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MODIFIED METHOD OF SIMPLEST EQUATION FOR OBTAINING EXACT SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS: HISTORY, RECENT DEVELOPMENTS OF THE METHODOLOGY AND STUDIED CLASSES OF EQUATIONS

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ABSTRACT: We present a short review about the story of the methodology of the Method of simplest equation for obtaining exact particular solutions of nonlinear partial differential equations (NPDEs) with emphasis on the recent extension of a version of this methodology called Modified method of simplest equation (MMSE). The extension of MMSE allows the use of more than one simplest equation and makes the methodology capable to lead to solutions of nonlinear partial differential equations that are more complicated than a single solitary wave. We list the classes of equations that have been studied up to now by the MMSE and mention as an example one of the theorem proved in the course of the application of the methodology.

KEY WORDS: modified method of simplest equation, exact solutions, nonlinear partial differential equations, solitary waves.

1 INTRODUCTION

Differential equations occur frequently in the mathematical modeling of many problems from natural and social sciences, e.g., fluid mechanics, plasma physics, atmospheric and ocean sciences, chemistry, materials science, biology, economics, social dynamics, etc. [1–17]. Often the model equations are nonlinear partial differential equations and by means of the exact solutions of these equations one : (i) can understand complex nonlinear phenomena such as existence and change of different regimes of functioning of complex systems, transfer processes, etc. or (ii) can test computer programs for numerical simulations by comparison of the obtained numerical results to the corresponding exact solutions. Because of the above the exact solutions of NPDEs are studied very intensively [18–26].

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The research on the methodology for obtaining exact solutions of NPDEs started by search for transformations that transform the solved nonlinear partial differential equation to a linear differential equation. The Hopf-Cole transformation [27, 28] transforms the nonlinear Burgers equation to the linear heat equation and the numerous attempts for obtaining such transformations led to the *Method of Inverse Scattering Transform* [18, 21]. Almost parallel to this Hirota developed a direct method for obtaining of exact solutions of NPDEs – *Hirota method* [1, 29] that is based on bilinearization of the solved nonlinear partial differential equation by means of appropriate transformations. Truncated Painleve expansions may lead to many of these appropriate transformations [26, 30–33]. The line of research of interest for us below emerged when Kudryashov [34] studied the possibility for obtaining exact solutions of NPDEs by a truncated Painleve expansion where the truncation happens after the "constant term" (i.e., the constant term is kept in the expansion). Kudryashov formulated the *Method of Simplest Equation (MSE)* [35] based on determination of singularity order n of the solved NPDE and searching of particular solution of this equation as series containing powers of solutions of a simpler equation called *simplest equation*. Kudryashov and Loguinova [36] extended the methodology and applied it for obtaining traveling wave solutions of NPDEs. (for several examples see [37–43]).

Below we shall review the results of our contribution to the method of simplest equation from our first research on the use of the ordinary differential equation of Bernoulli as simplest equation [44] and by application of the method to ecology and population dynamics [45] where we have used the concept of the balance equation to our last version of the method based on more than one simplest equations and eventually to more than one balance equation. We note that the version called called *Modified Method of Simplest Equation – MMSE* [46, 47] based on determination of the kind of the simplest equation and truncation of the series of solutions of the simplest equation by means of application of a balance equation is equivalent of the *Method of simplest equation*. Up to now our contributions to the methodology and its application are connected to the *MMSE* [48–54]. We note especially the article [55] where we have extended the methodology of the *MMSE* to simplest equations of the class

$$\left(\frac{d^k g}{d\xi^k}\right)^l = \sum_{j=0}^{m} d_j g^j \,, \tag{1}$$

where $k = 1, \dots, l = 1, \dots,$ and m and d_j are parameters. The solution of Eq. (1) defines a special function that contains as particular cases, e.g.,: (i) trigonometric functions; (ii) hyperbolic functions; (iii) elliptic functions of Jacobi; (iv) elliptic function of Weierstrass.

