

MECHANICAL SYSTEMS OF COSSERAT–ZHILIN

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ABSTRACT: Mechanical systems of Cosserat–Zhilin are introduced as the main object of rational (non-relativistic) mechanics on the base of new notions of vector calculus — sliders and screw measures (bi-measures).

KEY WORDS: classical mechanics, continuum mechanics, mechanical measures, foundations of mechanics, screw theory.

1 INTRODUCTION

*‘The ancients considered mechanics in a twofold respect; as rational, which proceeds accurately by demonstration; and practical. . . Rational mechanics will be the science of motions resulting from any forces whatsoever; and of the forces required to produce any motions, accurately proposed and demonstrated’* [1].

The progress of rational mechanics (and physics of XVIII–XIX centuries) is primarily based on working out its mathematical aspects. In 1687, I. Newton published *Philosophiae Naturalis Principia Mathematica*, where the leading role of mathematics in rational mechanics is directly pointed out in its title. Newton deliberately almost never used mathematical analysis: use new and unusual methods would jeopardize the credibility of his results. But already in 1736, in *Mechanics*, L. Euler explicitly stressed that *‘full understanding mechanics can be achieved only through mathematical analysis’*, thus, emphasizing that mathematics should be put at the forefront and the consideration of the physical aspects only is insufficient.

The necessity of mathematization<sup>1</sup> of mechanics was marked by D’Alembert in 1743: *‘Rational mechanics, like geometry, must be based upon axioms which are*

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<sup>1</sup> J.L. Lagrange continued the further mathematization of mechanics. He wrote in 1788: *‘Ceux qui aiment l’Analyse, verront avec plaisir la Mécanique en devenir une nouvelle branche. . .’* Although the Lagrangian and Hamiltonian approaches have proven critical to the study of classical mechanical problems and quantum mechanical systems and were used in electromagnetism, quantum mechanics, quantum relativity theory, and quantum field theory, *‘Lagrangian mechanics is a rather poor subclass of Newtonian one and it can not be considered as self-sufficient science about natural phenomena. It follows, e.g., from the fact that the fundamental concepts like space, time, forces (moments), energy and etc., are not discussed and can not be introduced into consideration in Lagrangian mechanics, where all these concepts are used, but are not determined. . . In many cases, such as for open systems, Lagrangian mechanics is not applicable’* [2] (see also [3]).

*obviously true*' [4] (see also [5]). The first system of axioms in mechanics was introduced by I. Newton. Now we have many systems of axioms at hand. The present paper gives one more to represent rational mechanics as a mathematical science and to determine the widest class of mechanical systems covered by it.

*'...the dynamics of a continuous system must clearly include as a limiting case (corresponding to a medium of density everywhere zero except in one very small region) the mechanics of a single material particle. This at once shows that it is absolutely necessary that the postulates introduced for the mechanics of a continuous system should be brought into harmony with the modifications accepted above in the mechanics of the material particle'* [6].

In other words, we must consider the various branches of general mechanics from a unified point of view.

## 2 PRIMITIVE CONCEPTS OF RATIONAL MECHANICS

The following '*experimental facts*' lie at the foundation of rational mechanics [7]:

1. All natural phenomena occur in *space* and *time*.
2. Galileo's principle of relativity: '*there exist coordinate systems (called inertial) possessing the following two properties:*
  - *all the laws of nature at all moments of time are the same in all inertial systems;*
  - *all coordinate systems in uniform rectilinear motion with respect to an inertial one are themselves inertial*'.
3. Newton's principle of determinacy: *the initial state of a mechanical system uniquely determines all of its motion.*

It is easy to see that all of the above is nothing more than constants of the science language. That is why it is necessary to clear up their mathematical essence. In particular, we must answer the following questions:

- *What are the laws of nature, about of which Galileo's principle says?*
- *What is the group of transformations w.r.t. of which the laws are invariant'?*
- *What do we mean under 'mechanical system'?*
- *What main types of mechanical systems do we have?*

**Galilean space-time structure.** In what follows, we shall use  $n$ -dimensional affine space  $\mathbf{A}^n$  modeled on  $n$ -dimensional vector space  $\mathbf{V}^n$ ,  $n$ -dimensional Euclidean space  $\mathbf{R}^n$  ( $\mathbf{R} = \mathbf{R}^1$  is the set of all real numbers).

Define the Galilean space-time structure as the quadruple  $\mathbf{G} = \{\mathbf{V}^n, \mathbf{A}^n, \tau, g\}$  where [7]

- $\tau : \mathbf{V}^n \rightarrow \mathbf{R}$  is a surjective linear mapping called time one, and
- $g = \langle \cdot, \cdot \rangle$  is an inner product on  $\ker\{\tau\}$  ( $= \mathbf{V}^{n-1}$ ).

The space  $\mathbf{A}^n$  with the Galilean space-time structure is called *Galilean*<sup>2</sup>.

Time is a linear mapping  $\tau : \mathbf{V}^n \rightarrow \mathbf{R}$  from the vector space of parallel displacements of  $\mathbf{A}^n$  to the real ‘time axis’. We shall denote the range of  $\tau$  by  $\mathbf{T} \subseteq \mathbf{R}$ . The time interval from event  $a \in \mathbf{A}^n$  to event  $b \in \mathbf{A}^n$  is the number  $\tau(b - a)$  (it is plain that  $b - a \in \mathbf{V}^n$ ). If  $\tau(b - a) = 0$ , then the events  $a$  and  $b$  are called simultaneous.

