

NUMERICAL SOLUTIONS FOR NONLINEAR HIGH DAMPING
RUBBER BEARING ISOLATORS: NEWMARK'S METHOD WITH
NETWON-RAPHSON ITERATION REVISITED

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ABSTRACT: Numerical methods for the solution of dynamical problems in engineering go back to 1950. The most famous and widely-used time stepping algorithm was developed by Newmark in 1959. In the present study, for the first time, the Newmark algorithm is developed for the case of the trilinear hysteretic model, a model that was used to describe the shear behaviour of high damping rubber bearings. This model is calibrated against free-vibration field tests implemented on a hybrid base isolated building, namely the Solarino project in Italy, as well as against laboratory experiments. A single-degree-of-freedom system is used to describe the behaviour of a low-rise building isolated with a hybrid system comprising high damping rubber bearings and low friction sliding bearings. The behaviour of the high damping rubber bearings is simulated by the trilinear hysteretic model, while the description of the behaviour of the low friction sliding bearings is modeled by a linear Coulomb friction model. In order to prove the effectiveness of the numerical method we compare the analytically solved trilinear hysteretic model calibrated from free-vibration field tests (Solarino project) against the same model solved with the Newmark method with Netwon-Raphson iteration. Almost perfect agreement is observed between the semi-analytical solution and the fully numerical solution with Newmark's time integration algorithm. This will allow for extension of the trilinear mechanical models to bidirectional horizontal motion, to time-varying vertical loads, to multi-degree-of-freedom-systems, as well to generalized models connected in parallel, where only numerical solutions are possible.

KEY WORDS: High damping rubber bearings; mechanical models; trilinear hysteretic model; shear behaviour; nonlinear response; base isolation; Newmark's method.

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1. INTRODUCTION

The first time stepping algorithm was constructed by Houbolt [1] within the context of solving for the dynamic response of aircraft frames using matrix methods of analysis. It was basically nothing more than a simple central difference method, whereby velocities and accelerations were written in terms of displacements computed in the immediately previous time steps. Within the next few years, Newmark [2] published his landmark paper on what became Newmark's method for the transient response of buildings to earthquake-induced loads. His method was versatile because there were two artificial parameters (β, γ) that could be 'tweaked' to produce time stepping algorithms with different properties. For instance, there was a range of values that yielded an unconditionally stable numerical integration algorithm, not to mention that the central difference algorithm could also be recovered as a special case. Since then, the terminology 'Newmark-beta' has been used to demarcate what are explicit algorithms, in contrast to the newer development of implicit algorithms that overshoot the immediately next time step and then move backwards in order to correct it. A comprehensive review of the truly explosive amount of work done on deriving a plethora of numerical integration algorithms in structural mechanics as early as the 1980's can be found in the edited book by Belytschko and Hughes [3].

In here, we focus on the nonlinear response of a class of 'soft' base isolators for earthquake protection of conventional buildings known as high damping rubber bearings (HDRB). These are becoming widely popular in Europe and there are currently two large firms that mass produce them according to specifications set by the client. The same holds true in Japan, while in the United States a different category of base isolators known as friction pendulums are popular. Recently, the authors have developed a generalized trilinear hysteretic model (THM) that best describes the mechanical response of HDRB specimens under harmonic excitation of variable frequency that leads to induced strains of up to 200% [4]. In contrast with the basic THM and its predecessor, the bilinear hysteretic model (BHM), which can be solved analytically, this is no longer the case with the generalized THM. Therefore, it is imperative to develop variants of the Newmark-beta method that are specially tailored for highly nonlinear response of a single-degree-of-freedom representation. This algorithm needs to be calibrated against experimental results so that its accuracy can be established.

More specifically, in the next section, we present the physical model developed to describe the behaviour of a low-rise building, namely the Solarino building [5], isolated with hybrid base isolation system (HBIS). Next, the mechanical models for the isolation system are presented. In the following section, the Newmark method with the Newton-Raphson iteration scheme is presented for the case of the THM,

while in the subsequent section, the comparison between the numerical and semi-analytical solution is shown and finally in the last section the conclusions are drawn.