2 MMSE and Its Version Based on a Single Simplest Equation – MMSE1 and on More Than One Simplest Equation – MMSEn

The methodology of MMSE1 is as follows. Let us consider a nonlinear partial differential equation

$$\mathcal{N}(u, \dots) = 0\,, \tag{2}$$

where $\mathcal{N}(u, \dots)$ depends on the function $u(x,t)$ and some of its derivatives participate in (u can be a function of more than 1 spatial coordinate). The steps of the methodology of the modified method of simplest equation for obtaining particular traveling wave solutions of a NPDE are:

1. By means of the traveling wave ansatz $\xi = \alpha x + betat$ (α and β are parameters) $u(x,t)$ is transformed to $u(\xi)$ which is represented as a function of other function f that is solution of some ordinary differential equation (the simplest equation). The form of the function $F(f)$ is can be different. One example is

$$u = \sum_{i=-M}^{N} \mu_n f^n \tag{3}$$

   μ is a parameter. In the most cases one uses $M = 0$.

2. The application of Eq. (3) to Eq. (2) transforms the left-hand side of this equation. Let the result of this transformation be a function that is a sum of terms where each term contains some function multiplied by a coefficient. This coefficient contains some of the parameters of the solved equation and some of the parameters of the solution. In the most cases a balance procedure must be applied in order to ensure that the above-mentioned relationships for the coefficients contain more than one term (e.g., if the result of the transformation is a polynomial then the balance procedure has to ensure that the coefficient of each term of the polynomial is a relationship that contains at least two terms). This balance procedure leads to one relationship among the parameters of the solved equation and parameters of the solution. The relationship is called *balance equation*.

3. We may obtain a nontrivial solution of Eq. (2) if all coefficients mentioned above are set to 0. This condition usually leads to a system of nonlinear algebraic equations for the coefficients of the solved nonlinear PDE and for the coefficients of the solution. Any nontrivial solution of this algebraic system leads to a solution the studied nonlinear partial differential equation. Usually

the above system of algebraic equations contains many equations that have to be solved with the help of a computer algebra system.

One can mention use of elements of MMSE1 in some of our first publications [56–60]. Our interest in the research on the exact solutions of nonlinear PDEs was revived when we have encountered such an equation as model of traveling waves in population dynamics [6]. We analyzed the equation

$$\frac{\partial Q}{\partial t} - D\frac{\partial^2 Q}{\partial x^2} = EQ^4 + FQ^3 + GQ^2 + HQ\,, \tag{4}$$

where D, E, F, G, H are parameters. In order to obtain the exact traveling wave solution of this equation we have used representation of the solution as finite series of power of solutions of Bernoulli equation and the concept of balance equation. The methodology used in [6] was used in [7] for the case of traveling waves in a system of two interacting populations. The next step was to search for exact solutions of more complicated equations. This was made for the class of equations [44]

$$\sum_{p=1}^{N_1} \alpha_p \frac{\partial^p Q}{\partial t^p} + \sum_{q=1}^{N_2} \beta_q \frac{\partial^q Q}{\partial x^q} + \sum_{m=1}^{M} \mu_m Q^m = 0\,, \tag{5}$$

where α_p, β_q and μ_m are parameters. Bernoulli and Riccati equations are used as simplest equations and the balance equation connected to the class of equations (5) is obtained. The developed theory is used for obtaining exact solutions of equations of Ablowitz and Zepetella, Huxley equation for gene propagation, equation connected to biological invasions, and more complicated equations.

Further application of the methodology was to the equations of the reaction-diffusion class [45]

$$\frac{\partial Q}{\partial t} + \frac{dD}{dQ}\left(\frac{\partial Q}{\partial x}\right)^2 + D(Q)\frac{\partial^2 Q}{\partial x^2} + F(Q) = 0 \tag{6}$$

and to reaction-telegraph class

$$\frac{\partial Q}{\partial t} - \alpha\frac{\partial^2 Q}{\partial t^2} - \beta\frac{\partial^2 Q}{\partial x^2} - \gamma\frac{dF}{qQ}\frac{\partial Q}{\partial t} - F(Q) = 0\,. \tag{7}$$

Above α, β, γ are parameters and D and F depend on the population density Q. As simplest equations are used the equations of Bernoulli and Riccati. As first result here the corresponding balance equations for (6) and (7) are obtained. Then numerous solutions are obtained for the two classes of studied equations (6) and (7).

