearity, and applying the index parity principle (See equation (6)), we will obtain the linear non-homogeneous system:

$$(19) \quad \begin{cases} q(a + \lambda)\theta'_3 + C\theta_3/k^2 = -64qk^4(\theta'_3 + 3q\theta''_3 + q^2\theta'''_3) + 16qk^2(\theta'_3 + q\theta''_3) \\ q(a + \lambda)\theta'_2 + C\theta_2/k^2 = -64qk^4(\theta'_2 + 3q\theta''_2 + q^2\theta'''_2) + 16qk^2(\theta'_2 + q\theta''_2) \end{cases}$$

The system (19) has a unique solution, since its determinant  $\Delta$  is:

$$\Delta = qW(\theta_2, \theta_3)/k^2 \neq 0,$$

where by  $W(\theta_2, \theta_3)$  we have denoted the Wronskian from  $\theta_2 = \theta_2(0, q^2)$  and  $\theta_3 = \theta_3(0, q^2)$ , i.e.  $W(\theta_2, \theta_3) = \theta_2\theta_3' - \theta_2'\theta_3$ . These solutions are (See Appendix B):

$$(20) \quad a = 16k^2W_0(q) - 64k^4W_1(q) - \lambda,$$

$$(21) \quad C = -\frac{16q^2k^4}{W(\theta_2, \theta_3)} [(12k^2 + 1)W(\theta_2', \theta_3') + k^2qW'(\theta_2', \theta_3')],$$

where for convenience by  $W_0(q)$  and  $W_1(q)$  we have denoted the expressions:

$$W_0(q) = 1 + \frac{W'(\theta_2, \theta_3)}{W(\theta_2, \theta_3)}; \quad W_1(q) = 3W_0(q) + q^2 \frac{\theta_2\theta_3'''' - \theta_3\theta_2''''}{W(\theta_2, \theta_3)} - 2.$$

By comparing the right sides of the equations (18) and (20) we derive the characteristic equation for the wave number  $k$ :

$$(22) \quad k^2 = \frac{W_0(q) - 3a_1(q)/2 \pm \sqrt{[W_0(q) - 3a_1(q)/2]^2 - \lambda[W_1(q) + 15a_0(q)/4]}}{8W_1(q) + 30a_0(q)},$$

which shows that it is functionally tied with the perturbation parameter  $q$ . All parameters found so far like  $a$ ,  $\omega$ ,  $B_0$ ,  $C$ , which depend on the wave number  $k$ , will be associated with the characteristic equation (22).

The last residual equation of the system (6)–(9) is equation (7). If we assume that the non-zero free parameter  $\mu$  possesses the structure:

$$(23) \quad \mu = \sum_{m=-\infty}^{\infty} \mu_m,$$

and taking into account (18) and (22), then (7) is reduced to the infinite chain of simple algebraic equations for  $\mu_m$ , i.e.:

$$-\mu_m k^2 \sum_{n=-\infty}^{\infty} (2n - 3m)^2 q^{2n^2 - 6mn + 5m^2} = a_m(q) \sum_{n=-\infty}^{\infty} q^{2n^2 - 6mn + 5m^2},$$

$$m = 0, \pm 1, \pm 2, \dots$$

We get from the last equality:

$$(24) \quad \mu_m = -\frac{a_m(q)a_2(m, q)}{k^2 a_3(m, q)}, \quad m = 0, \pm 1, \pm 2, \dots,$$

where by  $a_j(m, q)$ ,  $j = 2, 3$  we have denoted the sums of the absolutely convergent series (according to hypothesis (11)):

$$(25) \quad a_2(m, q) = \sum_{n=-\infty}^{\infty} q^{2n^2-6mn+5m^2};$$

$$a_3(m, q) = \sum_{n=-\infty}^{\infty} (2n-3m)^2 q^{2n^2-6mn+5m^2}, \quad m = 0, \pm 1, \pm 2, \dots$$