2. PHYSICAL MODEL

In 2004 in Solarino, Eastern Sicily two reinforced concrete buildings were retrofitted by using a HBIS and subsequently tested under free vibration excitation, where the accelerations of the structure were recorded, [5], see Fig. 1. The hybrid isolation system comprises HDRB with low friction sliding bearings (LFSB), see Fig. 2. The recorded accelerations were used in order the physical parameters of the system to be identified, [6]. Specifically, the physical model of the structure is a single-degree-of-freedom (SDOF) system describing the behaviour of the Solarino building is shown in Fig. 3. THM describes the shear behaviour of HDRB (see Fig. 4), a linear Coulomb friction model (LDFM) describes the shear behaviour of LFSB (see Fig. 5) and a linear viscous damping (LVD) accounts for any additional source of damping.

Table 1. Physical parameters of the SDOF system of the Solarino building

HDRB					LDSB		LVD	Mass
k_e	k_{h1}	k_{h2}	f_s	u_c	f_f	k_f	c	m
[kN/m]	[kN/m]	[kN/m]	[kN]	[m]	[kN]	[kN/m]	[kNs/m]	[kNs ² /m]
14770	21920	13745	172	0.0285	34	35	197	1284



Fig. 1. The Solarino building.



Fig. 2. (a) High damping rubber bearing (HDRB) and (b) low friction sliding bearing (LFSB) of the Solarino building.

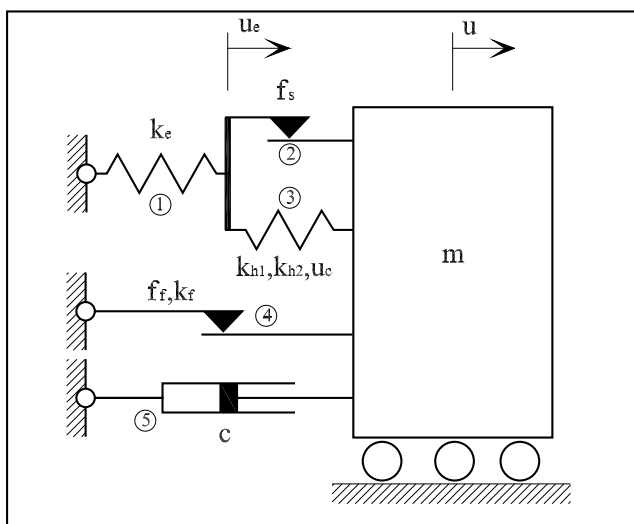


Fig. 3. Single-degree-of-freedom (SDOF) system of the Solarino building.

The compatibility, equilibrium and constitutive equations of THM are presented in Table 2, where f_T denotes the force in THM. The constitutive equation of the LCFM is given by the following expression:

$$(2.1) \quad f_F = (f_f + k_f |u|) \text{sign}(\dot{u}),$$

where f_F denotes the force in the LCFM, u denotes the displacement, \dot{u} the velocity, f_f denotes the force at zero displacement and k_f the slope, see Fig. 5. The relationships between the mechanical parameters shown in Fig. 4(a) and (e) of the THM

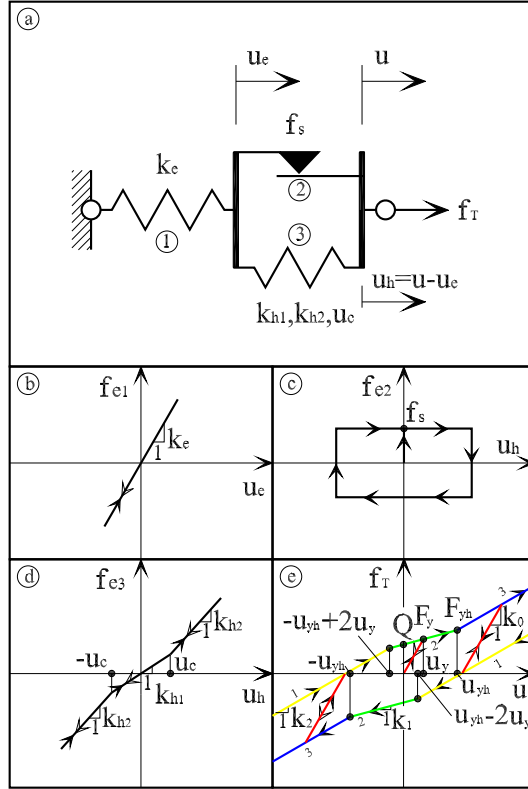


Fig. 4. Trilinear hysteretic model (THM) for the description of the shear behaviour of high damping rubber bearings: (a) mechanical model, (b) force-displacement graph of element 1, (c) force-displacement graph of element 2, (d) force-displacement graph of element 3 and (e) force-displacement graph of THM.

