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EXACT TRAVELING WAVE SOLUTIONS OF A GENERALIZED KAWAHARA EQUATION

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ABSTRACT: We apply the modified method of simplest equation for obtaining exact solutions of nonlinear partial differential equations (PDEs) from the kind of the generalized Kawahara equation (gKE). The ordinary differential equation of Abel of first kind is used as the simplest equation and two exact solutions of the studied equation are obtained. These solutions are expressed by the special function discussed in [1]. Particular cases of one of the obtained solutions are visualized. For these cases the special function is reduced to elementary functions and the corresponding solutions describe a kink and a solitary wave.

KEY WORDS: Kawahara equation, exact analytical solution, modified method of simplest equation, kink and solitary waves.

1 INTRODUCTION

Numerous models of natural and social systems are based on differential equations [2–11]. Exact solutions of these model equations are important as they give us useful information for understanding the states and processes in the modeled systems. In addition the exact solutions can be used to test computer programs for numerical simulations. Often the model equations are nonlinear partial differential equations and usually these model equations have traveling–wave solutions that are studied very intensively [12–22]. Powerful methods exist for obtaining exact traveling–wave solutions of nonlinear partial differential equations, e.g. the method of inverse scattering transform or the method of Hirota [14, 23, 24]. These methods are appropriate for the case of integrable nonlinear PDEs. Other approaches for obtaining exact special solutions of nonlintegrable nonlinear PDEs have been developed in the recent years (for examples see [25–29]). Below we shall discuss the method of simplest equation

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and our focus will be on a version of the method called modified method of simplest equation [1, 30–34]. The method of simplest equation is based on a procedure analogous to the first step of the test for the Painleve property [35–37]. In the version of the method called modified method of simplest equation [38–40], [41–43] this procedure is substituted by the concept for the balance equations. We shall mention below the possibility of use of more than one simplest equation for obtaining exact analytical solutions of the studied nonlinear partial differential equation [44, 45]. In addition the modified method of simplest equation may lead to more than one balance equation. The modified method of simplest equation is already applied to obtain exact traveling wave solutions of the generalized Kuramoto–Sivashinsky equation, the reaction-diffusion equation, the reaction-telegraph equation [33], the generalized Swift–Hohenberg equation, the generalized Rayleigh equation [34], the generalized Fisher equation, the generalized Huxley equation [38], the generalized Degasperis–Procesi equation and b-equation [40], the extended Korteweg-de Vries equation [41–43, 46–50], etc.

2 Methodology and Problem Formulation

We are going to apply the modified method of simplest equation. The last version of the methodology used by our research group is based on the possibility of use of more than one simplest equation (see [51] for application of the method based on two simplest equations). The steps of the methodology are as follows:

1. By means of appropriate ansätze (e.g, one or several traveling-wave ansätze such as $\xi = \alpha x + \beta t$; $\zeta = \gamma x + \delta t, \dots$) the studied nonlinear PDEs becomes a differential equation DE, containing derivatives of one or several functions

$$DE\,[a(\xi), a_\xi, a_{\xi\xi}, \dots, b(\zeta), b_\zeta, b_{\zeta\zeta}, \dots] = 0\,. \tag{1}$$

2. The functions $a(\xi)$, $b(\zeta)$, etc., are searched as some function of another functions, e.g., $g(\xi)$, $\phi(\zeta)$, etc., i.e.

$$a(\xi) = G[g(\xi)]; \;\; b(\zeta) = F[\phi(\zeta)]; \dots \tag{2}$$

3. Note that the kind of the functions F, $G, \dots$ is not prescribed. One uses a finite-series relationship, e.g.,

$$a(\xi) = \sum_{\mu_1=-\nu_1}^{\nu_2} q_{\mu_1}[g(\xi)]^{\mu_1}; \quad b(\zeta) = \sum_{\mu_2=-\nu_3}^{\nu_4} r_{\mu_2}[\phi(\zeta)]^{\mu_2}, \dots, \tag{3}$$

where $q_{\mu_1}, r_{\mu_2}, \dots$ are coefficients. The functions $g(\xi)$, $\phi(\zeta)$ are solutions of simpler ordinary differential equations called simplest equations. The idea to use of more than one simplest equation can be traced back at least to the early articles of Martinov and Vitanov [52–54].