Even more complicated class of equations was studied in [46], namely

$$\sum_{p=1}^{N_1} A_p(Q)\frac{\partial^p Q}{\partial t^p} + \sum_{r=2}^{N_2} B_r(Q)\Big(\frac{\partial Q}{\partial t}\Big)^r + \sum_{1=1}^{N_3} C_a(Q)\frac{\partial^s Q}{\partial x^s} + \sum_{u=2}^{N_4} D_u(Q)\Big(\frac{\partial Q}{\partial x}\Big)^u + F(Q) = 0\,, \tag{8}$$

which contains as particular cases the reaction-diffusion and the reaction-telegraph equations. As simplest equations authors in [46] use the equations of Bernoulli and Riccati. Any of these two simplest equations leads for corresponding balance equation for the class of equations (8).

The capabilities of the MMSE1 have been demonstrated in [47] for the class of equations

$$\sum_{i,j=0}^{m} A_{ij}(u)\frac{\partial^i u}{\partial t^i}\Big(\frac{\partial u}{\partial t}\Big)^j + \sum_{k,l=0}^{n} B_{ij}(u)\frac{\partial^k u}{\partial x^k}\Big(\frac{\partial u}{\partial x}\Big)^l = 0\,. \tag{9}$$

Equations of Bernoulli and Riccati and their particular case (extended tahn - function equation) are used as simplest equations and corresponding balance equations are obtained. Exact solutions of two particular cases of Eq. (9) (Swift–Hohenberg equation and generalized Rayleigh equation) are obtained.

The class of nonlinear PDEs that can be treated by the Modified method of simplest equation was extended in [48]. This class of equations is

$$\sum_{i_1=0}^{\overline{n}_1}\sum_{i_2=0}^{n_1^*}\sum_{j=0}^{n_2}\sum_{k_1=0}^{\overline{n}_3}\sum_{k_2=0}^{n_3^*}\sum_{l=0}^{n_4}\sum_{p_1=0}^{\overline{n}_5}\sum_{p_2=0}^{n_5^*}\sum_{q=0}^{n_6}\sum_{r_1=0}^{\overline{n}_7}\sum_{r_2=0}^{n_7^*}\sum_{s=0}^{n_8}\Big(\frac{\partial^{i_1+i_2}u}{\partial x^{i_1}\partial t^{i_2}}\Big)^j\Big(\frac{\partial^{k_1+k_2}u}{\partial x^{k_1}\partial t^{k_2}}\Big)^j \times \Big(\frac{\partial^{p_1+p_2}u}{\partial x^{p_1}\partial t^{p_2}}\Big)^j\Big(\frac{\partial^{r_1+r_2}u}{\partial x^{t_1}\partial t^{r_2}}\Big)^j A_{i_1,i_2,j,k_1,k_2,l,p_1,p_2,q,r_1,r_2,s}(u) = G(u)\,, \tag{10}$$

where it was assumed that

$$\frac{\partial^0 u}{\partial x^0} = \frac{\partial^0 u}{\partial t^0} = \frac{\partial^0 u}{\partial x^0 \partial t^0} = 0$$

and $G(u)$ and $A(u)$ are polynomials

$$G(u) = \sum_{\epsilon=0}^{\kappa} g_\epsilon u^\epsilon;$$

$$A_{i_1,i_2,j,k_1,k_2,l,p_1,p_2,q,r_1,r_2,s}(u) = \sum_{m=0}^{h_{i_1,i_2,j,k_1,k_2,l,p_1,p_2,q,r_1,r_2,s,m}} a_{i_1,i_2,j,k_1,k_2,l,p_1,p_2,q,r_1,r_2,s,m}u^m. \tag{11}$$

The MMSE1 was applied to Eqs.(10) and (11) and the balance equations are obtained for the case when the solution is searched as power series constructed by solutions of a simplest equation. As illustration of the methodology exact solutions of the b-equations and of the generalized Degasperis–Processi equations are obtained.

