

MODIFIED METHOD OF SIMPLEST EQUATION APPLIED TO
THE NONLINEAR SCHRÖDINGER EQUATION

NIKOLAY K. VITANOV^{1*}, ZLATINKA I. DIMITROVA²

¹*Institute of Mechanics, Bulgarian Academy of Sciences,
Akad. G. Bonchev Str., Bl. 4, 1113 Sofia, Bulgaria*

²*G. Nadjakov Institute of Solid State Physics, Bulgarian Academy of Sciences,
Blvd. Tzarigradsko Chaussee 72, 1784 Sofia, Bulgaria*

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ABSTRACT: We consider an extension of the methodology of the modified method of simplest equation to the case of use of two simplest equations. The extended methodology is applied for obtaining exact solutions of model nonlinear partial differential equations for deep water waves: the nonlinear Schrödinger equation. It is shown that the methodology works also for other equations of the nonlinear Schrödinger kind.

KEY WORDS: Nonlinear partial differential equations, exact solutions, nonlinear Schroedinger equation, method of simplest equation.

1. INTRODUCTION

We have observed several decades of a fast growth of research on nonlinear phenomena [1–5]. Today this is well established research area and the research on nonlinear waves finds its significant place within this area. One of the most studied models in the research on waves are the models of water waves. The nonlinear Schrödinger equation (NSE) is a model equation for nonlinear waves in a deep water. Thus, the traveling wave solutions of this equation are of large interest for fluid mechanics. Below, we discuss the nonlinear Schrödinger equation from the point of view of the modified method of simplest equation [6–8] for obtaining exact analytical solutions of nonlinear partial differential equations. The goal is to refine the existing methodology in order to make this methodology capable to obtain exact traveling wave solutions of the NSE.

Numerous models of natural and social systems are based on differential equations [9–13]. Often, the model equations are nonlinear partial differential equations that have traveling-wave solutions. These traveling wave solutions are studied very intensively [14–19]. Powerful methods exist for obtaining exact traveling-wave solutions of nonlinear partial differential equations, e.g., the method of inverse scattering

*Corresponding author e-mail: vitanov@imbm.bas.bg

transform or the method of Hirota [20, 21]. These methods work very well for the case of integrable nonlinear PDEs. Many other approaches for obtaining exact special solutions of nonintegrable nonlinear PDEs have been developed in the recent years (for examples see [22–24]). Below, we shall consider the method of simplest equation and our focus will be on a version of this method called, modified method of simplest equation [6–8, 25, 26]. The method of simplest equation is based on a procedure analogous to the first step of the test for the Painleve property [25–28]. In the version of the method called modified method of simplest equation [6–8], this procedure is substituted by the concept for the balance equation. We are going to discuss below the possibility of use of more than one simplest equation for obtaining exact analytical solution of the solved nonlinear partial differential equation. The modified method of simplest equation has many applications, e.g., obtaining exact traveling wave solutions of generalized Kuramoto - Sivashinsky equation, reaction - diffusion equation, reaction - telegraph equation [6, 29], generalized Swift - Hohenberg equation and generalized Rayleigh equation [7], generalized Fisher equation, generalized Huxley equation [30], generalized Degasperis - Procesi equation and b-equation [31], extended Korteweg-de Vries equation [32], etc. [33, 34].

2. MODIFIED METHOD OF SIMPLEST EQUATION

2.1. THE CLASSIC VERSION OF THE METHODOLOGY

The methodology of the modified method of simplest equation used up to now is appropriate for solving model nonlinear PDEs, e.g., from the research area of shallow water waves. These equations contain real functions of a coordinate and of the time and because of this one simplest equation is sufficient for application of the methodology. The model equations for the deep water waves contain complex functions. Because of this, we have to extend the methodology of the modified method of simplest equation to the case of use of two simplest equations. The classic version of the modified method of simplest equation follows below. The extension of the methodology is presented in the following subsection.