Table 2. Compatibility, equilibrium and constitutive equations of THM

Compatibility	$u = u_e + u_h$
Equilibrium	$f_T = f_{e1} = f_{e2} + f_{e3}$
Constitutive law	$f_{e1} = k_e u_e$
	$f_{e2}(\dot{u}_h \neq 0) = f_s \text{sgn}(\dot{u}_h)$
	$f_{e2}(\dot{u}_h = 0) = f_{e1} - f_{e3}$
	$f_{e3}(u_h \leq u_c) = k_{h1} u_h$
	$f_{e3}(u_h > u_c) = (k_{h1} u_c + k_{h2} (u_h - u_c)) \text{sgn}(u_h)$

Table 3. Relationships between mechanical and mathematical parameters of the THM, see Figs. 4(a),(e)

$$k_e = k_0; \quad k_{h1} = k_1 \frac{k_0}{k_0 - k_1}; \quad k_{h2} = k_2 \frac{k_0}{k_0 - k_2}; \quad f_s = k_0 u_y; \quad u_c = (u_3 - u_y) \frac{k_0 - k_1}{k_0}$$

are given in Table 3. The constitutive equation of the LVD is given by the following expression:

$$(2.2) \quad f_V = c\dot{u},$$

where f_V denotes the force in the LVD and c denotes the damping coefficient.

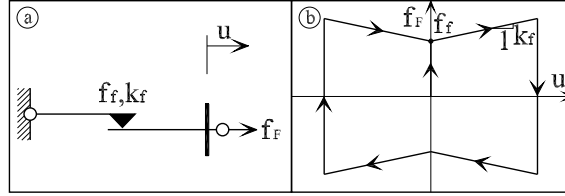


Fig. 5. Linear Coulomb friction model (LCFM) for the description of the shear behaviour of low friction sliding bearings: (a) mechanical model and (b) force-displacement graph of LCFM.

3. NEWMARK'S METHOD WITH NEWTON-RAPHON ITERATION

Newmark's method with Newton-Raphson iteration belongs to the group of implicit time-stepping procedures for nonlinear systems that are based on assumed variation of the acceleration, see [7]. The original formulation assumes either a constant average or a linear variation of the acceleration within a time increment Δt . The general form of the method is based on a Taylor series expansion of the velocities and displacements, respectively, at time step $i + 1$ in terms of the accelerations as follows:

$$(3.1) \quad \dot{u}_{i+1} = \dot{u}_i + [(1 - \gamma)\Delta t] \ddot{u}_i + (\gamma\Delta t) \ddot{u}_{i+1},$$

$$(3.2) \quad u_{i+1} = u_i + (\Delta t)\dot{u}_i + [(0.5 - \beta)(\Delta t)^2] \ddot{u}_i + [\beta(\Delta t)^2] \ddot{u}_{i+1}.$$

For the constant average acceleration method, the parameters introduced by Newmark are set as $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$, while for the linear acceleration method these are set as $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$. The above equations combined with the equilibrium equation at the end of the time step i will permit us to calculate the revised values

$u_{i+1}, \dot{u}_{i+1}, \ddot{u}_{i+1}$. In particular, the equilibrium equation of the SDOF system of the Solarino building can be written as follows:

$$(3.3) \quad m\ddot{u} + f_V + f_T + f_F = p,$$

where p denotes the external force. Equation 3.3 can be written in incremental form

$$(3.4) \quad m\ddot{u}_{i+1} + c\dot{u}_{i+1} + (f_T)_{i+1} + (f_F)_{i+1} = p_{i+1}.$$

