Journal of Theoretical and Applied Mechanics, Sofia, 2014, vol. 44, No. 4, pp. 69–82

DE GRUYTER
OPEN

DOI: 10.2478/jtam-2014-0023

VIBRATION-BASED METHODS FOR DETECTING A CRACK IN A SIMPLY SUPPORTED BEAM

Dimitrina Kindova-Petrova
Department of Technical Mechanics,
University of Architecture, Civil Engineering and Geodesy,
1 Hristo Smirnenski Blvd, 1046, Sofia, Bulgaria,
e-mail: dkindova@abv.bg

[Received 19 June 2014. Accepted 17 November 2014]

Abstract. Variable loadings cause fatigue to the structural elements leading to crack initiation and propagation, which results in a decrease in the fatigue life of the elements. In particular, cracks decrease the stiffness and the natural frequency thus, causing specimens to fail under normal working conditions. This paper presents the application of the vibration-based technique for detecting the location and the size of a fatigue crack in structures. The two presented methods are based on the measurements of the natural frequencies of the cracked beam. The crack is modelled by a rotational spring. The predicted crack depth and location are compared with the actual data obtained from finite element models.
Key words: vibration-based method, crack, rotational spring, simply supported beam.

1. Introduction

Cracks are one of the main causes of structural failure. In order to reduce or eliminate the sudden failure of structures, they should be regularly checked for cracks. Non-destructive testing methods such as ultrasonic testing, lamb wave, X-ray, acoustic emission, etc. are used for localized inspection and monitoring of damages in structures. These methods are efficient but laborious, time consuming, expensive and inconvenient for inaccessible structural elements. Vibration-based methods of detecting crack size and location have certain advantages. The test equipment is relatively cheap. The vibration data used can be collected from a single point or mostly several points on the component.

The crack present in the component imparts local flexibility to the element, which leads to reduction in natural frequencies and mode shapes. It is possible to estimate the location and the size of a crack by measuring changes in the vibration parameters (modal or structural). The modal parameters include natural frequencies and mode shapes, and structural parameters are stiffness, mass, flexibility and damping matrices of the system.

Adams *et al* [1] have presented a method for damage detection in a one-dimensional component using the natural frequencies of longitudinal vibrations. They modelled the damage by a linear spring. Petroski [2] has proposed a technique by which the section modulus is appropriately reduced to model a crack. Chondros and Dimarogonas [3] have used the concept of a rotational spring to model the crack and proposed a method to identify cracks in welded joints. Rizos *et al* [4] have applied this technique to a cantilever beam and detected the crack location through the measurement of amplitudes at two points of component vibrating at one of its natural modes. Liang *et al* [5] have proposed a similar method, but it required measurements of the three fundamental frequencies of the beam. Dimarogonas and Paipetis [6] calculated the rotational spring constant of a beam of a rectangular cross section from the crack strain energy function. Ostachowicz and Krawczuk [7] obtained the relationships between the reduced stiffness of the cracked section and the crack size of a beam of rectangular cross section from the decrease in the elastic deformation energy of the crack expressed in terms of the stress intensity factor. Barad *et al* [8] used relations between the rotational spring constant, crack size and location and proposed a method for detecting a crack in a cantilever beam. In most cases, the proposed methods are applied to a cantilever beam although the simply supported beam is commonly used.

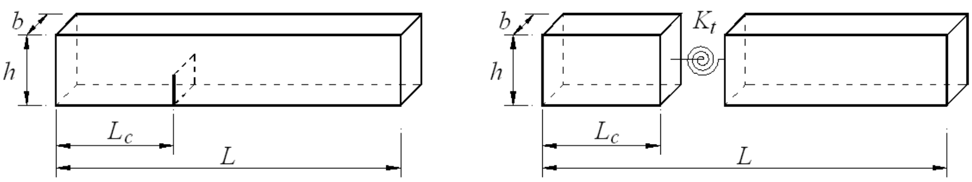

Fig. 1. A beam with a crack and representation of the crack with a rotational spring

The aim of the present paper is to study the effectiveness and the accuracy of the methods, based on the rotational spring through a number of numerical experiments. Two different methods are applied to a simply supported beam with a rectangular cross section. Both methods operate with measured natural frequencies of cracked components. The first method presented [4, 5] involves (a) obtaining plots of the rotational spring stiffness with the crack location for the first three natural modes through the characteristic equation and (b) computing the crack size. The other method [8] uses graphs of crack depth variation with crack location for the first two natural frequencies. The results obtained are compared with the data obtained by the finite element model.