4. After the substitution Eq. (2) in Eq. (1) we obtain a polynomial containing $g(\xi)$, $\phi(\zeta)$, ... Then a balance procedure is applied. It has to ensure that all the coefficients of the obtained polynomial of $g(\xi)$ and $\phi(\zeta)$ contain more than one term. This procedure leads to one or more balance equations relating some of the parameters of the solved equation and some of the parameters of the solution. Note that the coefficients of all powers of the polynomials have to be balanced (and not only the coefficient of the largest power). This kind of balance may require more than one balance equation.

5. Eqs. (2) represent a candidate for solution of Eq. (1) if all coefficients of the obtained polynomial of are equal to 0. This condition leads to a system of nonlinear algebraic equations for the coefficients of the solved nonlinear PDE and for the coefficients of the solution. Any nontrivial solution of this algebraic system leads to a solution of the studied nonlinear PDE.

We shall apply a particular case of the above methodology (based on one simplest equation [55]) to the generalized Kawahara equation [56–59]

$$\gamma u_t + \alpha u^n u_x + \beta u_{xxx} - \delta u_{xxxxx} = 0\,, \tag{4}$$

where γ, α, β and δ are parameters and n is a non-negative natural number. For the cases $n = 1$ and $n = 2$ Eq. (4) is reduced to the Kawahara equation (KE) and the modified Kawahara equation (mKE). The Kawahara equation has numerous applications, e.g., for modeling shallow water waves on water surface or under the ice cover [60,61]. We introduce the ansatz of a traveling-wave variable, i.e. $\zeta = \theta x + \nu t$ in Eq. (4) and we search for a solution of Eq. (4) of the kind:

$$u = u(\zeta) = \sum_{q=0}^{r} a_q g^q; \qquad \frac{dg}{d\zeta} = \sum_{m=0}^{l} b_m g^m\,, \tag{5}$$

where a_q and b_m are parameters, and $g(\zeta)$ is a solution of simplest equation. Next step is to obtain the balance equation by balancing the powers of g arising from the terms in Eq. (4). The balance equation is $qn = 4m - 4$. In this study we shall use the ordinary differential equation of Abel of first kind as a simplest equation

$$\frac{dg}{d\zeta} = b_0 + b_1 g + b_2 g^2 + b_3 g^3\,. \tag{6}$$

3 Several Solutions of Equations of the Class of Eq. (4)

3.1 Case $n = 1$

Let us firstly consider the case $n = 1$. Then Eq.(4) becomes

$$\gamma u_t + \alpha u u_x + \beta u_{xxx} - \delta u_{xxxxx} = 0 \,. \tag{7}$$

In this case the balance equation becomes $q = 4m - 4$. For the case of use of Abel equation of first kind as a simplest equation the solution of Eq. (7) can be presented as

$$u(\zeta) = a_0 + a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4 + a_5 g^5 + a_6 g^6 + a_7 g^7 + a_8 g^8 \,. \tag{8}$$

The solution of the differential equation of Abel can be written as

$$\left(\frac{dg}{d\zeta}\right)^2 = c_0 + c_1 g + c_2 g^2 + c_3 g^3 + c_4 g^4 + c_5 g^5 + c_6 g^6 \,, \tag{9}$$

where

$$\begin{aligned} c_0 &= b_0^2; \quad c_1 = 2b_0 b_1; \quad c_2 = 2b_0 b_2 + b_1^2; \quad c_3 = 2b_0 b_3 + 2b_1 b_2; \\ c_4 &= 2b_1 b_3 + b_2^2; \quad c_5 = 2b_2 b_3; \quad c_6 = b_3^2 \end{aligned} \tag{10}$$