The role of the simplest equation in the methodology of the MMSE1 is discussed in [49]. As examples of simplest equations are discussed the equations of Bernoulli, Riccati and the differential equation for the elliptic functions of Jacobi. It is shown how the choice of the simplest equation influences the balance equation as well as the system of algebraic equations that is obtained by the application of the methodology of the modified method of simplest equation. It is shown that any nontrivial solution of certain system of nonlinear algebraic equations leads to exact traveling wave solution of a nonlinear partial differential equation. As examples for obtaining exact solutions on the basis of Riccati equation as simplest equation one discussed the nonlinear partial differential equations of Newell–Whitehead and FitzHugh–Nagumo. The general algorithm for obtaining differential equations that have exact traveling wave solutions constructed as power series of solutions of selected simplest equation is presented. The case of use of differential equations for the Jacobi elliptic functions as simplest equations was discussed further in [50]. Special attention to the exact traveling wave solutions of the nonlinear equations that are models for nonlinear water waves is given in [51] where exact traveling wave solutions are obtained for the extended Korteweg-de Vries equation

$$2\frac{\partial \eta}{\partial \tau} + 3\eta \frac{\partial \eta}{\partial \xi} + \frac{1}{3}\delta^2 \frac{\partial^3 \eta}{\partial \xi^3} - \frac{3}{4}\epsilon \eta^2 \frac{\partial \eta}{\partial \xi} = -\frac{1}{12}\epsilon \delta^2 \left(23 \frac{\partial \eta}{\partial \xi} + 10\eta \frac{\partial^3 \eta}{\partial \xi^3} \right) \tag{12}$$

(where ϵ and δ are small parameters called amplitude parameter and shallowness parameter) and for the generalized Camassa-Holm equation

$$\frac{\partial u}{\partial t} + p_1 \frac{\partial u}{\partial x} + \frac{p_4}{2}\frac{\partial}{\partial x} g(u) - p_2 \frac{\partial^3 u}{\partial^2 x \partial t} - 2p_3 \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - p_3 u \frac{\partial^3 u}{\partial x^3} = 0 \tag{13}$$

for the case when the equation of Bernoulli and Riccati are used for simplest equations and for several forms of the polynomial $g(u)$.

Several interesting theorems have been proved in connection with the application of the Modified method of simplest equation. In [53] the following theorem was proved:

Theorem: *Let $\mathcal{P}$ be a polynomial of the function $u(x,t)$ and its derivatives. $u(x,t)$ belongs to the differentiability class C^k, where k is the highest order of derivative participating in $\mathcal{P}$. $\mathcal{P}$ can contain some or all of the following parts: **(A)** polynomial of u; **(B)** monomials that contain derivatives of u with respect to x and/or products* *of such derivatives. Each such monomial can be multiplied by a polynomial of u;* ***(C)*** *monomials that contain derivatives of u with respect to t and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of u;* ***(D)*** *monomials that contain mixed derivatives of u with respect to x and t and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of u;* ***(E)*** *monomials that contain products of derivatives of u with respect to x and derivatives of u with respect to t. Each such monomial can be multiplied by a polynomial of u;* ***(F)*** *monomials that contain products of derivatives of u with respect to x and mixed derivatives of u with respect to x and t. Each such monomial can be multiplied by a polynomial of u;* ***(G)*** *monomials that contain products of derivatives of u with respect to t and mixed derivatives of u with respect to x and t. Each such monomial can be multiplied by a polynomial of u;* ***(H)*** *monomials that contain products of derivatives of u with respect to x, derivatives of u with respect to t and mixed derivatives of u with respect to x and t. Each such monomial can be multiplied by a polynomial of u.*

Let us consider the nonlinear partial differential equation

$$\mathcal{P} = 0. \tag{14}$$

We search for solutions of this equation of the kind $u(\xi) = u(\alpha x + \beta t) = u(\xi) = \gamma f(\xi)$, where γ is a parameter and $f(\xi)$ is solution of the simplest equation $f_\xi^2 = 4(f^2 - f^3)$. The substitution of this solution in Eq. (14) leads to a relationship $\mathcal{R}$ of the kind

$$\mathcal{R} = \sum_{i=0}^{N} C_i f(\xi)^i + f_\xi \left(\sum_{j=0}^{M} D_j f(\xi)^j \right), \tag{15}$$

where N and M are natural numbers depending on the form of the polynomial $\mathcal{P}$. The coefficients C_i and D_j depend on the parameters of Eq.(14) and on α, β and γ. Then each nontrivial solution of the nonlinear algebraic system

$$C_i = 0, i = 1, \dots, N; \quad D_j = 0, j = 1, \dots, M \tag{16}$$

leads to solitary wave solution of the nonlinear partial differential equation (14).