The set of events simultaneous with a given event forms  $n - 1$ -dimensional affine space  $\mathbf{A}^{n-1} \subset \mathbf{A}^n$  modeled on  $\ker\{\tau\}$ . It is called *space of simultaneous events*.

There is the group of affine transformations of the space  $\mathbf{A}^n$  which preserve the Galilean time-space structure. The elements of this group are called *Galilean transformations*. They preserve intervals of time and the distance between simultaneous events [7].

**Theorem 1.** *Each Galilean transformation is movement of the space of simultaneous events, accompanied by a shift of the origin of time [7, 8].*

The inner product  $\langle \cdot, \cdot \rangle$  (in Galilean space-time) enables one to pass from the affine space  $\mathbf{A}^n$  to *Euclidean* space  $\mathbf{R}^n$  with the distance  $\rho(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}$  between points  $x$  and  $y \in \mathbf{A}^n$ .

The bijective map  $\mathbf{A}^n \rightarrow \mathbf{R}^{n-1} \times \mathbf{T}$  is called *frame of reference* [7, 8] (here  $\mathbf{R}^{n-1}$  is also a space of simultaneous events). For each frame of reference the corresponding space  $\mathbf{R}^{n-1} \times \mathbf{T}$  is Galilean. That is why we shall call any frame of reference Galilean, too.

Define *world-line* as a curve in  $\mathbf{A}^n$  whose image in  $\mathbf{R}^{n-1} \times \mathbf{T}$  associates one point  $x(t) \in \mathbf{R}^{n-1}$  to each instant<sup>3</sup>  $t \in \mathbf{T}$ . A collection of non-intersectional world-lines forms *world-tube* (see also [9]).

Let us fix a world-tube  $\tilde{\Lambda} \subset \mathbf{A}^n$  and call it *universe*<sup>4</sup>. A given world-tube  $\Lambda \subset \tilde{\Lambda}$ , the world-tube  $\Lambda^e = \tilde{\Lambda} \setminus \Lambda$  is called *environment* of  $\Lambda$  in the universe.

<sup>2</sup> Note that the term ‘Galilean’ is merely traditional and should not be regarded as an attribution to Galileo.

<sup>3</sup> It means that a world-line does not have simultaneous points.

<sup>4</sup> As well as in *probability theory* [10], any universe is separately specified for every problem under consideration.

The universe  $\tilde{\Lambda}$  defines the family  $\{\tilde{\Lambda}_t \subset \mathbf{R}^{n-1}, t \in \mathbf{T}\}$ . For any world-tube  $\Lambda \subset \tilde{\Lambda}$  we have the family  $\{\Lambda_t \subset \tilde{\Lambda}_t, t \in \mathbf{T}\}$ .

**Main measures.** Let  $\sigma_{n-1}$  be  $\sigma$ -algebra on the space  $\mathbf{R}^{n-1}$ . We shall use the following Borel measure

$$(1) \quad \mu_{n-1}(A) = \mu_{ac}(A) + \mu_{pp}(A), \quad A \in \sigma_{n-1},$$

where  $\mu_{ac}(A)$  is the absolutely continuous component w.r.t. Lebesgue measure and  $\mu_{pp}(A)$  is the pure point (discrete) component presented as  $\mu_{pp}(A) = \sum_k \mu_k(x_k)$  for points  $x_k \in A$  whose are called *pure*, the others being called *continuous* [11].

**Remark 1.** We shall assume that if a point of  $\tilde{\Lambda}_t \subset \mathbf{R}^{n-1}$  is pure (or continuous) at some time instant  $t$ , all other points of the corresponding world-line ( $\forall t \in \mathbf{T}$ ) are also pure (or continuous), too.

Hereinafter we shall use the new notions of vector calculus — sliders and screw measures (bi-measures) (see Appendix 1). Define the following signed field measure<sup>5</sup>  $\Pi$  as a screw, having values

$$\Pi(A) = \int \chi_A l^{p_x, q_x} \mu_{n-1}(dx),$$

where the slider  $l^{p_x, q_x}$  is a (see Appendix 1) is assumed to be  $\mu_{n-1}$ -integrable,  $\chi_A$  is the characteristic function of  $A \in \sigma_{n-1}$ . Introduce also the notion of *signed field bi-measure*: a function  $\Xi(A, B)$  is called *signed field bi-measure* if it is a signed field measure by each of arguments  $A$  and  $B \in \sigma_{n-1}$  (see also [13]). A bi-measure  $\Xi$  is called *skew* if  $\Xi(A, B) = -\Xi(B, A)$ .

**Motion.** For the sake of brevity, below we shall only consider the **three-dimensional case**.

With any point  $x(t) \in \tilde{\Lambda}_t$  we associate the position radius-vector  $r_x(t) = \overrightarrow{(O, x(t))}$  w.r.t. a point  $O \in \mathbf{R}^3$ . Introduce the translation vector  $r_{y,x} \in \mathbf{V}_3$  from a point  $y \in \mathbf{R}_3$  to  $x$ . Define sliders  $l^{p_x, q_x}$  by the following relation (see also Appendix 1)

$$p_y = p_x \in \mathbf{V}^3, \quad q_y = q_x - r_{y,x} \times p_x \in \mathbf{V}^3, \quad y \in \mathbf{R}^3.$$

It is plain that the corresponding space of screws is six-dimensional (see also [14] and Appendix 1).