and its solution is given by the special function $V_{c_0,c_1,c_2,c_3,c_4,c_5,c_6}(\zeta; 1, 2, 6)$ [1]. In general the special function $V_{c_0,c_1;\dots,c_m}(\zeta; k, l, m)$ is the solution of the ordinary differential equation

$$\left(\frac{d^k g}{d\zeta^k}\right)^l = \sum_{j=0}^{m} c_j g^j \,. \tag{11}$$

Then the solution of Eq. (9) is

$$g(\zeta) = V_{b_0^2, 2b_0 b_1, 2b_0 b_2 + b_1^2, 2b_0 b_3 + 2b_1 b_2, 2b_1 b_3 + b_2^2, 2b_2 b_3, b_3^2}(\zeta; 1, 2, 6) \,. \tag{12}$$

The relationships among coefficients of the solution and coefficients of the model are obtained by solving a system of eighteen algebraic equations. This system is very large and we shall not write it here. The process of solution of the algebraic system leads to the following relationships (we note that below the division in all ratios is made and thus they are written as real numbers):

$$\begin{aligned} a_0 = \Big(& -27 \times 10^2 \theta^5 \delta^2 b_2^{10} - 15 \times 10^2 \theta^5 \delta^2 b_2^8 M_2 - 260 \theta b_3^4 b_2^2 \beta^2 + \\ & + 38.5 \times 10^2 \theta^5 \delta^2 b_2^6 M_2^2 + 0.08 \theta^2 b_2^4 M_1^3 \delta^{-1} - 3.5 \times 10^{-6} \theta b_2^2 M_1^4 \delta^{-2} \end{aligned} \tag{13}$$

$$+ 2\theta^3 b_2^4 M_1^2 M_2 - 0.03\theta^3 M_1^2 b_2^2 M_2^2 - 35\times 10^2\theta^3 b_3^2\delta b_2^4\beta M_2$$
$$+ 70\theta^2 b_3^2 b_2^4\beta M_1 + 6.5\times 10^{-6} M_1^5\delta^{-3} - 13\times 10^2\theta^3 b_3^2\delta b_2^6\beta$$
$$+ 225\theta^4\delta b_2^8 M_1 + 6\times 10^2\theta^4\delta b_2^6 M_1 M_2 - 42.5\times 10^2\theta^3 b_2^6 M_1^2$$
$$+ 1.6\theta^5\delta^2 b_2^4 M_2^3 + 2b_3^4 M_1\gamma\nu\theta^{-1} - 42.5 b_3^4\delta b_2^2\gamma - 1.3\theta b_3^2 b_2^2\beta M_1^2\delta^{-1}$$
$$+ 115\theta^4\delta b_2^4 M_1 M_2^2 + 6.5\theta^2 b_3^2 b_2^2\beta M_1 M_2 + 1.6 b_3^4 M_1\beta^2\delta^{-1}$$
$$+ 3\times 10^{-3} b_3^2 M_1^3\beta\delta^{-2} - 0.06\theta^2 b_2^2 M_1^3 M_2\delta^{-1}\Big)/\theta b_3^4\delta\alpha M_3\,.$$