Finally, we can already make the conclusion that the evolution equation of Nikolaevskii (2) generates a localized periodic solution in the two-dimensional semi-infinite stripe  $\Omega$ , having the structure:

$$(26) \quad u(x, t) = \sum_{m=-\infty}^{\infty} \left\{ a_m(q) + k^2 \mu_m(q) \frac{\partial}{\partial \xi} \left[ \frac{\dot{\theta}_4(\xi, q)}{\theta_4(\xi, q)} \right] \right\},$$

where by  $\dot{\theta}_4$  we have denoted the derivative of theta 4 with respect to the phase variable  $\xi$ . The spatial displacements  $a_m(q)$ ,  $m = 0, \pm 1, \pm 2, \dots$  are defined by equalities (16) and (17), while  $\mu_m(q)$  are defined by (24). The phase frequency  $\omega$  and the wave number  $k$  are defined, correspondingly by (14) and (22), and the integration constant  $B_0$  and differential constant  $C$  are the same as in equalities (14) and (21), respectively.

### 3. Real periodic solutions and analyticity conditions

The values of the meromorphic function  $u(x, t)$  from (26) are complex in the general case. This inconvenience is further intensified by the twofold poles of this solution at the zero points of the function  $\theta_4(\xi, q)$ , i.e. in the lattice  $\xi_{mn} = m + i(n + 1/2)\text{Im}(\tau)$ ,  $m, n \in \mathbb{Z}$ .

In order to increase the practical applicability of the periodic solution (26), it is necessary to choose the free parameters  $q$ ,  $\delta$  ( $|q| \in (0, 1)$ ) in such a way that the solution  $u(x, t)$  from (26) to have real values and would not have any singularities. For this purpose, let:

$$(27) \quad \tau = i\varepsilon, \quad \text{where } \varepsilon > 0 \ (\varepsilon \in \mathbb{R}), \quad \text{i.e. } 0 < q = -\pi\varepsilon < 1.$$

In this way, the perturbation parameter  $q$  accepts real values within the interval  $(0, 1)$ . If we restrict the phase variable within the horizontal stripe:

$$(28) \quad -\pi\varepsilon < \text{Im}(\xi) < \pi\varepsilon,$$

we will “avoid” the singularities for the function  $u(x, t)$  from (26). Moreover, this horizontal stripe is also a domain of analyticity for the periodic solution  $u(x, t)$ . Let the parameter  $\alpha > 0$  be chosen so that the characteristic equation



(22) defines  $k^2 > 0$ , i.e.  $k > 0$ . In view of (17),  $a(q)$  is also a real number, and hence it follows that  $\xi = kx + \omega t + \delta$  will accept real or complex values depending on the phase shift  $\delta$ . Let  $\delta \in \mathbb{R}$ , and then the analyticity condition (28) will be fulfilled and if we take advantage of the formula for the second logarithmic derivative (See Appendix B), i.e.:

$$\frac{\partial^2}{\partial x^2} [\ln \theta_4(\xi, q)] = k^2 \alpha_0^2 cn^2(\xi_0, \beta_0) - b(\varepsilon)k^2,$$

where:  $\xi = kx + \omega t + \delta$ ;  $\xi_0 = \pi \theta_3^2(0, q)(kx + \omega t + \delta)$ ;  $\alpha_0^2 = \pi^2 \theta_2^4(0, q)$ ;

$$\beta_0 = \theta_2^2(0, q)/\theta_3^2(0, q); \quad q = e^{-\pi\varepsilon}; \quad b(\varepsilon) = \frac{1}{3\pi^2 \theta_2^4(0, q)} \left[ \frac{\theta_2^4(0, q)}{\theta_3^4(0, q)} - \frac{\theta_1'''(0, q)}{\theta_1'(0, q)} - 1 \right],$$

then we can obtain the following periodic-cnoidal solution of equation (2), based on (26):

$$(29) \quad u(x, t) = \sum_{m=-\infty}^{\infty} u_m(\varepsilon) \left[ 1 + k^2 b(\varepsilon) \frac{a_2(m, \varepsilon)}{a_3(m, \varepsilon)} \right] - \alpha_0^2 k^2 \sum_{m=-\infty}^{\infty} \frac{a_m(\varepsilon) a_2(m, \varepsilon)}{a_3(m, \varepsilon)} cn^2(\xi_0, \beta_0).$$