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<sup>5</sup> Signed measure is the generalization of the notion of measure allowing it to have negative values [12].

Define the translation velocity<sup>6</sup>  $v_x = \dot{r}_x(t)$  and introduce an orthonormal basis  $e_x$  at the point  $x(t) \in \mathbf{R}^3$ . Rotation of the basis  $e_x$  in  $\mathbf{R}^3$  can be characterized by the torque  $\mu_x$  (angular velocity — spin, angular momentum vector of mass unit or rotation tensor, *etc.*).

**Definition 1.** *The map of  $\mathbf{T}$  into the set  $\Lambda_t$  and the vectors  $v_x$  and  $\mu_x$ , associated with it, is called **motion**.*

**Main axioms.** With a common understanding the motion of bodies is caused by their mechanical interaction which is described in the terms of mass, force and torque of force (linear and angular momenta) as well as constraints on bodies and their parts, *etc.*

Let us introduce the measure  $\mathcal{P}$  as a screw having values

$$(2) \quad \mathcal{P}(\Lambda_t) = \int \chi_{\Lambda_t} l^{p_x, q_x} \mu_3(dx), \quad \Lambda_t \subset \tilde{\Lambda}_t, \quad t \in \mathbf{T},$$

where the slider  $l^{p_x, q_x}$  is defined by the following relation (see also [15])

$$(3) \quad \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \theta_x \begin{pmatrix} v_x \\ \mu_x \end{pmatrix}.$$

Here  $\theta_x$  is a non-negative defined, symmetric 2nd-order tensor.

For any two world-tubes  $\Lambda$  and  $\Lambda'$  with  $\Lambda_t$  and  $\Lambda'_t$  in the space of simultaneous events, respectively, define the skew signed field bi-measure  $\Phi$  as a screw (by each argument) having values  $\Phi(\Lambda_t, \Lambda'_t)$ .

**Axiom 1.** *There exist the measures  $\mathcal{P}$  and  $\Phi$  which express the mechanical interaction.*

**Definition 2.** *These measures are called kinetic and dynamic measures<sup>7</sup> of motion, respectively.*

**Axiom 2.** *The tensor  $\theta_x$  and the measure  $\Phi$  do not depend on frames of reference (see also [9]).*

Nevertheless, there are special frames of reference that help us to single out the kinetic and dynamic measures among different measures and bi-measures.

<sup>6</sup> We shall denote full derivatives by  $t$  with the help of the superscript  $\dot{\phantom{x}}$ , *e.g.*, for any function  $f = f(x(t), t)$  we have  $f^\dot{\phantom{x}} = \frac{\partial}{\partial t} f + (\text{div } f) x^\dot{\phantom{x}}(t)$ .

<sup>7</sup> Introducing the kinetic and dynamic measures as screws we follow L. Euler who has opened a new era in developing of Newtonian mechanics: two independent Laws of Dynamics are stated for the first time in ‘New method of determination of motion of rigid bodies’ [16].

**Axiom 3.** *There exists a Galilean frame of reference where the kinetic and dynamic measures of motion are connected by the following relation (see also [1, 14–17])*

$$(4) \quad \frac{d}{dt}\mathcal{P}(\Lambda_t) = \mathcal{F}(\Lambda_t), \quad \mathcal{F}(\Lambda_t) \stackrel{\text{def}}{=} \Phi(\Lambda_t, \Lambda_t^e), \quad \Lambda_t \subset \tilde{\Lambda}_t, \quad t \in \mathbf{T}.$$

As result we may introduce the following notions:

1. *the above mentioned frame of reference is called inertial;*
2. *the aggregate  $\alpha = \{\Lambda_t \subset \tilde{\Lambda}_t, \mu_3, \theta_x, x \in \Lambda_t, \mathcal{P}, \mathcal{F}, \forall t \in \mathbf{T}\}$  is called mechanical system of Cosserat–Zhilin;*
3. *the points of  $\Lambda_t$  and the vectors  $v_x$  and  $\mu_x$ , associated with them, constitute state of a mechanical system;*
4.  *$\mathcal{F}$  is called measure of impressed action of the environment  $\Lambda^e$  on  $\Lambda$  or force<sup>8</sup>;*
5. *relation (4) is called motion equation<sup>9</sup>;*
6. *the set  $\Lambda_t$  is called (actual) shape undergone by the mechanical system at  $t \in \mathbf{T}$ .*

**Remark 2.** *In the rational mechanics, the views of Aristotle were dominant for over two millennia, as long as Galileo did not introduce his principle of inertia:*

‘any isolated (lonely in the world) material point preserves its present state, whether it be of rest or of moving uniformly forward in a straight line in the absolute space’.

*Since it is impossible to determine the motion of an isolated point (body) relative to the absolute space, we could use the concept of reference frame as a reference body, equipped with a clock: (see, e.g., [15]):*

‘A body of reference with respect to which trajectories of an isolated (lonely in the world) particle are straightforward or a point, called inertial reference one’.