Solving Eq. 3.1 for \ddot{u}_{i+1} , we get a relationship with u_{i+1}

$$(3.5) \quad \ddot{u}_{i+1} = \frac{1}{\beta(\Delta t)^2}(u_{i+1} - u_i) - \frac{1}{\beta\Delta t}\dot{u}_i - \left(\frac{1}{2\beta} - 1\right)\ddot{u}_i.$$

Now substituting Eq. 3.5 to Eq. 3.1, we get

$$(3.6) \quad \dot{u}_{i+1} = \frac{\gamma}{\beta\Delta t}(u_{i+1} - u_i) + \left(1 - \frac{\gamma}{\beta}\right)\dot{u}_i + \Delta t \left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_i.$$

Equation 3.4 can also be written as

$$(3.7) \quad (f_S)_{i+1} = p_{i+1},$$

where

$$(3.8) \quad (f_S)_{i+1} = m\ddot{u}_{i+1} + c\dot{u}_{i+1} + (f_T)_{i+1} + (f_F)_{i+1}.$$

By using the Taylor series expansion to expand the force $(f_S)_{i+1}$ about the displacement u_{i+1} , which is unknown, and by dropping the higher-order terms, we get

$$(3.9) \quad (f_S)_{i+1}^{(j+1)} \simeq (f_S)_{i+1}^{(j)} + \frac{(\partial f_S)_{i+1}^{(j)}}{\partial u_{i+1}} \Delta u^{(j)} = p_{i+1}.$$

By differentiating Eq. 3.8 with respect to the unknown displacement $u_{i+1}^{(j)}$, we get

$$(3.10) \quad \frac{\partial (f_S)_{i+1}}{\partial u_{i+1}} = m \frac{\partial \ddot{u}_{i+1}}{\partial u_{i+1}} + c \frac{\partial \dot{u}_{i+1}}{\partial u_{i+1}} + \frac{\partial (f_T)_{i+1}}{\partial u_{i+1}} + \frac{\partial (f_F)_{i+1}}{\partial u_{i+1}}.$$

The derivative of \ddot{u} can be calculated from Eq. 3.5 and the derivative of \dot{u} can be calculated from Eq. 3.6 we recover the relations

$$(3.11) \quad \frac{\partial \ddot{u}_{i+1}}{\partial u_{i+1}} = \frac{1}{\beta(\Delta t)^2}; \quad \frac{\partial \dot{u}_{i+1}}{\partial u_{i+1}} = \frac{\gamma}{\beta\Delta t}.$$

Next, from the above two equations, we get the tangent stiffness of the SDOF as

$$(3.12) \quad (k_S)_{i+1}^{(j)} = \frac{1}{\beta(\Delta t)^2} m + \frac{\gamma}{\beta \Delta t} c + (k_T)_{i+1}^{(j)} + (k_F)_{i+1}^{(j)},$$

where

$$(3.13) \quad (k_T)_{i+1}^{(j)} = \left(\frac{\partial (f_T)_{i+1}}{\partial u_{i+1}} \right)^{(j)} ; \quad (k_F)_{i+1}^{(j)} = \left(\frac{\partial (f_F)_{i+1}}{\partial u_{i+1}} \right)^{(j)}.$$

Note that k_T can be set equal to k_0 , which is the largest stiffness in the THM, see Fig. 4, while k_F can be set equal to k_f , see Fig. 5. Equation 3.9 can be written as

$$(3.14) \quad (k_S)_{i+1}^{(j)} \Delta u^{(j)} = p_{i+1} - (f_S)_{i+1}^{(j)} \equiv R_{i+1}^{(j)}.$$

Substituting Eq. 3.5, 3.6 in Eq. 3.8 and then in Eq. 3.13, we get

$$(3.15) \quad R_{i+1}^{(j)} = p_{i+1} - (f_T)_{i+1}^{(j)} - (f_F)_{i+1}^{(j)} - \left[\frac{1}{\beta(\Delta t)^2} m + \frac{\gamma}{\beta \Delta t} c \right] (u_{i+1}^{(j)} - u_i) \\ + \left[\frac{1}{\beta \Delta t} m + \left(\frac{\gamma}{\beta} - 1 \right) c \right] \dot{u}_i + \left[\left(\frac{1}{2\beta} - 1 \right) m + \Delta t \left(\frac{\gamma}{2\beta} - 1 \right) c \right] \ddot{u}_i.$$

The whole procedure of programming the process of the Newmark method is presented in Table 4 by following [7].