The methodology based on the use of one simplest equation is described, e.g., in [35]. We note that cases of use of more than one simpler differential equation for obtaining exact analytical solutions of more complicated partial differential equation can be found in the literature even 25 years ago, e.g., [36–39]. First of all, by means of an appropriate ansatz, the solved nonlinear partial differential equation is reduced to a nonlinear differential equation, containing derivatives of a function:

$$(1) \quad P(u(\xi), u_\xi, u_{\xi\xi}, \dots) = 0.$$

Then, the function $u(\xi)$ is searched as some function of another function, e.g., $g(\xi)$,

i.e.,

$$(2) \quad u(\xi) = G[g(\xi)].$$

Note, that the kind of the function G is not prescribed. Often one uses a finite-series relationship, e.g.,

$$(3) \quad u(\xi) = \sum_{\mu_1=-\nu_1}^{\nu_2} p_{\mu_1} [g(\xi)]^{\mu_1},$$

where p_{μ_1} are coefficients and ν_1 and ν_2 are natural numbers. The function $g(\xi)$ is solution of a simpler ordinary differential equation called simplest equation. Equation (2) is substituted in Eq. (1) and let the result of this substitution be a polynomial containing $g(\xi)$. What follows is application of a balance procedure. This procedure has to ensure that all of the coefficients of the obtained polynomial of $g(\xi)$ contain more than one term. The procedure leads to a balance equation for some of the parameters of the solved equation and for some of the parameters of the solution. Equation (2) describes a candidate for solution of Eq. (1) if all coefficients of the obtained polynomial are equal to 0. This condition leads to a system of nonlinear algebraic equations for the coefficients of the solved nonlinear PDE and for the coefficients of the solution. Any nontrivial solution of this algebraic system leads to a solution of the studied nonlinear partial differential equation.

2.2. EXTENSION OF THE METHODOLOGY TO THE CASE OF USE OF TWO SIMPLEST EQUATIONS

Now we consider an extension of the methodology to the case where the idea of use of two simplest equations is incorporated in the methodology. First of all by means of an appropriate ansätze (for an example two traveling-wave ansätze) the solved nonlinear partial differential equation is reduced to a nonlinear differential equation, containing derivatives of two functions:

$$(4) \quad P(u(\xi), u_\xi, u_{\xi\xi}, \dots, w(\zeta), w_\zeta, w_{\zeta\zeta}, \dots) = 0.$$

Then, the functions $u(\xi)$, $w(\zeta)$, are searched as some function of another functions, e.g., $g(\xi)$ and $\phi(\zeta)$, i.e.

$$(5) \quad u(\xi) = G[g(\xi)]; \quad w(\zeta) = F[\phi(\zeta)]; \dots$$

Note, that the kind of the functions F and G is not prescribed. Often one uses a finite-series relationship, e.g.,

$$(6) \quad u(\xi) = \sum_{\mu_1=-\nu_1}^{\nu_2} p_{\mu_1} [g(\xi)]^{\mu_1}; \quad w(\zeta) = \sum_{\mu_2=-\nu_3}^{\nu_4} p_{\mu_2} [\phi(\zeta)]^{\mu_2},$$

where p_{μ_1}, p_{μ_2} are coefficients and $\nu_{1,2,3,4}$ are natural numbers. The functions $g(\xi)$, $\phi(\zeta)$ are solutions of simpler ordinary differential equations called simplest equations. Equation (5) is substituted in Eq. (4) and let the result of this substitution be a polynomial containing $g(\xi)$ and $\phi(\zeta)$. Then, a balance procedure is applied. This procedure has to ensure that all of the coefficients of the obtained polynomial of $g(\xi)$ and $\phi(\zeta)$ contain more than one term. The procedure leads to one or more balance equations for some of the parameters of the solved equation and for some of the parameters of the solution. Equations (5) could describe a candidate for solution of Eq. (4) if all coefficients of the obtained polynomial are equal to 0. This condition leads to a system of nonlinear algebraic equations for the coefficients of the solved nonlinear PDE and for the coefficients of the solution. Any nontrivial solution of this algebraic system leads to a solution the studied nonlinear partial differential equation.

The organization of the text below is as follows. In Sect. 3 we discuss an application of the extension of the modified method of simplest equation described above to the classic nonlinear Schrödinger equation and to another equation of the same class. Several concluding remarks are summarized in Sect. 4.