We note hat the simplest equation from the above theorem, namely $f_\xi^2 = 4(f^2 - f^3)$ has the solution $f(\xi) = 1/\cosh^2(\xi)$, where $\xi = \alpha x + \beta t$. In other words the theorem gives us an information about the solitary wave solutions of large class of nonlinear partial differential equations.

An extension of the methodology with respect to more general simplest equation was made in [55]. There was proposed the use of the simplest equation with polynomial nonlinearity (1). For the particular case $k = 2$ in (1) and for a large class of nonlinear PDE with polynomial nonlinearity the Modified method of simplest equation is formulated in terms of calculation of sequences of polynomials. The methodology is applied to the generalized Kortweg-de Vries equation and to second order Korteweg-de Vries equation (called also Olver equation).

In [54] the research presented in [53] was extended for the case of simplest equation $f_\xi^2 = n^2(f^2 - f^(2n+2)/n)$ which has the solution $f(\xi) = 1/\cosh^n(\xi)$, where $\xi = \alpha x + \beta t$.

The methodology of the modified method of simplest equation based on one simplest equation was applied in the last years for studying propagation of waves in artery with aneurism [61–64].

The last modification of the modified method of simplest equation is connected to the possibility of use of more than one simplest equation that was applied in [65]. This modification (MMSEn – Fig. 1) is as follows.

Let us consider a nonlinear partial differential equation (2) where $\mathcal{N}(u, \dots)$ depends on the function $u(x, ..., t)$ and some of its derivatives participate in (u can be a function of more than 1 spatial coordinate). The 7 steps of the methodology of the modified method of simplest equation are as follows.

1.) We perform a transformation

$$u(x, \dots, t) = G(F(x, \dots, t)), \tag{17}$$

where $G(F)$ is some function of another function F. In general $F(x, \dots, t)$ is a function of the spatial variables as well as of the time. The transformation $G(F)$ may be the Painleve expansion [1, 23, 34] or another transformation, e.g., $u(x,t) = 4\tan^{-1}[F(x,t)]$ for the case of the sine-Gordon equation or $u(x,t) = 4\tanh^{-1}[F(x,t)]$ for the case of sh-Gordon (Poisson–Boltzmann equation) (for applications of the last two transformations see, e.g. [56–60]). In many particular cases one may skip this step (then we have just $u(x, \dots, t) = F(x, \dots, t)$) but in numerous cases the step is necessary for obtaining a solution of the studied nonlinear PDE. The application of Eq. (17) to Eq. (2) leads to a nonlinear PDE for the function $F(x, \dots, t)$.

2.) The function $F(x, \dots, t)$ is represented as a function of other functions $f_1, \dots, f_N$ that are connected to solutions of some differential equations (these equations can be partial or ordinary differential equations) that are more simple than Eq. (2). We note that the possible values of N are $N = 1, 2, \dots$ (there may be infinite number of functions f too). The forms of the function $F(f_1, \dots, f_N)$ can be different. One example is