We obtain for the parameter a_1

$$a_1 = \Big(-0.3\gamma\nu\delta - 7.8\times 10^{-5}\theta b_3^2\beta M_1^2\delta^{-1} + 1.2\theta^3 b_3^2\beta\delta b_2^4 \tag{14}$$
$$+ 0.3\theta^3 b_3^2\beta b_2^2\delta M_2 - 1.7\times 10^{-8}\theta M_1^4\delta^{-2} + 1.8\times 10^{-4}\theta^2 b_2^2 M_1^3\delta^{-1}$$
$$- 10^{-4}\theta^3 b_2^4 M_1^2 - 0.2\theta b_3^4\beta^2 + 0.8\theta^5\delta^2 b_2^8 - 0.2\theta^2 b_3^2\beta b_2^2 M_1$$
$$+ 0.1\theta^4\delta b_2^4 M_1 M_2 + 15\theta^5\delta^2 b_2^4 M_2^2 + 0.04\theta^4\delta b_2^6 M_1^2$$
$$+ 8\times 10^{-4}\theta^3 b_2^2 M_1^2 M_2 + 0.08\theta^4\delta M_1 b_2^2 M_2^2 + 15\theta^5\delta^2 b_2^6 M_2$$
$$+ \Big(-7.4\times 10^3\theta^5\delta^2 b_2^{10} - 4\times 10^2\theta^5\delta^2 b_2^8 M_2 - 70\theta b_3^4 b_2^2\beta^2$$
$$+ 10^4\theta^5\delta^2 b_2^6 M_2^2 + 0.02\theta^2 b_2^4 M_1^3\delta^{-1} - 9.6\times 10^{-5}\theta b_2^2 M_1^4\delta^2$$
$$+ 0.5\theta^3 b_2^4 M_1^2 M_2 - 8\times 10^{-3}\theta^3 M_1^2 b_2^2 M_2^2 - 952\theta^3 b_3^2\delta b_2^4\beta M_2$$
$$+ 19\theta^2 b_3^2 b_2^4\beta M_1 + 1.7\times 10^{-7} M_1^5\delta^3 + 16.7\theta^4\delta b2^6 M_1 M_2$$
$$- 369\theta^3 b_3^2\delta b_2^6\beta + 61\theta^4\delta b_2^8 M_1 + 0.18\theta^3 b_2^6 M_1^2 - 4.8\theta^5\delta^2 b_2^4 M_2^3$$
$$+ 2.7 b_3^4 M_1\gamma\nu\theta^{-1} - 10^{-4} b_3^4\delta b_2^2\gamma\nu^{-2} - 0.3\theta b_3^2 b_2^2\beta M_1^2\delta^{-1}$$
$$+ 31.7\theta^4\delta b_2^4 M_1 M_2^2 + 1.7\theta^2 b_3^2 b_2^2\beta M_1 M_2 + 0.2 b_3^4 M_1\beta^2\delta^{-1}$$
$$+ 8\times 10^{-4} b_3^2 M_1^3\beta\delta^{-2} - 2.10^{-3}\theta^2 b_2^2 M_1^3 M_2\delta^{-1}\Big)/M_3\Big)/\theta b_3^3\delta\alpha b_2\,;$$

$$a_2 = \theta^2\Big(109.6\theta^2\delta b_2^4 M_2 - 0.6 b_3^2\beta M_1\delta^{-1}\theta^{-1} - 0.4 b_3^2\beta b_2^2 - 1.6\theta b_2^4 M_1 \tag{15}$$
$$+ 22\theta^2\delta b_2^6 + 6.1\theta^2\delta b_2^2 M_2^2 + 0.1 b_2^2 M_1^2\delta^{-1} + 10^{-3} M_1^3\theta^{-1}\delta^{-2}$$
$$+ 8.4\theta b_2^2 M_1 M_2\Big)/b_3^2\alpha\,;$$

$$a_3 = \theta^2\Big(1.1\theta M_1 b_2 M_2 - 11.5\beta b_3^2 b_2 + 0.1 b_2 M_1^2\delta^{-1} \tag{16}$$
$$- 3.1\theta^2\delta b_2^5 + 120\theta^2\delta b_2^3 M_2 + 3\theta b_2^3 M_1\Big)/b_3\alpha\,;$$

$$a_4 = \theta^2\Big(2.6\times10^3\theta^2\delta b_2^2M_2 + 2.9\times10^3\theta b_2^2M_1 - 86\beta b_3^2 + 3.2\theta^2\delta b_2^4 + 1.3\delta^{-1}M_1^2\Big)/\alpha\,; \tag{17}$$