3. APPLICATION TO NONLINEAR SCHRÖDINGER EQUATION AND SIMILAR EQUATIONS

The classic nonlinear Schrödinger equation is

$$(7) \quad iq_t + aq_{xx} + bq|q|^2 = 0,$$

where $i = \sqrt{-1}$ and a and b are parameters. The solution of Eq. (7) will be searched as

$$(8) \quad q(x, t) = g(\xi)h(x, t),$$

where $g(\xi)$ is a real function ($\xi = \alpha x + \beta t$) and $h(x, t)$ is a complex function. The two simplest equations will be for the functions $g(\xi)$ and $h(x, t)$ respectively. Let us denote as $h^*(x, t)$ the complex conjugate function of $h(x, t)$. The substitution of Eq. (8) in Eq. (7) leads to (we denote g_ξ as g')

$$(9) \quad i\beta g'h + igh_t + \alpha^2 ag''h + 2\alpha ag'h_x + agh_{xx} + bg^3h^2h^* = 0.$$

The first simplest equation is for the function $h(x, t)$. Taking into an account the presence of h and its derivatives in Eq. (9) as well as the presence of the term hh^* there and aiming to choose such simplest equation that will lead to reduction of Eq. (9) to an equation for $g(\xi)$, we arrive at the simplest equation

$$(10) \quad h_\zeta = ih, \quad \zeta = \kappa x + \omega t + \theta,$$

which solution is

$$(11) \quad h(\zeta) = \exp(i\zeta) = \exp[i(\kappa x + \omega t + \theta)].$$

The substitution of Eq. (10) in Eq. (9) reduces Eq. (9) to an equation for the function $g(\xi)$

$$(12) \quad \alpha^2 a g'' + (2\alpha\kappa a + \beta) i g' - (\omega + \kappa^2 a) g + b g^3 = 0$$

$g(\xi)$ has to be a real function and then

$$(13) \quad \beta = -2\alpha\kappa a.$$

The substitution of Eq. (13) in Eq. (12) followed by multiplication of the result by g' and integration with respect to ξ leads to the equation

$$(14) \quad \alpha^2 a g'^2 - (\omega + \kappa^2 a) g^2 - c + \frac{b g^4}{2} = 0,$$

where c is a constant of integration. Further we set

$$(15) \quad u = g^\sigma,$$

where σ is a parameter that will be determined below. The substitution of Eq. (15) in Eq. (14) leads to the following equation for $u(\xi)$:

$$(16) \quad u'^2 = \frac{\sigma^2(\omega + \kappa^2 a)}{\alpha^2 a} u^2 + \frac{\sigma^2 c}{\alpha^2 a} u^{2(\sigma-1)/\sigma} - \frac{b\sigma^2}{2\alpha^2 a} u^{2(1+\sigma)/\sigma}.$$

Two cases are possible here: $\sigma = 1$ and $\sigma = 2$. For the case $\sigma = 1$, Eq. (16) becomes

$$(17) \quad u'^2 = \frac{c}{\alpha^2 a} + \frac{(\omega + \kappa^2 a)}{\alpha^2 a} u^2 - \frac{b}{2\alpha^2 a} u^4.$$

This equation contains as particular case the equation for the elliptic functions of Jacobi

$$(18) \quad u'^2 = p + q u^2 + r u^4$$

(for $\text{sn}(\xi; k^*)$: $p = 1, q = -(1 + k^{*2}), r = k^{*2}$; for $\text{cn}(\xi; k^*)$: $p = 1 - k^{*2}, q = 2k^{*2} - 1, r = -k^{*2}$; for $\text{dn}(\xi; k^*)$: $p = -(1 - k^{*2}), q = 2 - k^{*2}, r = -1$; k^* is the modulus of the corresponding elliptic function of Jacobi).

For the case $\sigma = 2$, Eq. (16) becomes

$$(19) \quad u'^2 = \frac{4c}{\alpha^2 a} u + \frac{4(\omega + \kappa^2 a)}{\alpha^2 a} u^2 - \frac{2b}{\alpha^2 a} u^3.$$

This equation contains as particular case the equation for the elliptic functions of Weierstrass $\wp(\xi, \gamma_2, \gamma_3)$

$$(20) \quad \wp'^2 = 4\wp^3 - \gamma_2\wp - \gamma_3,$$

where γ_2 and γ_3 are parameters.

Let us first consider Eq. (17) and write a solution on the basis of the Jacobi elliptic function $\text{cn}(\xi, k^*)$. For this case, we have to solve the system of equations

$$(21) \quad p = \frac{c}{\alpha^2 a} = 1 - k^{*2}; \quad q = \frac{\omega + \kappa^2 a}{\alpha^2 a} = 2k^{*2} - 1; \quad r = -\frac{b}{2\alpha^2 a} = -k^{*2}$$