*At the same time, forgetting how it is possible to talk about the single particle in the world, when another body — the body of reference — is entered.*

<sup>8</sup> According to Glossary, Earth Observatory, NASA: force is any external agent that causes a change in the motion of a mechanical system, or that causes stress in a fixed mechanical system.

<sup>9</sup> It is the motion equation that constitutes the definition of each mechanical system [7].

*But the trouble does not come alone: straight lines are passed in straight lines under an arbitrary affine transformation, i.e., according to the above definition, new reference frames will be also inertial reference ones. If some new frame is accepted as the original one, we see that Newton’s Second Law can not be executed in this frame of reference. Thus such a classical definition of reference frames as well as Newton’s First Law proves to be unsatisfactory (see also [18]).*

**Remark 3.** Note that tensor  $\theta_x$  defines tensor measure of inertia  $\Theta$  having values

$$\Theta(\Lambda_t) = \int \chi_{\Lambda_t} \theta_x \mu_3(dx), \quad \Lambda_t \subset \tilde{\Lambda}_t, \quad t \in \mathbf{T}$$

and measure of kinetic energy (scalar measure of motion)  $\mathcal{T}$  having values (see also [19])

$$\mathcal{T}(\Lambda_t) = \int \chi_{\Lambda_t} \begin{pmatrix} v_x \\ \mu_x \end{pmatrix}^T \theta_x \begin{pmatrix} v_x \\ \mu_x \end{pmatrix} \mu_3(dx), \quad \Lambda_t \subset \tilde{\Lambda}_t, \quad t \in \mathbf{T}.$$

Hereinafter we assume that frames of reference are inertial.

### 3 THE LAW OF UNIVERSAL GRAVITY

If the first two laws of motion, Newton had predecessors (the authorship of their own, he did not claim — see page 71 (50) Russian translation of Newton’s Principia). The third law is wholly owned by Newton (predecessors to date nobody has been able to specify). Without it, there would be no law of universal gravity<sup>10</sup> (see page 13 of the Russian translation of Newton’s Principia):

*‘A particle attracts every other particle in the universe using a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them’.*

Below we shall consider systems where

$$(5) \quad \theta_x \stackrel{\text{def}}{=} \rho_x \begin{bmatrix} I & A \\ A^T & B \end{bmatrix}.$$

Here  $I$  is the unit tensor;  $A$  and  $B$  are 2nd-order tensors;  $\rho_x$  is a non-negative function for any  $x \in \Lambda_t \subset \tilde{\Lambda}_t$ ,  $t \in \mathbf{T}$  (see also [15]).

The function  $\rho_x$  generates the measure  $\mathcal{M}$ , having the values

$$\mathcal{M}(\Lambda_t) = \int \chi_{\Lambda_t} \rho_x \mu_3(dx), \quad \Lambda_t \subset \tilde{\Lambda}_t, \quad t \in \mathbf{T}.$$

<sup>10</sup> The equation of rigid body motion would not be without Newton’s Third Law, too (see [20]).

The measure  $\mathcal{M}$  is called *mass*. The set  $\tilde{\Lambda}_t^c \subset \tilde{\Lambda}_t$  is called *set of concentration* of the measure  $\mathcal{M}$  if  $\mathcal{M}(\Lambda_t) = 0$  for any set  $\Lambda_t \subset \tilde{\Lambda}_t \setminus \tilde{\Lambda}_t^c$ . We shall assume that relation (4) is true only for  $\Lambda_t \subset \tilde{\Lambda}_t^c, t \in \mathbf{T}$ .

The measure  $\mathcal{M}$  is introduced as an integral defined over actual shapes  $\Lambda_t \subset \tilde{\Lambda}_t, t \in \mathbf{T}$ , undergone by a mechanical system. That is why there is the following relation (see Appendix 2)

$$\frac{d}{dt}\mathcal{M}(\Lambda_t) = \int \chi_{\Lambda_t^{ac}} \left[ \frac{d}{dt} \rho_x + (\operatorname{div} v_x) \rho_x \right] \mu_{ac}(dx) + \sum_k \left( \frac{d}{dt} \rho_k \right) \mu_{pp}(x_k).$$

We shall assume that the function  $\rho_x = \rho(x, t)$  is defined by the *continuity equation* (see also Appendix 2)

$$\frac{d}{dt} \rho_x + (\operatorname{div} v_x) \rho_x = \frac{\partial}{\partial t} \rho_x + \operatorname{div} (v_x \rho_x) = \nu_x$$

in continuous points and  $\frac{d}{dt} \rho_k = \nu_k$  in pure points (here we may referred terms such as generation ( $\nu_x > 0$ ) or re-movement ( $\nu_x < 0$ ) to ‘sources’ and ‘sinks’, respectively). Then

$$\frac{d}{dt} \mathcal{M}(\Lambda_t) = \Delta \mathcal{M}(\Lambda_t) \stackrel{\text{def}}{=} \int \chi_{\Lambda_t^{ac}} \nu_x \mu_{ac}(dx) + \sum_k \nu_k \mu_{pp}(x_k), \Lambda_t \subset \tilde{\Lambda}_t, t \in \mathbf{T}.$$

We shall give a screw version of the law of universal gravity. To this end let us define the measure  $\Gamma$ , having the values

$$\Gamma(\Lambda_t, \Lambda_t^e) = \int \chi_{\Lambda_t} l^{\gamma_x} \rho_x \mu_3(dx), \quad \gamma_x = \gamma \int \chi_{\Lambda_t^e} r_{x,y} \frac{\rho_y \mu_3(dy)}{\|r_{x,y}\|^3}$$

on the sets  $\Lambda_t \in \sigma_3$  (here  $\gamma$  is the positive constant).