2. Formulation

The governing differential equation of the free transverse vibration of a beam with uniform cross section, manufactured of homogeneous linear elastic material and without any crack, can be presented as [7, 8]:

$$\frac{\partial^4 w(x,t)}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 w(x,t)}{\partial t^2} = 0, \tag{1}$$

where $w(x,t)$ is the function of the transverse displacements, $E$is Young's modulus of elasticity, I is the area moment of inertia, ρ is mass density, A is the cross-sectional area, t is time and x varies along the beam axis from left to right.

We look for solution of Eq. (1) of the type:

$$w(x,t) = W(x)\ e^{j\omega t}, \tag{2}$$

where $j = \sqrt{-1}$, ω is the natural frequency of free vibration, $W(x)$ is the function of the transverse displacements, which does not depend on time. Taking into account Eq. (2), then Eq. (1) can be written in the following form:

$$\frac{d^4 W(x)}{dx^4} - \frac{\rho A \omega^2}{EI} W(x) = 0. \tag{3}$$

Equation (3) can be written as:

$$\frac{d^4 W(\beta)}{d\beta^4} - \lambda^4 W(\beta) = 0, \tag{4}$$

where $\beta = x/L$, $\beta_c = L_c/L$ is normalized crack location, L is the length of the beam and λ is the non-dimensional frequency parameter, given by:

$$\lambda^4 = \frac{\rho A L^4}{EI}\omega^2. \tag{5}$$

The general solution of Eq. (4) is:

$$W(\beta) = A\cos\lambda\beta + B\sin\lambda\beta + C\cosh\lambda\beta + D\sinh\lambda\beta. \tag{6}$$

The crack is presented by a rotational spring with stiffness $K_t = KEI/L$ and is located at a distance $\beta_c(L_c)$ from the left end of the beam.

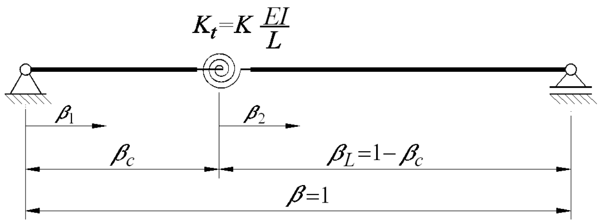

Fig. 2. Representation of the crack with a rotational spring in a simply supported beam

The equations for the two segments of the beam are, as follows:

$$\frac{d^4 W_i(\beta_i)}{d\beta_i^4} - \lambda^4 W_i(\beta_i) = 0, \tag{7}$$

where $i = 1$ for the left segment, $i = 2$ for the right segment, $0 \le \beta_1 \le \beta_c = L_c/L$ and $0 \le \beta_2 \le \beta_L = 1 - \beta_c$.

The general solutions of equations (7) are given by:

$$W_1(\beta_1) = A_1\cos\lambda\beta_1 + B_1\sin\lambda\beta_1 + C_1\cosh\lambda\beta_1 + D_1\sinh\lambda\beta_1, \tag{8}$$

$$W_2(\beta_2) = A_2\cos\lambda\beta_2 + B_2\sin\lambda\beta_2 + C_2\cosh\lambda\beta_2 + D_2\sinh\lambda\beta_2. \tag{9}$$

The boundary conditions at the ends of the beams are:

$$W_1(0) = 0, \tag{10}$$

$$W_1''(0) = 0, \tag{11}$$

$$W_2(\beta_L) = 0, \tag{12}$$

$$W_1''(\beta_L) = 0. \tag{13}$$

The continuity of displacements, moments and shear forces at the crack location ($\beta_1 = \beta_c$, $\beta_2 = 0$), can be written as:

$$W_1(\beta_e) = W_2(0), \tag{14}$$

$$W_1''(\beta_c) = W_2''(0), \tag{15}$$

$$W_1'''(\beta_c) = W_2'''(0). \tag{16}$$

At the same beam location a crack jump occurs toward the slope [4], which in terms of β_i can be formulated as:

$$K\left[W_1'(\beta_c) - W_2'(0)\right] + W_1''(\beta_c) = 0, \tag{17}$$

where $K = K_t L/(EI)$ is the non-dimensional stiffness of the rotational spring.