$$a_5 = \theta^4\delta b_3\Big(1.6\times10^3 b_2M_2 + 6\times10^2 b_2M_1\delta^{-1}\theta^{-1} + 2.3\times10^4 b_2^3\Big)/\alpha\,; \tag{18}$$

$$a_6 = \theta^4 b_3^2\delta\Big(6.6\times10^4 b_2^2 + 3.4\times10^2 M_1\delta^{-1}\theta^{-1}\Big)/\alpha\,; \tag{19}$$

$$a_7 = 7\times10^4\delta\theta^4 b_3^3 b_2/\alpha\,; \tag{20}$$

$$a_8 = 2.6\times10^4\delta\theta^4 b_3^4/\alpha\,; \tag{21}$$

$$b_0 = 4.10^{-2} b_2M_2/b_3^2\,;\quad b_1 = 6\times10^{-3}M_1/\delta b_3\theta\,, \tag{22}$$

where

$$M_1 = 52\delta b_2^2\theta + 6\sqrt{13\beta\delta}\,b_3\,, \tag{23}$$

$$M_2 = 2b_2^2 + 0,06M_1/\delta\theta\,, \tag{24}$$

$$M_3 = -4358b_2^2 + 10,31M_1/\delta\theta \tag{25}$$

and b_2 and b_3 are free parameters. Thus the solution of Eq. (7) becomes

$$\begin{aligned} u(\zeta) = {} & a_0 + a_1V_{b_0^2,2b_0b_1,2b_0b_2+b_1^2,2b_0b_3+2b_1b_2,2b_1b_3+b_2^2,2b_2b_3,b_3^2}(\zeta;1,2,6) \\ & + a_2V^2_{b_0^2,2b_0b_1,2b_0b_2+b_1^2,2b_0b_3+2b_1b_2,2b_1b_3+b_2^2,2b_2b_3,b_3^2}(\zeta;1,2,6) \\ & + a_3V^3_{b_0^2,2b_0b_1,2b_0b_2+b_1^2,2b_0b_3+2b_1b_2,2b_1b_3+b_2^2,2b_2b_3,b_3^2}(\zeta;1,2,6) \\ & + a_4V^4_{b_0^2,2b_0b_1,2b_0b_2+b_1^2,2b_0b_3+2b_1b_2,2b_1b_3+b_2^2,2b_2b_3,b_3^2}(\zeta;1,2,6) \\ & + a_5V^5_{b_0^2,2b_0b_1,2b_0b_2+b_1^2,2b_0b_3+2b_1b_2,2b_1b_3+b_2^2,2b_2b_3,b_3^2}(\zeta;1,2,6)+ \\ & + a_6V^6_{b_0^2,2b_0b_1,2b_0b_2+b_1^2,2b_0b_3+2b_1b_2,2b_1b_3+b_2^2,2b_2b_3,b_3^2}(\zeta;1,2,6)+ \\ & + a_7V^7_{b_0^2,2b_0b_1,2b_0b_2+b_1^2,2b_0b_3+2b_1b_2,2b_1b_3+b_2^2,2b_2b_3,b_3^2}(\zeta;1,2,6) \\ & + a_8V^8_{b_0^2,2b_0b_1,2b_0b_2+b_1^2,2b_0b_3+2b_1b_2,2b_1b_3+b_2^2,2b_2b_3,b_3^2}(\zeta;1,2,6) \end{aligned} \tag{26}$$

We can obtain a particular case of this solution expressed by elementary functions for the special case $b_0 = \dfrac{b_2}{3b_3}\Big(b_1 - \dfrac{2b_2^2}{9b_3}\Big)$. Then the Abel equation has the following solution:

$$\begin{aligned} g(\zeta) &= V_{b_0^2,2b_0b_1,2b_0b_2+b_1^2,2b_0b_3+2b_1b_2,2b_1b_3+b_2^2,2b_2b_3,b_3^2}(\zeta;1,2,6) \\ &= \frac{\exp\left[\left(b_1 - \frac{b_2^2}{3b_3}\right)\zeta\right]}{\sqrt{C^* - b_3\exp\left[2\left(b_1 - \frac{b_2^2}{3b_3}\right)\zeta\right]}} - \frac{b_2}{3b_3} \end{aligned} \tag{27}$$

where C^* is a constant of integration. The substitution of Eqs. (13)–(25) in Eq.(26) leads to a particular case of the solution (26) expressed by elementary functions.