**Definition 3.** *The universal gravity is that is expressed by the measure<sup>11</sup>  $\mathcal{F}$  such that  $\mathcal{F}(\Lambda_t) \stackrel{\text{def}}{=} \Gamma(\Lambda_t, \Lambda_t^e)$ .*

**Definition 4.** *The mechanical interaction is that of the universal gravity.*

The concept of universal gravity is the great intellectual achievement that Newton represented in the most outstanding book in the history of science: *Philosophiae Naturalis Principia Mathematica* or, in modern language, *Mathematical Foundations of Physics*.

<sup>11</sup> The screw measure  $\mathcal{F}$  is called *measure of gravitating action* of  $\alpha^e$  upon  $\alpha$  [13].



Newton’s *Principia* formulated the laws of motion and universal gravity, which dominated scientists’ view of the physical universe for the next three centuries.

By deriving Kepler’s laws of planetary motion from his mathematical description of gravity, and then using the same principles to account for unknown before hyperbolic and parabolic orbits of celestial bodies, the tides, the precession of the equinoxes, and other phenomena, Newton demonstrated that the motion of objects on Earth and of celestial bodies could be described by the same principles<sup>12</sup>.

It is paradoxical, and even insulting to anyone who is familiar with the revolution produced by Newton in science, that, in the textbooks of theoretical mechanics, the law of universal gravity is not regarded. Sporadically, a particular case of the law, as the inverse square law, is derived from Kepler’s three laws.

#### 4 CONCEPT OF BODY

We shall use the following convention:

*body is that takes some shapes  $\Lambda_t \subset \tilde{\Lambda}_t$  in the space at some instants of time (cf. ‘every sensible body is in place’ — Aristotle, *Physics*, III, 4, 208b27).*

The concept of a body is the subject of various formalizations. For example, one may represent a body as a point-wise set, a differentiable manifold, a topological or measure space [13, 22] where a map into the space of shapes is considered<sup>13</sup>. But there is a small obstacle: we must also transfer masses and forces to body shapes. If we do it in some way then the construction — body with mass and force — loses the primitive nature. In order to work out a mathematical theory we have all the necessary: shapes with kinematic, kinetic and dynamic structures attributed by them. While the concept of a mechanical system has strict mathematical sense, the concept of body has only descriptive character, being a tribute of the very seminal tradition.

#### 5 GENERALIZATION OF THE MECHANICAL SYSTEM CONCEPT

The non-trivial nature of the mechanical system concept can be seen from the fact that we may postulate the following relation (see also [15])

$$(6) \quad \frac{d}{dt}\mathcal{P}(\Lambda_t) = \mathcal{F}(\Lambda_t) + \Delta\mathcal{P}(\Lambda_t) + \mathcal{R}(\Lambda_t), \quad \Lambda_t \subset \tilde{\Lambda}_t, \quad t \in \mathbf{T},$$

<sup>12</sup> From the above follows that the inertia appearance is due not to *inborn force of the matter, included in itself* but to mechanical systems belonging to the universe [21].

<sup>13</sup> *De facto*, these definitions follow from Plato’s idea on the existence of two worlds: the world of ideas (eidos) and the world of things, or forms. And then we have only a ‘photo’ of the body at each time [22] while the body itself is out of Plato’s cave.

where the signed field measure  $\Delta\mathcal{P}$  is so called *increment velocity* of the measure  $\mathcal{P}$ , the signed field measure  $\mathcal{R}$  is so called *constraint action*.

Below we shall assume that the measure  $\mathcal{R}$  is formed by internal and external constraints:  $\mathcal{R} = \mathcal{R}_{int} + \mathcal{R}_{ext}$ . Here the signed field measures  $\mathcal{R}_{int}$  and  $\mathcal{R}_{ext}$  have the values  $\mathcal{R}_{int}(\Lambda_t)$  and  $\mathcal{R}_{ext}(\Lambda_t)$  on the sets  $\Lambda_t$ , respectively.

#### APPENDIX 1: SLIDERS AND SCREWS

In mechanics there is mainly absent the understanding that motion of bodies and interaction between them can be described with the help of screws. It is considered as conventional that ‘*being very attractive representation of a system of forces and rigid body motions with the help motors and screws, nevertheless it has no essential practical value*’ [23] and that the screw calculus is not adapted for the description of continuum motion [24].

Contrary to this view, we shall demonstrate in this paper that screw calculus is rather useful and convenient tools in mechanics (see also [13] and author’s paper ‘*On Foundations of Newtonian Mechanics*’, arXiv:1012.3633).

**Sliders.** Due to the Great Soviet Encyclopedia, v. 5 (Moscow: Soviet Encyclopedia, 1971), ‘*screw calculus is the section of vector calculus in which operations over screws are studied. Here the screw is called the pair of vectors  $\{p, q\}$ , which is bounded at a point  $O$  and satisfied to conditions: at transition to a new point  $O'$  the vector  $p$  does not change, and the vector  $q$  is replaced with a vector  $q' = q - \overrightarrow{(O, O')} \times p$  where  $\times$  means cross-product. The notion of the screw is used in the mechanics (the resultant  $f$  of a force system and its main moment  $m$  form the screw  $\{f, m\}$ ), and also in geometry (in the theory of ruled surfaces)*’ (see also [14]).