Generally, applying boundary conditions (10–17) to equations (8) and (9), the system of eight characteristic equations can be obtained. For a non-trivial solution, the determinant of coefficients in the front of the unknowns A_i, B_i, C_i and D_i must be zero. In the case of a simply supported beam with a single crack, the system of the eight equations can be simplified and a single characteristic equation can be obtained:

$$K = \frac{\lambda}{2}\left(\frac{\cos(\lambda - 2\lambda\beta_c) - \cos\lambda}{2\sin\lambda} + \frac{\mathrm{ch}(\lambda - 2\lambda\beta_c) - \mathrm{ch}\lambda}{2\mathrm{sh}\lambda}\right). \tag{18}$$

According to Linear Elastic Fracture Mechanics (LEFM) Ostachowicz and Krawczuk [7], defined the rotational constant of a cracked section with edge crack of uniform depth:

$$K_t = \frac{Ebh^2}{72\pi f_1(\alpha)}, \tag{19}$$

where b is the width and h is the height of the beam cross section, $\alpha = a/h$, a is the crack depth and:

$$f_1(\alpha) = 0.6384\alpha^2 - 1.035\alpha^3 + 3.7201\alpha^4 - 5.1773\alpha^5 + \\ + 7.553\alpha^6 - 7.332\alpha^7 + 2.4909\alpha^8. \tag{20}$$

The relation between K and α can be written as:

$$K = \frac{L}{EI} K_t = \frac{L}{6\pi h f_1(\alpha)}. \tag{21}$$

Dimarogonas and Paipetis [6] defined K_t from the crack strain energy function:

$$K_t = \frac{EI}{5.346 h f_2(\alpha)}, \tag{22}$$

where $f_2(\alpha)$ is defined by:

$$f_2(\alpha) = 1.8624\alpha^2 - 3.95\alpha^3 + 16.375\alpha^4 - 37.226\alpha^5 + 76.81\alpha^6 - \\ - 126.9\alpha^7 + 172\alpha^8 - 143.97\alpha^9 + 66.56\alpha^{10}. \tag{23}$$

In this case parameter K is written as:

$$K = \frac{L}{EI} K_t = \frac{L}{5.346 h f_2(\alpha)}. \tag{24}$$

3. Crack detection in a simply supported beam of rectangular cross section

3.1. Method I

The first three transverse natural frequencies are measured for the cracked beam. The graph of variation of K with the location of the crack β_c, using Eq. (18), is drawn for each frequency. The three plots have to intersect in a single point, since the stiffness K_t (and K) is independent of the mode and the crack is only one. In the case of a simply supported beam, the intersection points are two, because of the symmetry of the beam [5, 9]. In order the three curves to possess a common point, modulus of elasticity E have to be calibrated [9]. The frequency parameter λ_{uct}, theoretically calculated for an uncracked beam, and λ_{ucm}, obtained using measured frequency of uncracked beam, have to be equal but in generally they differ. The expressions of λ_{uct} and λ_{ucm} are given by, respectively:

$$(\lambda_{uct})^4 = (\omega_{uct})^2 \frac{\rho A L^4}{EI}, \tag{25}$$

$$(\lambda_{ucm})^4 = (\omega_{ucm})^2 \frac{\rho A L^4}{EI}. \tag{26}$$

For each mode modulus of elasticity E is adjusted so that $\lambda_{uct} = \lambda_{ucm}$. The new value of modulus of elasticity is E_{eff} and has the following form:

$$E_{eff} = \frac{\rho A}{I} (\omega_{ucm})^2 \left(\frac{L}{\lambda_{uct}} \right)^4 . \tag{27}$$

This effective value of E_{eff} is used as an input parameter in Eq. (18). This procedure is called zero setting.

After obtaining K, then the crack size is the calculated using Eqs. (21) or (24).