In order to define the force in THM, upper and lower limit bounds need to be established, depending whether the velocity is positive or negative, see Fig. 6. When the velocity is positive the force cannot be larger than f_{lim}^{up} , while when the velocity is negative the force cannot be smaller than f_{lim}^{lo} . In Appendix the pseudo-code for the definition of f_{lim}^{up} and f_{lim}^{lo} is presented.

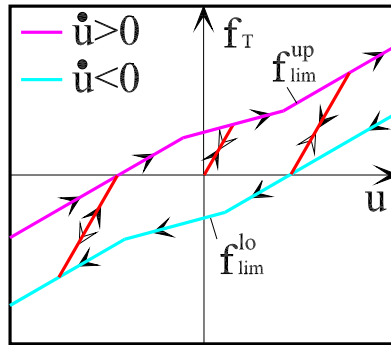


Fig. 6. Upper and lower limit forces for THM.

Table 4. Newmark method: for the SDOF system of Solarino building, following [7]

	<i>Special cases</i>
	(1) Constant average acceleration method ($\gamma = \frac{1}{2}, \beta = \frac{1}{4}$)
	(2) Linear acceleration method ($\gamma = 1/2, \beta = 1/6$)
1.0	<i>Initial conditions</i>
1.1	Check whether $ p_i > f_f \Rightarrow$ initiation of motion, set $i = 0$.
1.2	State determination $(f_T)_0, (k_T)_0, (f_F)_0$ and $(k_F)_0$.
1.3	Solve $m\ddot{u}_0 = p_0 - cu_0 - (k_T)_0 u_0 - (k_F)_0 u_0$.
1.4	Select Δt .
1.5	$a_1 = \frac{1}{\beta(\Delta t)^2} m + \frac{\gamma}{\beta \Delta t} c;$
	$a_2 = \frac{1}{\beta \Delta t} m + \left(\frac{\gamma}{\beta} - 1\right) c;$
	$a_3 = \left(\frac{1}{2\beta} - 1\right) m + \Delta t \left(\frac{\gamma}{2\beta} - 1\right) c.$
2.0	<i>Calculations for each time instant</i>
2.1	Initialize $j = 1,$
	$u_{i+1}^{(j)} = u_i, (f_T)_{i+1}^{(j)} = (f_T)_i^{(j)}, (f_F)_{i+1}^{(j)} = (f_F)_i^{(j)},$
	$(k_T)_{i+1}^{(j)} = (k_T)_i^{(j)}, (k_F)_{i+1}^{(j)} = (k_F)_i^{(j)}.$
2.2	$\hat{p}_{i+1} = p_{i+1} + a_1 u_i + a_2 \dot{u}_i + a_3 \ddot{u}_i.$
3.0	<i>For each iteration $j = 1, 2, 3, \dots$</i>
3.1	$R_{i+1}^{(j)} = \hat{p}_{i+1} - a_1 u_{i+1}^{(j)} - (f_T)_{i+1}^{(j)} - (f_F)_{i+1}^{(j)}.$
3.2	Check convergence;
	if the acceptance criteria are not satisfied, implement steps 3.3 to 3.7;
	otherwise go directly to step 4.0.
3.3	$(k_S)_{i+1}^{(j)} = a_1 + (k_T)_{i+1}^{(j)} + (k_F)_{i+1}^{(j)}.$
3.4	Solve $(k_S)_{i+1}^{(j)} \Delta u^{(j)} = R_{i+1}^{(j)} \Rightarrow \Delta u^{(j)}.$
3.5	$u_{i+1}^{(j+1)} = u_{i+1}^{(j)} + \Delta u^{(j)}.$
3.6	State determination $(f_T)_{i+1}^{(j+1)}, (k_T)_{i+1}^{(j+1)}, (f_F)_{i+1}^{(j+1)}$ and quad $(k_F)_{i+1}^{(j+1)}.$
3.7	Replace j by $j + 1$ and repeat steps 3.1 to 3.6; denote final $u_{i+1}.$
4.0	<i>Calculation for velocity and acceleration</i>
4.1	$\dot{u}_{i+1} = \frac{\gamma}{\beta \Delta t} (u_{i+1} - u_i) + \left(1 - \frac{\gamma}{\beta}\right) \dot{u}_i + \Delta t \left(1 - \frac{\gamma}{2\beta}\right) \ddot{u}_i.$
4.2	$\ddot{u}_{i+1} = \frac{1}{\beta(\Delta t)^2} (u_{i+1} - u_i) - \frac{1}{\beta \Delta t} \dot{u}_i - \left(\frac{1}{2\beta} - 1\right) \ddot{u}_i.$
5.0	<i>Repetition for next time step.</i>
	Replace i by $i + 1$ and
	implement steps 2.0 to 4.0 for the next time step.