The solution of this system is

$$(22) \quad \alpha^2 = \frac{b}{2ak^{*2}}; \quad c = \frac{b(1 - k^{*2})}{2k^{*2}}; \quad \omega = \alpha^2 a(2k^{*2} - 1) - \kappa^2 a,$$

and the corresponding solution of the nonlinear Schrödinger equation (7) is

$$(23) \quad q(x, t) = \text{cn} \left[\left(\frac{b}{2ak^{*2}} \right)^{1/2} x + \beta t; k^* \right] \\ \times \exp \left[i \left(\kappa x + a[\alpha^2(2k^{*2} - 1) - \kappa^2]t + \theta \right) \right].$$

When $k^* = 1$, then the Jacobi elliptic function cn is reduced to $\text{sech}(\xi)$. In this case, the solution (23) becomes

$$(24) \quad q(x, t) = \text{sech} \left[\left(\frac{b}{2a} \right)^{1/2} x + \beta t \right] \exp \left[i \left(\kappa x + a[\alpha^2 - \kappa^2]t + \theta \right) \right].$$

Let us now consider (19) for the particular case where the solution can be obtained on the basis of the elliptic function of Weierstrass (20). In this case $\gamma_3 = 0$; $\alpha = \pm[-b/(2a)]^{1/2}$; $\omega = -\kappa^2 a$. Thus the corresponding solution of the nonlinear equation of Schrödinger becomes

$$(25) \quad q(x, t) = \wp \left\{ \left[\pm \left(-\frac{b}{2a} \right)^{1/2} (x - 2\kappa a) \right]; \frac{8c}{b}, 0 \right\}^{1/2} \exp[i(\kappa x - \kappa^2 at + \theta)].$$

3.1. ANOTHER EXAMPLE FOR APPLICATION OF THE EXTENDED METHODOLOGY

Let us now consider more complicated equation of the nonlinear Schrödinger kind, i.e.,

$$(26) \quad iq_t + aq_{xx} + b_{-2}q |q|^{-4} + b_0q + b_1q |q|^2 + b_2q |q|^4 = 0,$$

where b_i are parameters. The application of the extended version of the modified method of simplest equation leads to Eqs. (10), (11) for $h(\zeta)$ and for $g(\xi)$ we obtain $g = u^{1/2}$ associated with the following (simplest) equation:

$$(27) \quad u'^2 = c_{-2} + c_0u^2 + c_1u^3 + c_2u^4,$$

where

$$c_{-2} = \frac{4b_{-2}}{\alpha^2 a}; \quad c_0 = \frac{4(\omega + \kappa^2 a - b_0)}{\alpha^2 a}; \quad c_k = -\frac{4b_k}{\alpha^2 a(k+1)}, \quad k = 1, 2$$

We shall consider the case $c_1 = 0$ and then Eq. (27) is reduced to the differential equation for the elliptic functions of Jacobi. Let us write the solution based on the Jacobi elliptic function $\text{sn}(\xi, k^*)$. k^* is the modulus of the Jacobi elliptic function and for the case of the sn function we have the following relationships among the modulus of the elliptic function and the coefficients in Eq. (27):

$$c_{-2} = 1; \quad c_0 = -(1 + k^{*2}); \quad c_2 = k^{*2}.$$

Then

$$(28) \quad k^{*2} = \frac{b_2}{3b_{-2}}; \quad \alpha^2 = \frac{4b_{-2}}{a}; \quad \omega = b_0 - \kappa^2 a - \frac{1}{3}(b_2 + 3b_{-2}).$$

Thus, the searched solution is

$$(29) \quad q(x, t) = \left\{ \text{sn} \left[\left(\frac{4b_{-2}}{a} \right)^{1/2} (x - 2\kappa at); k^* \right] \right\}^{1/2} \times \exp \left\{ i \left[\kappa x + \left(b_0 - \kappa^2 a - \frac{1}{3}(b_2 + 3b_{-2}) \right) t + \theta \right] \right\}.$$

4. DISCUSSION

As we have seen above, the methodology of the modified method of simplest equation is effective tool for obtaining exact particular solutions of nonlinear partial differential equations. The emphasis of this paper was on the extension of the methodology in order to include the model equations for deep water waves in the class of equation that can be treated by the modified method of simplest equation. In order to do this, we have to use more than one simplest equation. We note that the extended methodology can be applied to many equations of nonlinear Schrödinger kind. It is known that numerous equations of NSE kind have solitary wave solutions [40–43]. Thus, the perspectives are very good for obtaining new solutions of these model equation and for application of the extended methodology to more complicated equations from the area of water waves and optics.

Of course, the methodology can be extended to the case of use of more than two simplest equations simultaneously. This research and its results will be reported elsewhere.

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