3.2. Method II

This method uses only the first two natural frequencies of the cracked beam. For each mode, the plot of variation of crack location β_c with crack depth α is drawn [8], according Eqs (18) and (21) or (24). The two curves intersect in a point whose coordinates indicates crack location and depth.

4. Numerical examples

The numerical analysis is carried out for a simply supported beam of rectangular cross section with a single crack. The natural frequencies of transverse vibration at different crack location and size are calculated by the finite element method, using a computer program (SAP 2000). The beam is discretized by 8-node solid elements. The cross section is divided into 48 identical parts and the length into 20 parts. The dimensions of the beam are: length, $L = 3$ m, height, $h = 0.3$ m, width, $b = 0.18$ m. The material properties are: modulus of elasticity, $E = 2.1 \times 10^{11}$ N/m^2, density, $\rho = 7850$ kg/m^3, Poisson's coefficient, $\nu = 0.3$. The natural frequencies for the twenty studied cases are shown in Table 1.

For the uncracked beam, the first three transverse natural frequencies theoretically calculated using Eqs. (4) are $\omega_1 = 491.20$ rad/s, $\omega_2 = 1964.8$ rad/s and $\omega_3 = 4420.8$ rad/s. The frequencies obtained from the finite element model are given in Table 1. The errors in the computed natural frequencies are respectively -3.03%, 1.78% and 8.34%. They are set to zero by using E_{eff}.

Table 1. Crack location and size for the studied cases and natural frequencies from the finite element model

Case number	Crack location β_c	Crack size α	Natural frequencies [rad/s]		
			ω_1	ω_2	ω_3
uncracked			506.10	1929.8	4052.1
1	0.10	0.125	505.61	1923.2	4028.0
2		0.250	503.76	1896.0	3945.9
3		0.375	500.02	1825.1	3820.8
4		0.500	493.11	1690.2	3665.3
5	0.20	0.125	504.45	1914.1	4022.5
6		0.250	498.09	1854.0	3927.7
7		0.375	485.56	1738.0	3805.2
8		0.500	463.44	1567.0	3689.7
9	0.30	0.125	503.00	1914.3	4049.1
10		0.250	488.26	1846.5	4014.3
11		0.375	466.52	1749.2	3998.8
12		0.500	431.31	1615.9	3979.1
13	0.40	0.125	501.84	1923.9	4040.2
14		0.250	486.00	1901.9	3998.4
15		0.375	457.72	1861.2	3932.2
16		0.500	414.74	1796.1	3850.5
17	0.50	0.125	501.40	1929.8	4018.3
18		0.250	484.05	1929.8	3903.5
19		0.375	453.57	1929.4	3731.6
20		0.500	408.31	1928.2	3537.3

4.1. Method I

Typical plots of K vs. β_c for the three natural frequencies are presented in Fig. 2. After the procedure of zero settings, the three curves have to intersect at a common point but in many cases it does not happen. To avoid any subjective error in the graphical procedure, a computational procedure is used for finding the intersection point of the pair of curves corresponding to ω_1 and ω_2. In the same way, intersection points nearest to the first intersection point are calculated for the other pairs of the curves (ω_2 and ω_3; ω_3 and ω_1). The average of the three intersection points is taken as a prediction.