3.2 Case $n = 2$

Let us now consider the case $n = 2$. Then Eq. (4) is reduced to the mKE in the form

$$\gamma u_t + \alpha u^2 u_x + \beta u_{xxx} - \delta u_{xxxxx} = 0\,. \tag{28}$$

In this case the balance equation becomes $q = 2m - 2$ and we search for the solution of Eq. (28) in the form

$$u(\zeta) = a_0 + a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4\,, \tag{29}$$

where $g(\zeta)$ is a solution of the Abel differential equation of first kind (Eq. (9)). The relationships among the coefficients of the solution and the coefficients of the model are derived by solving a system of fourteen algebraic equations. The process of solution of the algebraic system leads to the following relationships:

$$a_0 = \sqrt{10\alpha\delta}\Big(- 0.03\theta^2 b_2^4 \delta L_2 + 0.03\theta^2 b_2^4 \delta L_2^2 + 0.01\theta^2 \delta b_2^4 + 0.03\theta^2 b_2^2 \delta (3b_2^2 L_2 - 2b_2^2) - 0.002\sqrt{60L_4} b_2^4 \theta^2 \beta L_1^{-2}\Big)/\sqrt{L_4} b_2^4 \theta^2 \alpha\delta\,, \tag{30}$$

$$a_1 = \sqrt{10\alpha\delta}\,\sqrt[4]{2.4} L_1 \Big(2.4 b_2^3 L_2 - 0.3 b_2^3 + 0.5 b2 (3b_2^2 L_2 - 2b2^2)\Big)/\sqrt[4]{L_4} b_2^2 \alpha\,, \tag{31}$$

$$a_2 = 8\theta^2\sqrt{10\alpha\delta}(b_2^2 + b_2^2 L_2)/\alpha\,, \tag{32}$$

$$a_3 = 7.1 b_2^3 \theta^2 \sqrt[4]{540 L_4}\sqrt{10\alpha\delta}/\alpha L_1\,, \tag{33}$$

$$a_4 = 3.6\theta^4\sqrt{10\alpha\delta}\,\sqrt[4]{L_4}/\alpha L_1\,, \tag{34}$$

$$b_0 = 0.004\sqrt{60} L_1^2 (3b_2^2 L_2 - 2b_2^2)/b_2^3 \theta^2 \sqrt{L_4}\,, \tag{35}$$

$$b_1 = 0.25\sqrt[4]{2,4} L_1 L_2/\theta\sqrt{L_4}\,, \tag{36}$$

$$b_3 = 0.2\sqrt[4]{540}\sqrt{L_4}/L_1\,, \tag{37}$$

where

$$L_1 = \beta^2\theta + 10\gamma\nu\delta\,, \qquad L_2 = -2 + \frac{1}{3}\sqrt{51}\,, \tag{38}$$

$$L_3 = -943 + 132\sqrt{51}\,, \quad L_4 = \theta\delta^2 L_3 L_1^3 \tag{39}$$

and b_2 is a free parameter. Thus the solution of Eq. (28) becomes

$$u(\zeta) = a_0 + a_1 V_{b_0^2, 2b_0 b_1, 2b_0 b_2 + b_1^2, 2b_0 b_3 + 2b_1 b_2, 2b_1 b_3 + b_2^2, 2b_2 b_3, b_3^2}(\zeta; 1, 2, 6) \tag{40}$$