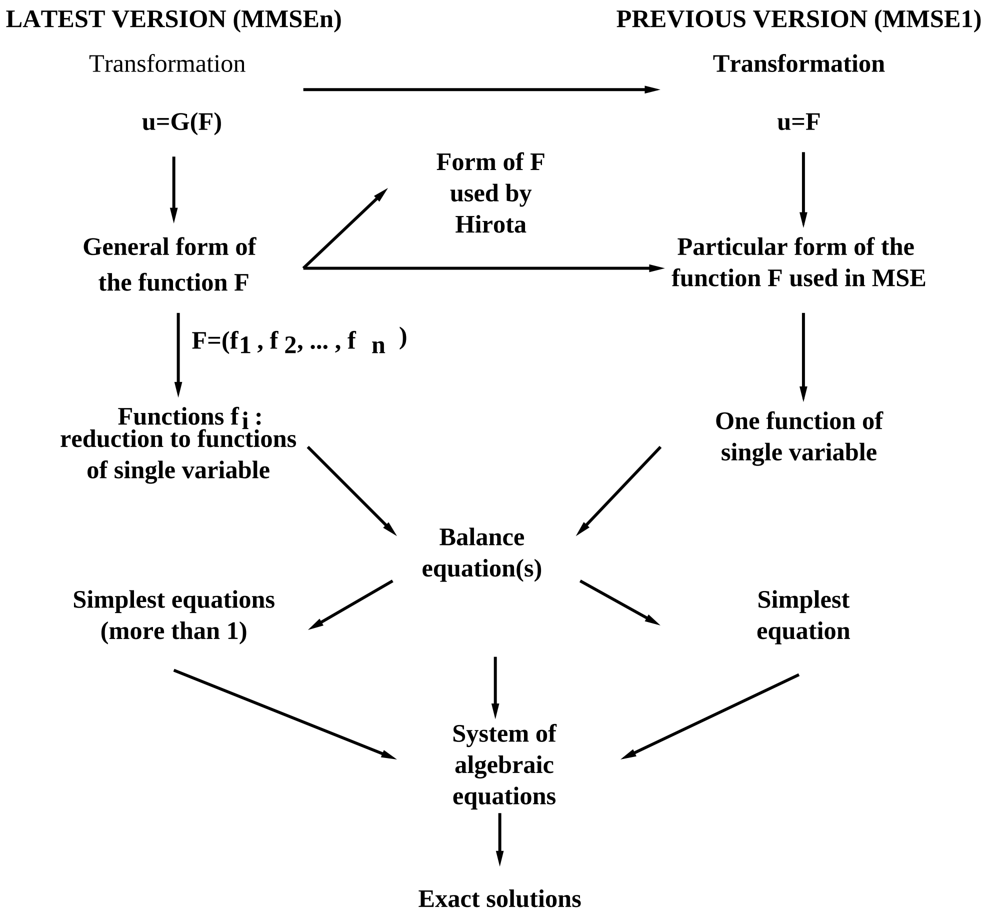

Fig. 1. Versions on the MMSE. MMSE1 is based on 1 simplest equation and 1 balance equation. MMSEn is based on n simplest equations ($n \geq 1$) and on one or more than one balance equation. Note that in the two versions of the method the relationship between the solution of the solved NPDE and the simplest equation(s) is different.

$$F = \alpha + \sum_{i_1=1}^{N} \beta_{i_1} f_{i_1} + \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \gamma_{i_1,i_2} f_{i_1} f_{i_2} + \dots + \sum_{i_1=1}^{N} \cdots \sum_{i_N=1}^{N} \sigma_{i_1,\dots,i_N} f_{i_1} \dots f_{i_N}\,, \tag{18}$$

where α, β_{i_1}, γ_{i_1,i_2}, $\sigma_{i_1,\dots,i_N}\dots$ are parameters. We shall use Eq. (18) below. Note that the relationship (18) contains as particular case the relationship used by Hirota [1]. The power series $\sum_{i=0}^{N} \mu_n f^n$ (where μ is a parameter) used in the previous versions of the methodology of the modified method of simplest equation are a particular case of the relationship (18) too.

3.) In general the functions $f_1, \dots, f_N$ are solutions of partial differential equations. By means of appropriate ansätze (e.g., traveling-wave ansätze such as $\xi = \hat{\alpha}x + \hat{\beta}t$; $\zeta = \hat{\gamma}x + \hat{\delta}t$, $\eta = \hat{\mu}y + \hat{\nu}t \dots$) the solved differential equations for $f_1, \dots, f_N$ may be reduced to differential equations E_l, containing derivatives of one or several functions

$$E_l\left[a(\xi), a_\xi, a_{\xi\xi}, \dots, b(\zeta), b_\zeta, b_{\zeta\zeta}, \dots\right] = 0; \quad l = 1, \dots, N\,. \tag{19}$$

In many cases (e.g, if the equations for the functions $f_1, \dots$ are ordinary differential equations) one may skip this step but the step may be necessary if the equations for $f_1, \dots$ are partial differential equations.