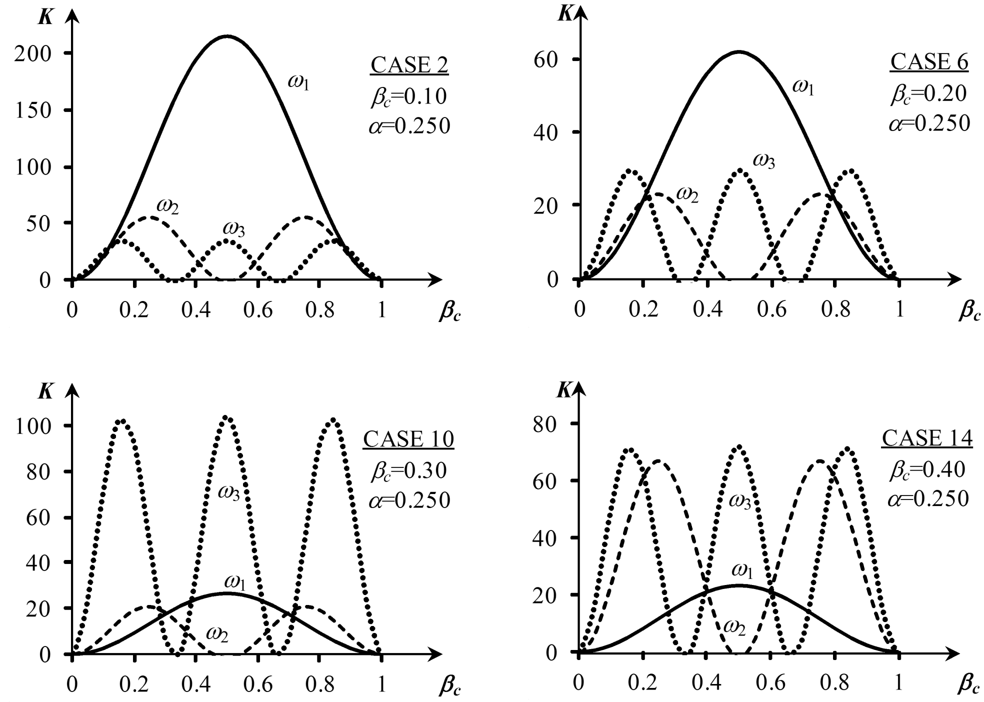

Fig. 3. Typical plots of K vs. β_c of a crack for three fundamental modes

The method gives high accuracy of prediction of the crack location. It can be seen in Table 2, that the maximum error of the crack position is 8.2% at $\beta_c = 0.10$ and $\alpha = 0.125$ and 3.7% for the other crack positions. In cases 3 and 4, the plots of K vs. β_c for the first and the second frequency do not have a common point. The results can be obtained by using the intersection point of plots for second and third frequency as well as third and first frequency. The failure of the method in cases 17 and 18 is due to the fact that the frequency of the cracked beam is equal to the frequency of the uncracked one, for the second mode of vibration. In these cases, the variation of K vs. β_c cannot be

Table 2. Comparison of predicted and actual crack location and size – method I

Case	Actual		Predicted						
						Ostachowicz & Krawczuk		Dimarogonas & Paipetis	
	Crack location β_c	Crack size α	Crack location β_c	error β_c %	Stiffness K	Crack size α	error α %	Crack size α	error α %
1	0.10	0.125	0.1082	8.2	116.2506	0.0893	–28.6	0.1004	–19.7
2		0.250	0.1022	2.2	22.0941	0.2091	–16.3	0.2360	–5.6
3		0.375	–	–	–	–	–	–	–
4		0.500	–	–	–	–	–	–	–
5	0.20	0.125	0.2068	3.4	112.2168	0.0909	−27.2	0.1022	−18.2
6		0.250	0.2075	3.7	22.1448	0.2089	−16.4	0.2357	−5.7
7		0.375	0.2049	2.4	7.7628	0.3417	−8.9	0.3809	1.6
8		0.500	0.2055	2.7	3.3182	0.4865	−2.7	0.5214	4.3
9	0.30	0.125	0.3083	2.8	106.6058	0.0934	−25.3	0.1050	-16.0
10		0.250	0.2928	−2.4	17.9377	0.2316	−7.3	0.2611	4.4
11		0.375	0.2941	−2.0	7.2549	0.3520	−6.1	0.3917	4.5
12		0.500	0.2921	−2.6	3.2433	0.4909	−1.8	0.5252	5.0
13	0.40	0.125	0.3989	−0.3	108.2011	0.0927	−25.9	0.1042	−16.6
14		0.250	0.4008	0.2	21.5112	0.2119	−15.2	0.2391	−4.4
15		0.375	0.3952	−1.2	8.182267	0.3339	−11.0	0.3726	−0.6
16		0.500	0.3913	−2.2	3.6144	0.4703	−5.9	0.5072	1.4
17	0.50	0.125	–	–	–	–	–	–	–
18		0.250	–	–	–	–	–	–	–
19		0.375	0.4891	−2.2	8.559433	0.3272	−12.7	0.3656	−2.5
20		0.500	0.4863	−2.7	4.01326	0.4510	−9.8	0.4896	−2.1

obtained by using another mode instead of the second one. Another way to overcome the problem is to use only the intersection point of curves for the first and the second frequency. The accuracy of the predicted crack sizes increases with increasing the crack depth. In most cases, Eq. (24) gives better results for predicting the crack depth than Eq. (21). The error for crack depth 12.5% of the section height is around 20%, but for other cases it is less than 5.7% using Eq. (24) and 16.4% using Eq. (21).