Taking into account the absolute convergence of the infinite series in formula (29) (for  $\varepsilon > 0$ ), we can derive the conclusion, that the obtained real one-parametric family of periodic-cnoidal waves of equation (2) is an aggregate of dispersive waves with group velocities:  $V(\varepsilon) = 512k^3 W_1(\varepsilon) - 96k^2 W_0(\varepsilon) + 2\lambda$  ( $\varepsilon > 0$ ) and lengths  $\sigma(\varepsilon) = 4K_1/k\pi\theta_3^2(0, \varepsilon)$ . Obviously, for a real wave number  $k$ , defined by means of (22), these dispersive waves with phase velocities:

$$V_0(\varepsilon) = \omega/k = -2a(\varepsilon) = 128k^4 W_1(\varepsilon) - 32k^2 W_0(\varepsilon) + 2\lambda,$$

will form non-dissipative waves. Such non-dissipative waves are related to gradual fading of the amplitudes of the separate harmonics with time. In the case of  $\omega$  being an imaginary number, these periodic waves are dissipative.

When  $\varepsilon \rightarrow 0$ , i.e.  $q \rightarrow 1$ , the general solution (29) is characterized by the small amplitude mode, where we can employ the Fourier expansion of the logarithmic derivative (See [13]):

$$\frac{\theta_4'(\xi, e^{-\varepsilon\pi})}{\theta_4(\xi, e^{-\varepsilon\pi})} = 2 \sum_{m=-\infty}^{\infty} \cos ech(\varepsilon\pi m) \sin(2m\xi),$$

by means of which we can represent the complex solution (29) in the form:

$$(30) \quad u(x, t) = \sum_{m=-\infty}^{\infty} a_m(\varepsilon) \left[ 1 - 4m \frac{a_2(m, \varepsilon)}{a_3(m, \varepsilon)} \cos ech(\varepsilon\pi m) \cos(2m\xi) \right].$$

The obtained localized solution (30) is a superposition of a one-parametric family (with parameter  $\varepsilon > 0$ ) of cosinusoidal waves, where each harmonic (for a fixed  $m$ ) has a separate spatial displacement  $a_m(\varepsilon)$  and a separate amplitude. In view of the condition  $k \in \mathbb{R}$ , which without any limitations is considered to be a positive number, i.e.  $k > 0$ . The last inequality, as well as (22), means that the positive parameter  $\varepsilon$  should be such that:

$$(31) \quad [W_0(\varepsilon) - 3a_1(\varepsilon)/2]^2 \geq \lambda [W_1(\varepsilon) + 15a_0(\varepsilon)/4].$$

Due to its slow convergence, the periodic cosinusoidal solution (30), taking into account the conditions (28) and (31), is practically invalid in the strongly non-linear zones, i.e. for  $\varepsilon \rightarrow \infty$ , i.e.  $q \rightarrow 0$ . We will perform a transformation of first degree for the function  $\theta_4(\xi, q)$ , to carry out a smooth boundary transition from  $\varepsilon \rightarrow 0$  to  $\varepsilon \rightarrow \infty$ . For this purpose, we will define a new perturbation parameter  $q_1$ , such that  $q_1 = e^{i\pi\tau_1} = e^{-\pi/\varepsilon}$ , i.e.  $\tau_1 = -1/\varepsilon$  or  $\tau_1 = i/\varepsilon$ . Thus, the boundary transition  $q \rightarrow 1$  is equivalent to the transition  $q_1 \rightarrow 0$ , and the transformation of first degree will be:  $\theta_4(\xi, q) = (-i\tau_1)^{1/2} e^{i\tau_1 \xi^2/\pi} \theta_1(\tau_1 \xi, q_1)$  (See [13]). After applying the formula for the second logarithmic derivative (See Appendix C):