The definition given above is not entirely satisfactory (the screw is not the pair of vectors  $\{p, q\}$ ), but it allows to focus on the following elements: three vectors  $p$ ,  $q$  and  $r$ , as well as a bi-linear antisymmetric (skew) map  $R(r, p) = r \times p$  and the relation defining the vector  $q'$  at the point  $O'$ . This observation permits us to avoid the geometric constructions, which are the starting point for the conventional screw theory (see, e.g., [24]), and to offer a simple (algebraic) version of the theory of screws.

Let  $\mathbf{P}$  and  $\mathbf{Q}$  be some (polar and pseudo, respectively) tensor fields on  $\mathbf{R}^n$ . A given point  $x \in \mathbf{R}^n$  let us define the translation vector  $r_{y,x} \in \mathbf{V}^n$  from a point  $y \in \mathbf{R}^n$  to  $x$ . Introduce a bi-linear map  $R(\mathbf{V}^n, \mathbf{P}) : \mathbf{V}^n \times \mathbf{P} \rightarrow \mathbf{Q}$  as well as the following relations:

$$(7) \quad p_y = p_x \in \mathbf{P}, \quad q_y = q_x - R(r_{y,x}, p_x) \in \mathbf{Q}, \quad y \in \mathbf{R}^n.$$

**Definition 5.** The element  $l^{p_x, q_x} = \{p_y \in \mathbf{P}, q_y \text{ and } R(r_{y,x}, p_x) \in \mathbf{Q}, \forall y \in \mathbf{R}^n\}$  is

called sliding tensor-function of  $x \in \mathbf{R}^n$  or, briefly, slider. The fields  $\mathbf{P}$  and  $\mathbf{Q}$  are usually called resultant and moment or torque ones, respectively [14].

A slider is called *homogeneous* if  $q_x = 0$ . In this case we shall use the notation  $l^{p_x}$ .

Denote some point of  $\mathbf{R}^n$  by  $z$ . Then from (7) follows that

$$p_z = p_x \in \mathbf{P}, \quad q_z = q_x - R(r_{x,z}, p_z) \in \mathbf{Q}$$

and<sup>14</sup>

$$p_y = p_z \in \mathbf{P}, \quad q_y = q_z - R(r_{y,z}, p_z) \in \mathbf{Q}, \quad \forall y \in \mathbf{R}^n.$$

For the purposes of computing we may introduce slider modifications

$$l^{p_x, q_x, wr} = \left\{ \begin{bmatrix} p_y \\ q_y \end{bmatrix}, p_y \in \mathbf{P}, q_y \text{ and } R(r_{y,x}, p_x) \in \mathbf{Q}, \forall y \in \mathbf{R}^n \right\} \text{ and}$$

$$l^{p_x, q_x, tw} = \left\{ \begin{bmatrix} q_y \\ p_y \end{bmatrix}, p_y \in \mathbf{P}, q_y \text{ and } R(r_{y,x}, p_x) \in \mathbf{Q}, \forall y \in \mathbf{R}^n \right\}$$

which are called *wrench* and *twist*, respectively. For any fixed  $y \in \mathbf{R}^n$ , the vectors  $\begin{bmatrix} p_y \\ q_y \end{bmatrix}$  and  $\begin{bmatrix} q_y \\ p_y \end{bmatrix}$  are called their *reductions* (at the *reduction point*  $y$ ).

**Screw measures (screws).** Below we shall use the notion of *signed field measure* being absolutely continuous w.r.t.  $\mu_n$  (see also Radon–Nikodym theorem).

**Definition 6.** Let  $\sigma_n$  be  $\sigma$ -algebra on the set  $\mathbf{R}^n$  and  $\mu_n(A)$  be Borel measure of (1)-type. Then the following signed field measure

$$(8) \quad \Pi(A) = \int \chi_A l^{p_x, q_x} \mu_n(dx), \quad p_x \in \mathbf{P}, \quad q_x \in \mathbf{Q}$$

is called screw one or, briefly, screw<sup>15</sup>.

We assume that all discussed below sliders are  $\mu_n$ -integrable.

**Remark 4.** From relation (8) we have

$$(9) \quad \int \chi_A q_y \mu_n(dx) = \int \chi_A q_x \mu_n(dx) - \int \chi_A R(r_{y,x}, f_x) \mu_n(dx), \quad y \in \mathbf{R}^n$$

<sup>14</sup> It means also that  $p_z$  and  $q_z$  given above can be used in order to restore sliders.

<sup>15</sup> The name *screw* will be used for surface integrals of sliders, too.

and

$$(10) \quad \int \chi_A q_y \mu_n(dx) = \int \chi_A q_z \mu_n(dx) - R(r_{y,z}, \int \chi_A f_x \mu_n(dx)), \quad y, z \in \mathbf{R}^n$$

where the tensor  $\int \chi_A q_x \mu_n(dx) \in \mathbf{Q}$  would have to be called intrinsic torque<sup>16</sup> of screw (8) while the tensor  $\int \chi_A R(r_{y,x}, f_x) \mu_n(dx) \in \mathbf{Q}$  is called resultant torque.