4. COMPARISON BETWEEN SEMI-ANALYTICAL AND NUMERICAL SOLUTION

The equation of motion, see Eq. 3.3, can be solved semi-analytically as it is shown in [6]. In this section, a comparison between the semi-analytical and the numerical solution, provided by the Newmark method, will be presented. The excitation p in Eq. 3.3 will be

$$(4.1) \quad p = -0.025mg \sin(2\pi f_r t),$$

where m is the mass of the structure (see Table 1), g is the gravitational acceleration and $f_r = 0.41$ Hz. It should also be noted that the time step used in the numerical simulation is $Dt = 0.005$ sec. The results shown in Fig. 7 show that the results are identical and the numerical solution is extremely accurate. The difference between the maximum displacements in the steady-state stage of the excitation is 0.2 mm (237 mm for the semi-analytical, 237.2 mm for the numerical). In Table 5 the time required for the solution by the semi-analytical and the numerical solution is presented. Finally, the results show that the numerical solution requires almost double implementation time compared to the semi-analytical one.

Table 5. Implementation time required for the solution with the semi-analytical solution and the numerical one

Semi-analytical [sec]	Numerical [sec]
1.85	3.44

5. CONCLUSIONS

For the first time, the numerical solution for the THM model is presented by using the Newmark method. The numerical solution is compared with the semi-analytical solution, provided by [6], for the SDOF system of the Solarino building. The results show that the agreement between numerical and semi-analytical solution is practically identical. The disadvantage for the numerical solution is that it requires almost double time to solve the problem. Most importantly, the numerical solution can be used for extension of the trilinear mechanical models to bidirectional horizontal motion, to time-varying vertical loads, to multi-degree-of-freedom-systems as well to generalized models connected in parallel, see [4], where only numerical solutions are possible.