$$+ a_2 V^2_{b_0^2, 2b_0b_1, 2b_0b_2+b_1^2, 2b_0b_3+2b_1b_2, 2b_1b_3+b_2^2, 2b_2b_3, b_3^2}(\zeta; 1, 2, 6)$$
$$+ a_3 V^3_{b_0^2, 2b_0b_1, 2b_0b_2+b_1^2, 2b_0b_3+2b_1b_2, 2b_1b_3+b_2^2, 2b_2b_3, b_3^2}(\zeta; 1, 2, 6)$$
$$+ a_4 V^4_{b_0^2, 2b_0b_1, 2b_0b_2+b_1^2, 2b_0b_3+2b_1b_2, 2b_1b_3+b_2^2, 2b_2b_3, b_3^2}(\zeta; 1, 2, 6)$$

For the special case $b_0 = \frac{b_2}{3b_3}\left(b_1 - \frac{2b_2^2}{9b_3}\right)$ the Abel equation has the solution given by (27) and then the solution given by Eq. (40) can be expressed by elementary functions.

4 Discussion

Figures 1 and 2 show two kinds of waves obtained when the solution (26) of Kawahara equation can be expressed by elementary functions (27). These two kinds of waves are: solitary wave (Fig. 1) and kink wave (Fig. 2) The waves are obtained by

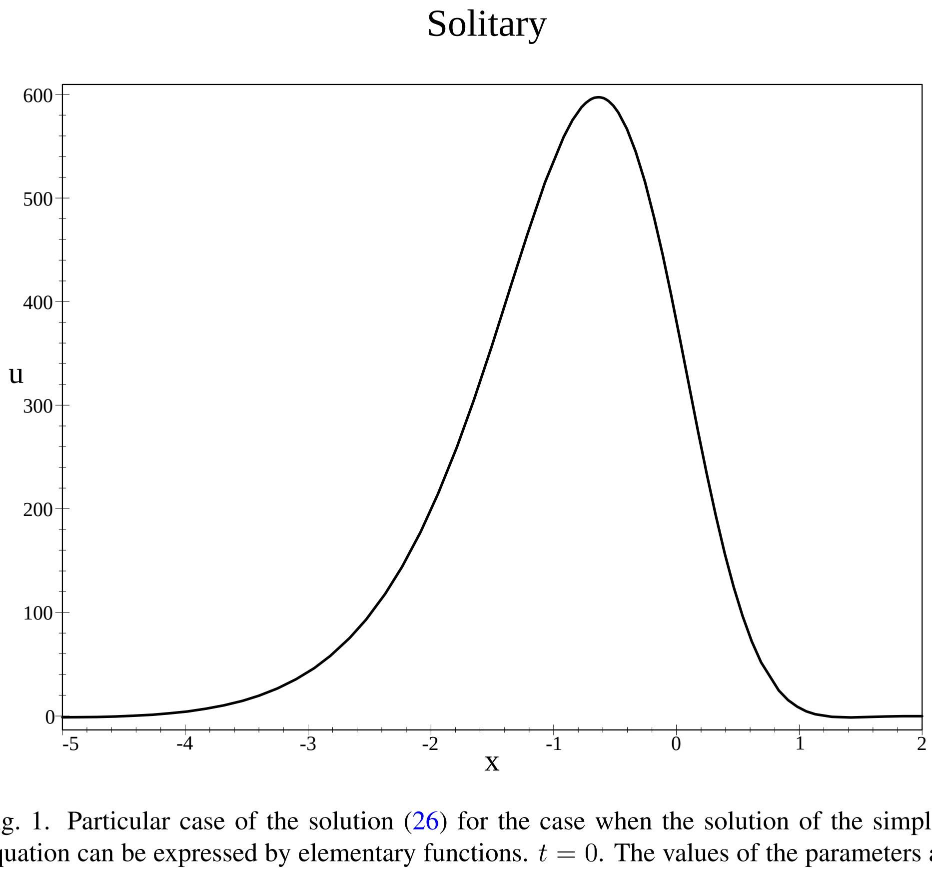

Fig. 1. Particular case of the solution (26) for the case when the solution of the simplest equation can be expressed by elementary functions. $t = 0$. The values of the parameters are $\theta = 1$, $\delta = 2$, $\beta = 2$, $\alpha = 1$, $C^* = 1$, $\gamma = 1$, $\nu = 1$. $b_2 = 1.5$, $b_3 = 0.666173$.