In the case where the fields  $\mathbf{P}$  and  $\mathbf{Q}$  may be considered as finite-dimensional vector spaces, the corresponding screws form a vector space, too — see also [14].

In addition, we assume that all screws used below have time-independent points of reduction.

**Sliders defined by alternants.** Concretize the slider notion in the case where  $n = 4$ ,  $\mathbf{P} = \mathbf{V}^4$ ,  $\mathbf{Q}$  is 2nd-order skew  $4 \times 4$ -pseudotensor field and  $R : \mathbf{V}^4 \times \mathbf{P} \rightarrow \mathbf{Q}$  is alternant:  $R(r_{y,x}, p_x) = p_x \otimes r_{y,x} - r_{y,x} \otimes p_x$  where  $\otimes$  means the tensor product [25].

**Definition 7.** Vectors  $\omega$  and  $\varpi$  are called dual to a given skew 2nd-order  $4 \times 4$ -tensor  $\Omega$  if there is the following representation

$$\Omega = \begin{bmatrix} I_{3 \times 3} & O_3 \\ O_3 & O_1 \end{bmatrix} \otimes \omega + \varpi \otimes e_4 - e_4 \otimes \varpi,$$

where  $I_{3 \times 3}$  is the unit tensor;  $O_1$  and  $O_3$  are null tensors (with corresponding orders), the vector  $\varpi$  is orthogonal with the vector  $e_4$  from the canonical basis  $\mathbf{e}_0 = \{e_1, e_2, e_3, e_4\}$ .

Let us show that these two vectors exist. Indeed, let introduce the following representations

$$\omega^0 = \text{col}\{\omega_1, \omega_2, \omega_3, 0\}, \quad \varpi^0 = \text{col}\{\varpi_1, \varpi_2, \varpi_3, 0\}, \quad e_4^0 = \text{col}\{0, 0, 0, 1\}$$

in the basis  $\mathbf{e}_0$ .

Then the tensor  $\Omega$  has the following representation (in the basis  $\mathbf{e}_0$ ) [25]

$$\Omega^0 = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & \varpi_1 \\ \omega_3 & 0 & -\omega_1 & \varpi_2 \\ -\omega_2 & \omega_1 & 0 & \varpi_3 \\ -\varpi_1 & -\varpi_2 & -\varpi_3 & 0 \end{bmatrix}.$$

Thus there exist the dual vectors  $\omega$  and  $\varpi$ , the latter being orthogonal with the vector  $e_4$  (note that in an other bases, one cannot guarantee that the coordinates  $\omega_4$

<sup>16</sup> There is no idea how the intrinsic torque can be defined if we might not know the tensor  $q_x$  at all points of  $A$ . Unlike the well-known property (10) — see also [14, 15], it is relation (9) that is a part of the screw definition.

and  $\varpi_4$  are null that is why the vectors  $\omega$  and  $\varpi$ , indeed, must be introduced as 4-dimensional).

It is plain that

$$R^0(r_{y,x}, p_x) = \begin{bmatrix} 0 & r_2p_1 - r_1p_2 & r_3p_1 - r_1p_3 & -r_1p_4r_4p_1 - r_1p_4 \\ r_1p_2 - r_2p_1 & 0 & r_3p_2 - r_2p_3 & r_4p_2 - r_2p_4 \\ r_1p_3 - r_3p_1 & r_2p_3 - r_3p_2 & 0 & r_4p_3 - r_3p_4 \\ r_1p_4 - r_4p_1 & r_2p_4 - r_4p_2 & r_3p_4 - r_4p_3 & 0 \end{bmatrix},$$

where  $p_i$  and  $r_i$  are the coordinates of  $p_x$  and  $r_{y,x} \in \mathbf{V}^4$  ( $i = \overline{1,4}$ ), respectively.

Hereinafter we shall denote any skew 2nd-order tensor with superscript  $\times$ , e.g.,  $q^\times$ .

Represent the tensors  $q_y^\times$  and  $q_x^\times \in \mathbf{Q}$  with the help of the matrices

$$q_y^{\times 0} = \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 & \bar{\alpha}_1 \\ \alpha_3 & 0 & -\alpha_1 & \bar{\alpha}_2 \\ -\alpha_2 & \alpha_1 & 0 & \bar{\alpha}_3 \\ -\bar{\alpha}_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 & 0 \end{bmatrix}, \quad q_x^{\times 0} = \begin{bmatrix} 0 & -\beta_3 & \beta_2 & \bar{\beta}_1 \\ \beta_3 & 0 & -\beta_1 & \bar{\beta}_2 \\ -\beta_2 & \beta_1 & 0 & \bar{\beta}_3 \\ -\bar{\beta}_1 & -\bar{\beta}_2 & -\bar{\beta}_3 & 0 \end{bmatrix}.$$