$$\frac{d^2}{dz^2} [\ln \theta_2(z, q)] = - \sum_{m=-\infty}^{\infty} \sec h^2 [i(z - m\pi\tau)],$$

and the convenient rescaling:  $k \rightarrow \varepsilon k$ ;  $\omega \rightarrow \varepsilon\omega$ ;  $\delta \rightarrow \varepsilon\delta$ ;  $\xi \rightarrow \varepsilon\xi$ , we will obtain the so-called solitary-wave form of the localized solution:

$$(32) \quad u(x, t) = \sum_{m=-\infty}^{\infty} [a_m(\varepsilon) - 2k^2\mu_m(\varepsilon)] + k^2\mu \sum_{m=-\infty}^{\infty} \sec h^2 \left( \xi - \frac{m\pi}{\varepsilon} \right).$$

The solution (32) is a one parameter<sup>2</sup> family of solitary-wave profiles of the type  $\sec h^2$ , having lengths  $2\pi/k\varepsilon$  and crests at the points  $\xi_m = 0, \pm 2\pi/\varepsilon, \pm 4\pi/\varepsilon, \dots, \pm 2\pi m/\varepsilon, \dots, m \in \mathbb{N}$ . In the zones of strong non-linearity

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<sup>2</sup>Formally, this is a two parameter family  $\varepsilon > 0, \delta \in C$ , but the phase shift  $\delta$  is a passive parameter.

$\varepsilon \rightarrow 0$ , due to the increasing lengths of the sinusoidal waves (30), a process of localization of the wave forms is initiated. In contrast to the integrable and semi-integrable models of nonlinear partial differential equations, they possess in the solitary-wave packages of the  $\text{sech}^2$  type individual and strictly defined spatial displacements for each separate harmonics. In this case they are:  $a_m(\varepsilon) - 2k^2\mu_m(\varepsilon)$  for a fixed  $m \in \mathbb{Z}$ .

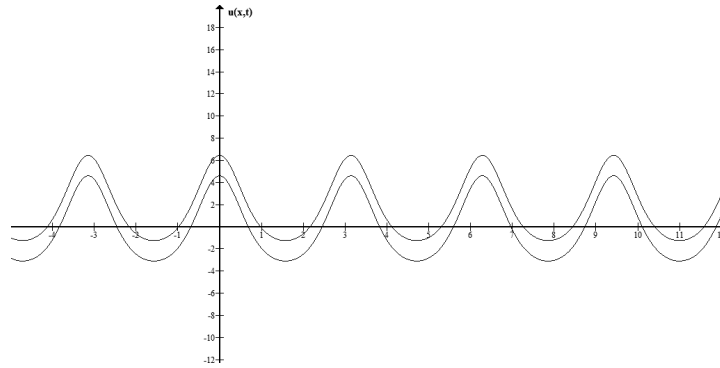


Fig. 1. Region of small amplitudes:  $q = 0, 1; k = 1/2$ . The lower graph shows the surface elevation without spatial displacements. The above graph shows the elevation with spatial displacements