4.2. Method II

Corresponding to Eqs. (18) and (21) or Eqs. (18) and (24), the variation of crack location β_c with crack depth α is plotted for the two natural frequencies. The graphs are given in Fig. 4 for some of the studied cases. Again, the intersection point of the two curves is calculated, not graphically obtained.

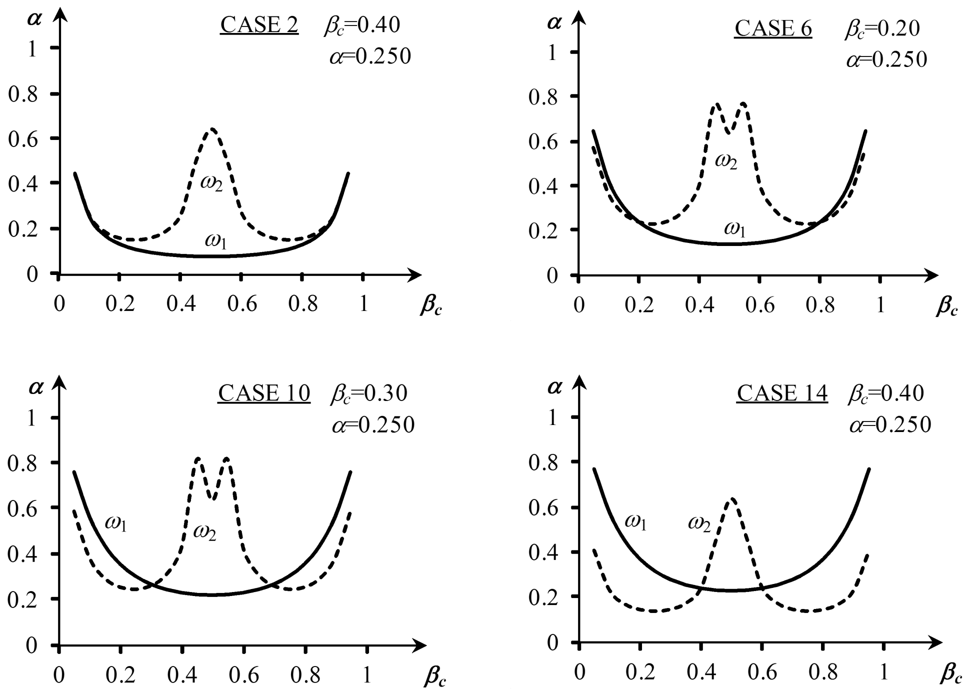

Fig. 4. Typical plots of α vs β_c for two fundamental modes

The method fails for cracks located at a distance of 10% from the support of the beam. At the other crack positions, the errors in prediction crack