4.) We assume that the functions $a(\xi)$, $b(\zeta)$, etc., are functions of other functions, e.g., $v(\xi)$, $w(\zeta)$, etc., i.e.

$$a(\xi) = A[v(\xi)]; \;\; b(\zeta) = B[w(\zeta)]; \dots \tag{20}$$

Note that the kinds of the functions A, B, $\dots$ are not prescribed. Often one uses a finite-series relationship, e.g.,

$$a(\xi) = \sum_{\mu_1=-\nu_1}^{\nu_2} q_{\mu_1}[v(\xi)]^{\mu_1}; \quad b(\zeta) = \sum_{\mu_2=-\nu_3}^{\nu_4} r_{\mu_2}[w(\zeta)]^{\mu_2}, \dots, \tag{21}$$

where q_{μ_1}, r_{μ_2}, $\dots$ are coefficients. However other kinds of relationships may be used too.

5.) The functions $v(\xi)$, $w(\zeta)$, $\dots$ are solutions of simpler ordinary differential equations called *simplest equations*. For several years the methodology of the modified method of simplest equation was based on use of one simplest equation. This version of the methodology allows for the use of more than one simplest equation.

6.) The application of the steps 1.) – 5.) to Eq. (2) transforms the left-hand side of this equation. Let the result of this transformation be a function that is a sum of terms where each term contains some function multiplied by a coefficient. This coefficient contains some of the parameters of the solved equation and some of the parameters of the solution. In the most cases a balance procedure must be applied in order to ensure that the above-mentioned relationships for the coefficients contain more than one term (e.g., if the result of the transformation is a polynomial then the balance procedure has to ensure that the coefficient of each term of the polynomial is a relationship that contains at least two terms). This balance procedure may lead to one or more additional relationships among the parameters of the solved equation and parameters of the solution. The last relationships are called *balance equations*.

7.) We may obtain a nontrivial solution of Eq. (2) if all coefficients mentioned in Step 6.) are set to 0. This condition usually leads to a system of nonlinear algebraic equations for the coefficients of the solved nonlinear PDE and for the coefficients of the solution. Any nontrivial solution of this algebraic system leads to a solution the studied nonlinear partial differential equation. Usually the above system of algebraic equations contains many equations that have to be solved with the help of a computer algebra system.

The above steps of the methodology are generalization of what was used in [65] to obtain exact traveling wave solutions of the nonlinear Schrödinger equation. We note Eq.(18) that represents the relationship among the solution of the solved NPFE and the solutions of the corresponding simplest equations. This relationship from MMSEn contains as particular case the corresponding relationship from MMSE1 as well as the relationship used in [1].

3 Concluding Remarks

There exists also research on the relation between the MMSE1 and other methods fro obtaining exact solutions of NPDEs. Results of this research can be found in [66–69]. This research shows that MMSE1 is powerful method for obtaining exact particular solutions of many NPDEs. The extension to MMSEn makes the methodology even more useful as it becomes capable to obtain solutions that are more complicated in comparison to solitary waves (if such solutions do exist). The main characteristic of MMSEn is the possibility of use of more than one simplest equation. MMSEn includes also a possibility for a transformation connected to the searched solution. In such a way the possibility for use of a Painleve expansion or other transformations is presented in the methodology of MMSEn. This possibility in combination with the possibility of use of more than one simplest equation adds the capability for obtaining multisolitons by the discussed version of the methodology of the modified method of simplest equation. In addition we consider the relationship (18) that is used to connect the solution of the solved nonlinear partial differential equation to solutions of more simple differential equations. The discussed version of the methodology allows for the use of more than one balance equation too. All above opens new horizons for application of MMSEn for obtaining exact solutions of complicated nonlinear partial differential equations.

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