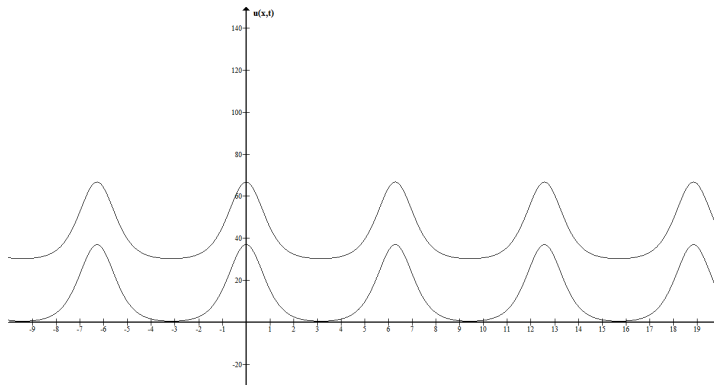


Fig. 2. The elevation for  $\varepsilon \rightarrow \infty (q \rightarrow 1)$ . The solitary - wave forms on the lower graph are in the case of absence of spatial displacements. The above graph is in the presence of spatial displacements

#### 4. Conclusion

Formally, the obtained periodic solutions (30) and (32) of the model equation of Nikolaevskii are dynamically equivalent – they both follow from the solution (26). On the one hand, they reflect the smooth transition between the boundary states of the dynamic parameter  $\varepsilon$ , i.e. from  $\varepsilon \rightarrow 0$  to  $\varepsilon \rightarrow +\infty$  and on the other hand, they suggest the possibility their phase velocities to coincide with the velocities of the soliton impulse or the solitary wave. From a purely physical aspect, this eventual coincidence is of particular importance as in this case the infinite sum of the solitary-wave profiles of the  $\sec h^2$  type in solution (32) can be interpreted as a real linear superposition of solitary waves. This means, that in this case the soliton impulse or solitary wave is like a shell of the periodic solitary-wave forms.

It is unusual for a non-integrable evolution equation such as equation (2) to break up into four residual equations (6)–(9). Moreover, these residual equations have a different bilinear structure: two of them have classic bilinearity with respect to the Hirota operators, while the other two (the second and the fourth) have a complicated nonlinear structure with respect to the same operators. It was a real challenge these residual equations to be satisfied by one and the same function and five parameters. The physical interpretation has various aspects, but the most interesting is the one related to the form of the separate solitary-wave harmonics (See (32)). The values  $\mu_m(\varepsilon)$  of the separate amplitudes also take part in forming the individual spatial displacements. A similar effect was observed for the first time in the case of non-integrable evolution partial differential equations. Typically, the individual spatial displacements are independent of the amplitudes of the solitary profiles  $\sec h^2$ . The explanation of this circumstance lies in the non-standard form of the evolution equation (2). There, the non-linear effects are balanced (in their major part) practically by only one dominating convective term:  $u_{xxxxx}$ . This reduces the role of the remaining convective terms with lower derivatives.

#### Appendix A

Logarithmic derivatives expressed by the Hirota's bilinear differential operators  $D_t$ ,  $D_x$  and identities with the Jacobi  $\theta$ -functions:

$$(\ln \zeta)_{xx} = \frac{D_x^2 \zeta \cdot \zeta}{2\zeta^2} \quad (\ln \zeta)_{xt} = \frac{D_t D_x \zeta \cdot \zeta}{2\zeta^2};$$

$$(\ln \zeta)_{xxxx} = \frac{D_x^4 \zeta \cdot \zeta}{2\zeta^2} - 6 \left( \frac{D_x^2 \zeta \cdot \zeta}{2\zeta^2} \right)^2;$$

$$(\ln \zeta)_{xxxxxx} = \frac{D_x^6 \zeta \cdot \zeta}{2\zeta^2} - 30 \left( \frac{D_x^2 \zeta \cdot \zeta}{2\zeta^2} \right) \left( \frac{D_x^4 \zeta \cdot \zeta}{2\zeta^2} \right) + 120 \left( \frac{D_x^2 \zeta \cdot \zeta}{2\zeta^2} \right)^3;$$