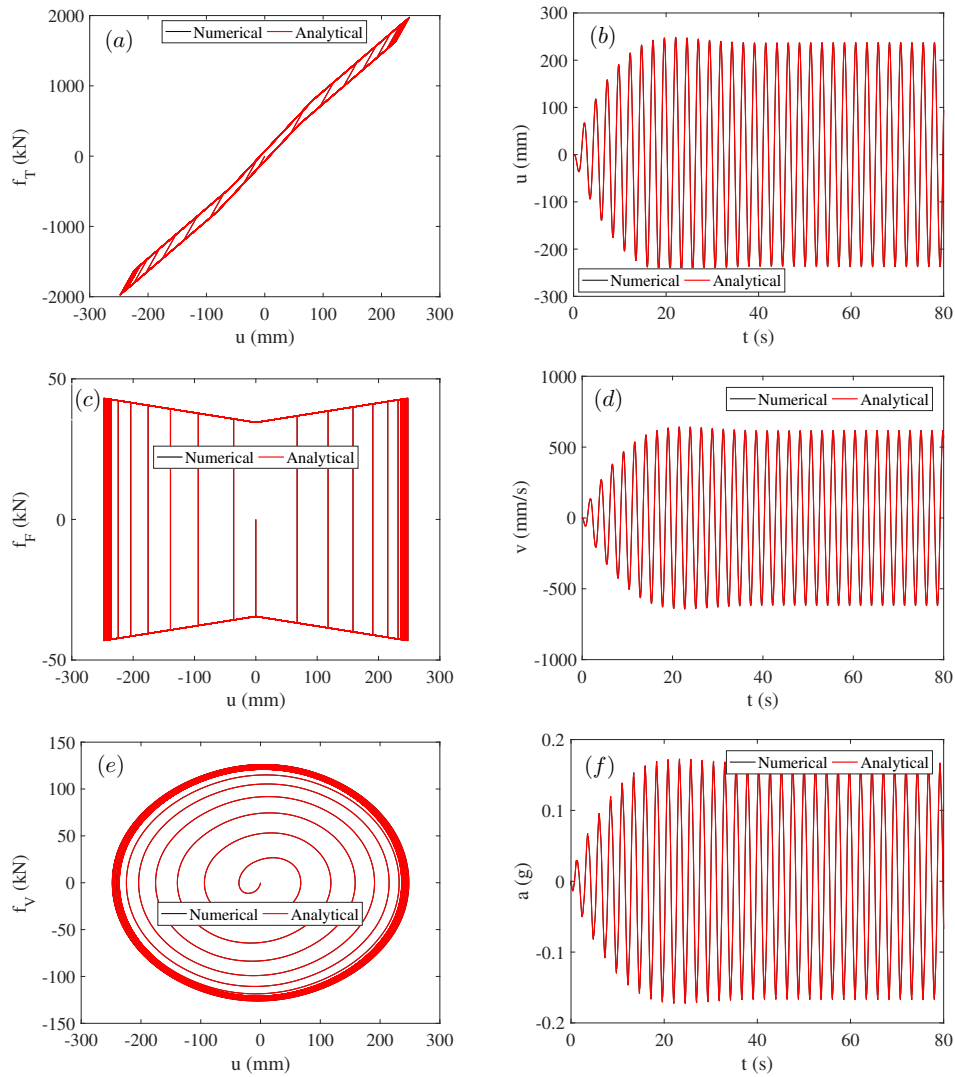


Fig. 7. (a) Force-displacement graph of THM, (b) displacement-time graph of SDOF system, (c) force-displacement graph of LCFM, (d) velocity-time graph of the SDOF system, (e) Force-displacement graph of LVD, (f) acceleration-time graph of the SDOF system under harmonic excitation.

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APPENDIX

In order to define the upper and lower limits of the force in the THM the conditions described in the following pseudo-code (in Matlab/Octave) need to be applied. Note that u_{jt} denotes the current displacement amplitude.

```
% Calculate flimup
if -uyh+2*uy <= ujt && ujt <= uyh
    if uyh>=2*uy
        flimup = Q + k1*ujt;
    elseif uyh< 2*uy
        flimup = Fy + k1*(ujt - uy);
    end
elseif ujt > uyh
    if uyh>=2*uy
        flimup = Q + k1*uyh + k2*(ujt - uyh);
    elseif uyh< 2*uy
        flimup = Fy + k1*(uyh -uy) + k2*(ujt - uyh);
    end
elseif ujt < -uyh + 2*uy
    if uyh>=2*uy
        flimup = Q + k1*(-uyh + 2*uy) + k2*(ujt - (-uyh + 2*uy));
```

```

        elseif uyh < 2*uy
            flimup = Fy - k1*(uyh - uy) + k2*(ujt - (-uyh + 2*uy));
        end
    end
% Calculate flimlo
if -uyh <= ujt && ujt <= uyh-2*uy
    if uyh >= 2*uy
        flimlo = -Q + k1*ujt;
    elseif uyh < 2*uy
        flimlo = -Fy + k1*(ujt + uy);
    end
elseif uyh-2*uy < ujt
    if uyh >= 2*uy
        flimlo = -Q + k1*(uyh - 2*uy) + k2*(ujt - (uyh - 2*uy) );
    elseif uyh < 2*uy
        flimlo = -Fy + k1*(uyh - uy) + k2*(ujt - (uyh - 2*uy) );
    end
elseif ujt < -uyh
    if uyh >= 2*uy
        flimlo = -Q - k1*uyh + k2*( ujt + uyh ) ;
    elseif uyh < 2*uy
        flimlo = -Fy - k1*(uyh -uy ) + k2*( ujt + uyh ) ;
    end
end
%

```