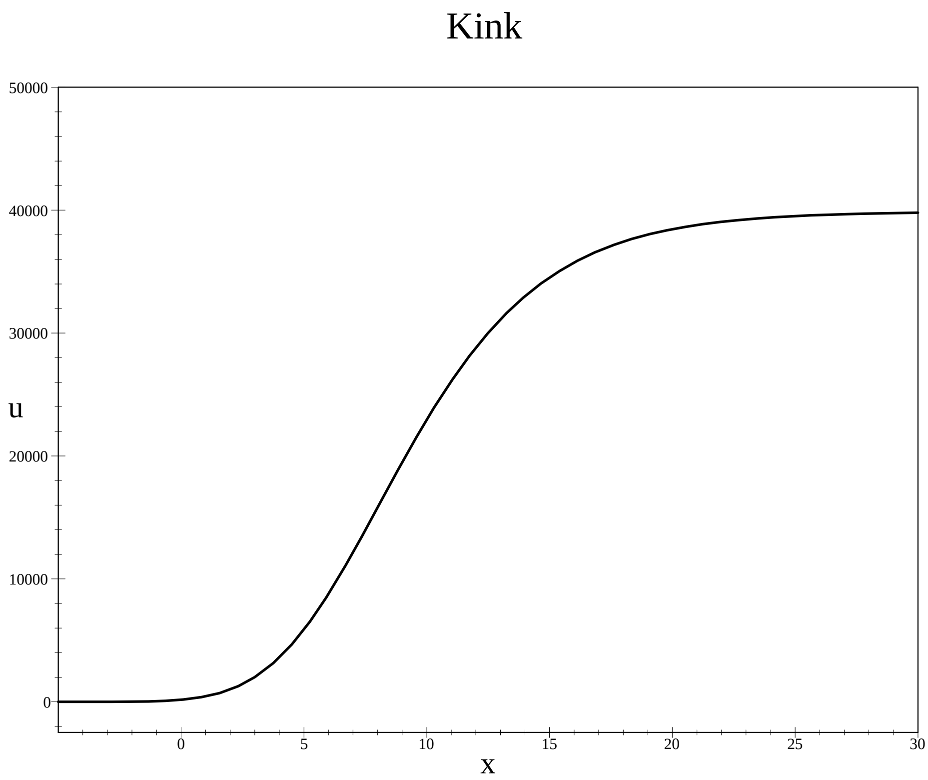

Fig. 2. Particular case of the solution (26) for the case when the solution of the simplest equation can be expressed by elementary functions. $t = 0$. The values of the parameters are $\theta = 1$, $\delta = 2$, $\beta = 2$, $\alpha = 1$, $C^* = 1$, $\gamma = 1$, $\nu = 1$. $b_2 = 1$, $b_3 = 2.347300$.

keeping constant values of the parameters $\alpha, \beta, \gamma, \delta$ of the Kawahara equation. The parameters of the Abel equation b_2 and b_3 are varied and this variation leads to change of the kinds of the solution from a solitary wave to a kink.

5 Concluding Remarks

In this paper we obtain exact traveling wave solutions of two nonlinear PDEs from the class (4) of generalized Kawahara equation. We present only particular analysis of the obtained solution of the generalized Kawahara equation, i.e., we show some forms of the solution only for the case $n = 1$ and for the values of parameters for which the solutions are a solitary wave or a kink. Extensive analysis of the obtained solutions can be made by means of variation of larger number of parameters and by means of numerical investigation of the solutions for general case when they are expressed by the special function V. In addition an application of the obtained solutions for modeling shallow water waves in different systems can be made. All these topics will be discussed elsewhere.

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