$$\sum_{n=-\infty}^{\infty} q^{2n^2} = \theta_3;$$

$$\theta_3 = \theta_3(0, q^2) \sum_{n=-\infty}^{\infty} q^{n^2+(n-1)^2} = q^{1/2} \theta_2; \quad \theta_2 = \theta_2(0, q^2);$$

$$\sum_{n=-\infty}^{\infty} n^2 q^{2n^2} = q \theta_3' / 2 \sum_{n=-\infty}^{\infty} (2n-1)^2 q^{n^2+(n-1)^2} = 2q^{3/2} \theta_2';$$

$$\sum_{n=-\infty}^{\infty} n^4 q^{2n^2} = q(\theta_3' + q\theta_3'')/4; \quad \sum_{n=-\infty}^{\infty} (2n-1)^4 q^{n^2+(n-1)^2} = 4q^{3/2}(\theta_2' + q\theta_2'');$$

$$\sum_{n=-\infty}^{\infty} n^6 q^{2n^2} = q(q\theta_3' + q^2\theta_3'')'/8; \quad \sum_{n=-\infty}^{\infty} (2n-1)^6 q^{n^2+(n-1)^2} = 8q^{3/2}(q\theta_2' + q^2\theta_2'')';$$

## Appendix B

The second logarithmic derivative formula of the fourth Jacobi  $\theta$ -function  $\theta_4(\xi, q)$ :

$$(B1) \quad \frac{\partial^2}{\partial x^2} [\ln \theta_4(\xi, q)] = \mu_0^2 k^2 \operatorname{cn}^2(\xi_0, \mu) - k_0(q),$$

where  $\xi = kx + \omega t + \delta$ ;  $\xi_0 = \pi \theta_3^2(0, q)(kx + \omega t + \delta)$ ;  $\mu_0^2 = \mu^2 \omega_1^2 = \pi^2 \theta_2^4(0, q)$ ;  $\mu = \theta_2^2(0, q)/\theta_3^2(0, q)$ ;  $k_0(q) = \frac{k^2}{3\pi^2 \theta_2^4(0, q)} \left( \frac{\theta_2^4(0, q)}{\theta_3^4(0, q)} - \frac{\theta_1'''(0, q)}{\theta_1'(0, q)} - 1 \right)$ .

**Proof.** If  $\omega_1 = 2K_1(\eta)/\sqrt{e_1 - e_3}$  and  $\omega_2 = 2iK_2(\eta)/\sqrt{e_1 - e_3}$  are the primitive periods of the basic Weierstrass elliptic function  $\wp(u, \omega_1, \omega_2)$  ( $\operatorname{Im}(\omega_2/\omega_1) > 0$ ), then the elliptic function  $\sigma_3(u, \omega_1, \omega_2)$  can be presented by the following two equalities (See [14]):

$$\sigma_3(z\omega_1) = e^{\eta z^2/\omega_1} \theta_4(u/\omega_1, q) \theta_4^{-1}(0, q);$$

$$\sigma_3(z\omega_1) = e^{\eta_2 u/2} \sigma(\omega_2/2 - u) \sigma^{-1}(\omega_2/2),$$

where  $\eta = \eta_1 + \eta_2$ ,  $\eta_j = \zeta(\omega_j/2)$ ,  $j = 1, 2$ ,  $d^2[\ln \sigma(u)]/du^2 = -\wp(u)$ . If we equalize the right sides of the last two equalities and double differentiate with respect to  $\xi = u/\omega_1 z$ , we will obtain:

$$(B2) \quad \frac{\partial^2}{\partial \xi^2} [\ln \theta_4(\xi, q)] = -\omega_1^2 \wp(\omega_2/2 - \xi\omega_1) - 2\eta\omega_1.$$