Let us define the vector  $q_y^0 = \text{col}\{\alpha_1, \alpha_2, \alpha_3, 0, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, 0\}$  and in the same way the vector  $q_x^0$  for the tensor  $q_x^\times$ . Introduce the following relations:

$$\begin{pmatrix} r_2p_1 - r_1p_2 \\ r_3p_1 - r_1p_3 \\ r_3p_2 - r_2p_3 \\ 0 \\ r_4p_1 - r_1p_4 \\ r_4p_2 - r_2p_4 \\ r_4p_3 - r_3p_4 \\ 0 \end{pmatrix} = R_{y,x}^0 \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}, \quad R_{y,x}^0 = \begin{bmatrix} 0 & -r_3 & r_2 & 0 \\ r_3 & 0 & -r_1 & 0 \\ -r_2 & r_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ r_4 & 0 & 0 & -r_1 \\ 0 & r_4 & 0 & -r_2 \\ 0 & 0 & r_4 & -r_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then due to (7)

$$p_y = p_x \in \mathbf{V}^4, \quad q_y = q_x - R_{y,x}p_x \in \mathbf{V}^8.$$

For  $n = 3$  any skew 2nd-order  $3 \times 3$ -tensor  $\Omega$  is defined as  $\omega^\times = I_{3 \times 3} \otimes \omega$ , where the vector  $\omega$  is dual to  $\Omega$ . In the canonical basis  $e_0$  we have  $\Omega^0 = \omega^{\times 0}$ , where

$$\omega^{\times 0} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \quad \omega^0 = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$$

Supposing that the tensor  $R(r_{y,x}, p_x)$  is an alternant we have

$$R^0(r_{y,x}, p_x) = \begin{bmatrix} 0 & r_2p_1 - r_1p_2 & r_3p_1 - r_1p_3 \\ r_1p_2 - r_2p_1 & 0 & r_3p_2 - r_2p_3 \\ r_1p_3 - r_3p_1 & r_2p_3 - r_3p_2 & 0 \end{bmatrix}$$

$$\begin{pmatrix} r_2 p_1 - r_1 p_2 \\ r_3 p_1 - r_1 p_3 \\ r_3 p_2 - r_2 p_3 \end{pmatrix} = R_{y,x}^0 \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \quad R_{y,x}^0 = r_{y,x}^{\times 0} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}$$

and<sup>17</sup>

$$p_y = p_x \in \mathbf{V}^3, \quad q_y = q_x - R_{y,x} p_x = q_x - r_{y,x} \times p_x \in \mathbf{V}^3.$$

From the above follows also that for  $n = 2$  we have

$$R^0(r_{y,x}, p_x) = \begin{bmatrix} 0 & r_2 p_1 - r_1 p_2 \\ r_1 p_2 - r_2 p_1 & 0 \end{bmatrix}$$

and

$$p_y = p_x \in \mathbf{V}^2, \quad q_y = q_x - R_{y,x} p_x \in \mathbf{V}^1,$$

where  $R_{y,x}^0 = [-r_2, \quad r_1]$ .

**Remark 5.** In the above we could use the 2nd-order tensor field  $\mathbf{P}$  instead of  $\mathbf{V}^4$ .

## APPENDIX 2: DERIVATIVES OF SOME INTEGRALS DEFINED OVER ACTUAL SHAPES $\Lambda_t$ OF MECHANICAL SYSTEMS

We shall assume that Stocks theorem is applicable and there are the following statements<sup>18</sup> (see also [22]).

**Lemma 1.** Let  $f(x, t)$  be a measurable function on some set  $G \in \mathbf{R}^3$ . If

$$\int \chi_V f(x, t) \mu_3(dx) = 0$$

for any subset  $V \in G$  and any  $t \in \mathbf{T}$ , then  $f(x, t) \equiv 0$  in  $G$ .

**Lemma 2.** For any measurable function  $f(x, t)$  on  $\Lambda_t \subset \tilde{\Lambda}_t$ ,  $t \in \mathbf{T}$ , we have

$$\begin{aligned} \frac{d}{dt} \int \chi_{\Lambda_t} f(x, t) \mu_3(dx) &= \int \chi_{\Lambda_t^{ac}} \left[ \frac{d}{dt} f(x, t) + f(x, t) \operatorname{div} v_x \right] \mu_{ac}(dx) + \\ &\quad \sum_k \frac{d}{dt} f(x_k, t) \mu_{pp}(x_k). \end{aligned}$$

**Lemma 3.** For any measurable function  $f(x, t)$  on  $\Lambda_t \subset \tilde{\Lambda}_t$ ,  $t \in \mathbf{T}$ , we have

$$\begin{aligned} \frac{d}{dt} \int \chi_{\Lambda_t} \rho_x f(x, t) \mu_3(dx) &= \int \chi_{\Lambda_t^{ac}} \left( \rho_x \frac{d}{dt} + \nu_x \right) f(x, t) \mu_{ac}(dx) + \\ &\quad \sum_k \left( \rho_k \frac{d}{dt} + \nu_k \right) f(x_k, t) \mu_{pp}(x_k). \end{aligned}$$

<sup>17</sup> Here we use the fact that the product  $r_{y,x}^{\times 0} p_x^0$  is the coordinate representation of the vector product  $r_{y,x} \times p_x$ .

<sup>18</sup> We assume that all integrals below have sense.

## 6 CONCLUSION

We formulated some axioms of rational mechanics. Further, the question of their implementation on the examples of main types of mechanical systems will be considered.

The author would be highly grateful with whoever would bring any element likely to be able to make progress the development, and thus the comprehension, of the paper. Any comments, reviews, critiques, or objections are kindly invited to be sent to the author by e-mail.

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