Table 3. Comparison of predicted and actual crack location and size – method II

Case	Actual		Predicted							
			Ostachowicz & Krawczuk				Dimarogonas & Paipetis			
	Crack location β_c	Crack size α	Crack location β_c	error β_c %	Crack size α	error α %	Crack location β_c	error β_c %	Crack size α	error α %
1	0.10	0.125	0.1115	11.5	0.0899	−28.1	0.1115	11.5	0.1011	−19.1
2		0.250	0.0753	−24.7	0.3086	23.4	0.0739	−26.1	0.3485	39.4
3		0.375	–	–	–	–	–	–	–	–
4		0.500	–	–	–	–	–	–	–	–
5	0.20	0.125	0.2076	3.8	0.0914	−26.9	0.2076	3.8	0.1027	−17.8
6		0.250	0.1996	−0.2	0.2126	−15.0	0.1996	−0.2	0.2399	−4.0
7		0.375	0.1825	−8.8	0.3632	−3.1	0.1818	−9.1	0.4038	7.7
8		0.500	0.1683	−15.9	0.5294	5.9	0.1666	−16.7	0.5579	11.6
9	0.30	0.125	0.3034	1.1	0.0931	−25.5	0.3034	1.1	0.1046	−16.3
10		0.250	0.3031	1.0	0.2318	−7.3	0.3031	1.0	0.2613	4.5
11		0.375	0.2928	−2.4	0.3526	−6.0	0.2928	−2.4	0.3923	4.6
12		0.500	0.2808	−6.4	0.4920	−1.6	0.2808	−6.4	0.5256	5.1
13	0.40	0.125	0.4011	0.3	0.0935	−25.2	0.4011	0.3	0.1052	−15.8
14		0.250	0.4003	0.1	0.2120	−15.2	0.4003	0.1	0.2392	−4.3
15		0.375	0.3951	−1.2	0.3357	−10.5	0.3952	−1.2	0.3746	−0.1
16		0.500	0.3848	−3.8	0.4709	−5.8	0.3860	−3.5	0.5071	1.4
17	0.50	0.125	–	–	–	–	–	–	–	–
18		0.250	–	–	–	–	–	–	–	–
19		0.375	0.4741	−5.2	0.3356	−10.5	0.4775	−4.5	0.3742	−0.2
20		0.500	0.4835	−3.3	0.4649	−7.0	0.4871	−2.6	0.5021	0.4

location is 15.9% using Eq. (21) and 16.7% using Eq. (24). The two equations for calculating the crack depth give approximately equal results for crack position, but different for its depth, as in most cases the Eq. (24) has higher accuracy. The error in prediction of crack size for crack depth 12.5% of the section height is around 20%, but for the other cases it is under 11.6%, using Eq. (24) and 15.2% using Eq. (21). It is possible to improve the accuracy of the method if the curve of the variation of crack location β_c with crack depth α for the third natural frequency is considered too, not only for the first and the second natural frequencies.

5. Conclusion

Two methods for crack detection in a simply supported beam of rectangular cross section have been presented. The accuracy of the methods depends predominantly on the change in the frequency of the cracked beam compared to the frequency of the uncracked one. The error increases when the crack is close to the supports of the beam and when the crack depth has a relatively small value. In general, the first method presented gives results with higher accuracy, than the results of the second method presented. The application of method II is more complicated because of the cooperative solution of equation (18) and (21) or (18) and (24). The Eq. (24) for crack depth obtained by Dimarogonas and Paipetis gives higher accuracy than Eq. (21) by Ostachowicz and Krawczuk for both methods.

REFERENCES

[1] Adams, R. D., P. Cawley, C. J. Pye, B. J. Stone. A Vibration Technique for Non-Destructively Assessing the Integrity of Structures. *Journal of Mechanical Engineering Science*, **20** (1978), 93–100.

[2] Petroski, H. J. Simple Static and Dynamic Models for the Cracked Elastic Beam. *International Journal of Fracture*, **17** (1981), 71–76.

[3] Chondros, T. G., A. D. Dimarogonas. Identication of Cracks in Welded Joints of Complex Structures. *Journal of Sound and Vibration*, **69** (1980), 531–538.

[4] Rizos, P. F., N. Aspragathos, A. D. Dimarogonas. Identification of Crack Location and Magnitude in a Cantilever Beam from the Vibration Modes. *Journal of Sound and Vibration*, **138** (1990), 381–388.

[5] Liang, R. Y., F. K. Choy, J. Hu. Detection of Cracks in Beam Structures Using Measurements of Natural Frequencies. *Journal of the Franklin Institute*, **328** (1991), 505–518.

[6] Dimarogonas, A. D., S. A. Paipetis. Analytical Methods in Rotor Dynamics, Elsevier Applied Science, 1983.

[7] Ostachowicz, W. M., M. Krawkczuk. Analysis of the Effect of Cracks on the Natural Frequencies of a Cantilever Beam. *Journal of Sound and Vibration*, **150** (1991), 191–201.

[8] Barad, K. H., D. S. Sharma, V. Vyas. Crack Detection in Cantilever Beam by Frequency based Method. *Procedia Engineering*, **51** (2013), 770–775.

[9] Nandwana, B. P., S. K. Maiti. Modelling of Vibration of Beam in Presence of Inclined Edge or Internal Crack for Its Possible Detection based on Frequency Measurements. *Engineering Fracture Mechanics*, **58** (1997), 193–205.