Taking into account the evenness of the function  $\wp(u)$  and the phase modulations:

$$\wp(u) = e_3 + (e_1 - e_3) \operatorname{sn}^{-2}(u\sqrt{e_1 - e_3}, \mu);$$

$$\operatorname{sn}(u + iK_2, \mu) = \mu \operatorname{sn}(u, \mu),$$

where  $\mu^2 = (e_2 - e_3)/(e_1 - e_3)$ . Usually is denoted  $\tau = \omega_2/\omega_1$ ,  $e_1 - e_2 = 1$ , whereat  $\omega_1 = \pi\theta_3^2(0, q)$ ;  $e_1 - e_2 = \theta_4^4(0, q)/\theta_3^4(0, q)$ ;  $e_2 - e_3 = \theta_2^4(0, q)/\theta_3^4(0, q) = \mu^2$ . Then (B2) takes the following more compact form:

$$(B3) \quad \frac{\partial^2}{\partial x^2} [\ln \theta_4(\xi, q)] = \omega_1^2 k^2 \operatorname{cn}^2(\xi\omega_1, \mu) - (e_2 + 2\eta\omega_2).$$

The residual term in (B3) is transformed, using the relations (See [14]):  $\theta_1'(0, q) = \pi\theta_2(0, q)\theta_3(0, q)\theta_4(0, q)$ ;  $2\eta\omega_1 = -\theta_1'''(0, q)/3\theta_1'(0, q)$  whereat  $k_0(q) = e_2 + 2\eta\omega_1 = \frac{1}{3} \left( \frac{\theta_2^4(0, q)}{\theta_3^4(0, q)} - \frac{\theta_1'''(0, q)}{\theta_1'(0, q)} - 1 \right)$  and formula (B3) confirms the equality (B1).

### Appendix C

Formula of the logarithmic derivative of  $\theta_2$ . The second Jacobi theta-function  $\theta_2(z, q)$ , ( $0 < |q| < 1$ ) can be presented (See [14]) in the form of an infinite product:

$$\theta_2(z, q) = 2H_0(q)q^{1/4} \cos z. \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2z + q^{4n}),$$

where  $H_0(q) = \prod_{k=1}^{\infty} (1 - q^{2k})$ ,  $q = e^{i\pi\tau}$   $\operatorname{Im}\tau > 0$ .

After differentiating with respect to  $z$ , we obtain for the logarithmic derivative of  $\theta_2(z, q)$ :

$$\begin{aligned}
\frac{\theta_2'(z, q)}{\theta_2(z, q)} &= -\operatorname{tg} z - 4 \sum_{n=1}^{\infty} \frac{q^{2n} \sin 2z}{1 + 2q^{2n} \cos 2z + q^{4n}} \\
&= -\operatorname{tg} z - 4 \sum_{n=1}^{\infty} \frac{\sin 2z}{(q^{2n} + q^{-2n}) + 2 \cos 2z} \\
&= -\operatorname{tg} z - 2 \sum_{n=-\infty}^{\infty} \frac{\sin 2z}{\cos 2n\pi\tau + \cos 2z} \\
&= -\operatorname{tg} z - \sum_{n=-\infty}^{\infty} \frac{\sin[(z + n\pi\tau) + (z - n\pi\tau)]}{\cos(z + n\pi\tau) \cos(z - n\pi\tau)} \\
&= -\operatorname{tg} z - \sum_{n=1}^{\infty} [\operatorname{tg}(z + n\pi\tau) + \operatorname{tg}(z - n\pi\tau)] \\
&= - \sum_{n=-\infty}^{\infty} \operatorname{tg}(z - n\pi\tau) = i \sum_{n=-\infty}^{\infty} \tanh[i(z - n\pi\tau)],
\end{aligned}$$

i.e.

$$\frac{d^2}{dz^2} [\ln \theta_2(z, q)] = - \sum_{n=-\infty}^{\infty} \operatorname{sech}^2 [i(z - n\pi\tau)],$$

or

$$\frac{\theta_2'(z, q)}{\theta_2(z, q)} = - \sum_{n=-\infty}^{\infty} \operatorname{sech}^2 [i(z - n\pi\tau)].$$

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