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INTERDISCIPLINARY TOPICS

SOME FEATURES OF DOUBLE PENDULUM SYSTEM – NUMERICAL AND SIMULATION STUDY

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ABSTRACT: An interesting double pendulum system is investigated in this work. The Lagrangian and the Hamiltonian equations were derived. Furthermore, we obtained the equations of motion describing the system. Numerical and simulation solution was obtained for the equations of motion for some specific initial conditions. MATLAB was used to obtain the numerical solution, especially some build in codes like the ode113.

KEY WORDS: Lagrangian-Hamiltonian mechanics, Euler-Lagrange equations, Hamilton's equations, double pendulum, numerical calculation, Runge-Kutta method, ode113, odeset.

1 INTRODUCTION

In nature there are many forms of motion (periodic, quasiperiodic, and chaotic). Some systems undergo only one form of these motions, while other systems can have more than one form of motion. Among these systems, arise two important physical systems: pendulums, and oscillators. In classical mechanics texts, one can find many interesting examples about pendulums, oscillators, and other systems for example one can refer to the classical texts [1–6].

Systems consisting of pendulum or oscillators can be studied by two methods: The first method depends on vector concepts (i.e. forces acting on the system). This method is mainly known as Newtonian mechanics, while the second method depends on scalar concepts (i.e. kinetic and potential energies of the system), and it is known as Lagrangian and Hamiltonian mechanics. In classical mechanics, the second approach is preferred and used widely to study different physical systems although they look complicated when trying to solve those using Newtonian mechanics. For more details about the two approaches those who are interested can refer to classical mechanics texts, and we recommend readers to refer to references [3–5].

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As one can see from literature, applying Lagrangian and Hamiltonian mechanics to any system leads in general to differential equations of second order known as Euler- Lagrange equations (equations of motions). In general, we aim to solve these differential equations analytically and for simple systems, solution is easy to obtain by simple integration with applying some given initial conditions. For this reason, one has to refer to a course on ordinary or even partial differential equations, for example one can refer to the following references [7–10].

On the other hand, when dealing with more complicated systems differentials equations obtained cannot be solved analytically so easily. Therefore, we have to look for numerical solutions. Many numerical techniques has been established in literature such as Euler method, Runge-Kutta method, finite difference method, and many other methods [11, 12]. MATLAB is a powerful and a robust language for technical computing and math modeling. Matlab plays an important tool in applying these methods to solve differential equations numerically. It has a built in codes that enables us to solve such differential equations as ordinary differential equation (ODE) solvers as ode45, ode113 and ode23s, dsolve and odeset function to create and customize option structure for input arguments to ode solvers [11, 13]. In addition to solve differential equations using built-in routines in MATLAB; it has graphics commands that illustrate and presents the results as plot and plot3 [11].

In this research, we study a very important system called double pendulum such that shown in Fig. 1 below. In spite of the simplicity of this system, it is of great interest for physicists, mathematicians, and engineers as it exhibits rich dynamic behavior. Robot arms, cranes segments, and control systems are of great similarity to double pendulum behavior [14, 15]. The motion and the trajectory of robot arms and segments are described by generalized coordinates and affected by angles between segments. The amazing and most interest is the reverse of this problem i.e. to obtain a specific and desired trajectory or movement by determining the angles between segments [16].

The double pendulum is of great important rich system, exhibiting conventional linear multi degree of freedom system behavior for small angles, but displaying chaotic behavior for large angle and very sensitive and affected by initial conditions. A chaotic system is a deterministic and very sensitive system to the initial conditions: the “butterfly” effect.

The rest of this research is arranged as follows: In Section 2 we derive the Lagrangian and the Hamiltonian of the system and as a result Euler-Lagrange equations in addition to Hamilton’s equations of motion were then derived, while in Section 3 we reviewed the basic tools used in our simulation and numerical method. Finally, we end our research by results and discussion part in Section 4.

2 THE PHYSICAL SYSTEM

2.1 GENERAL CASE

A double pendulum is a system consisting of two simple pendulums connected to each other as shown in Fig. 1 below. As it is clear, the system consists of two masses m_1 and m_2 attached by light rods with lengths (L_1, L_2).

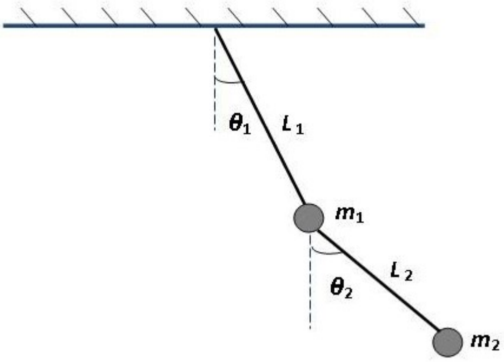

Fig. 1: Double pendulum system.

Here in this example we need just two generalized coordinates to describe the motion (θ_1, θ_2), and this is due to the fact that the length of each rod remains constant. Therefore, the kinetic and potential energies of the system respectively read:

$$T = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2\dot{\theta}_2^2 + m_2L_1L_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2\,, \tag{1}$$

$$V = -(m_1 + m_2)gL_1\cos\theta_1 - m_2gL_2\cos\theta_2\,. \tag{2}$$

Consequently, the classical Lagrangian $L = T - V$ takes the form

$$L = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2\dot{\theta}_2^2 + m_2L_1L_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + (m_1 + m_2)gL_1\cos\theta_1 + m_2gL_2\cos\theta_2\,. \tag{3}$$

Now, on applying the relation $\frac{\partial L}{\partial \theta_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} = 0$ to Eq. (3) for both (θ_1, θ_2), one get respectively the following two Classical Euler-Largange equations (CELE):

$$(m_1 + m_2)L_1\ddot{\theta}_1 + m_2L_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2L_2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + (m_1 + m_2)g\sin\theta_1 = 0\,, \tag{4}$$

$$L_2\ddot{\theta}_2 + L_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) - L_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + g\sin\theta_2 = 0\,. \tag{5}$$

Dividing Eq. (4) by $(m_1 + m_2)L_1$, and Eq. (5) by L_2, we can rewrite them as

$$\ddot{\theta}_1 + \beta_1(\theta_1, \theta_2)\ddot{\theta}_2 = F_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)\,, \tag{6}$$

$$\ddot{\theta}_2 + \beta_2(\theta_1, \theta_2)\ddot{\theta}_1 = F_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)\,, \tag{7}$$

where

$$\beta_1(\theta_1, \theta_2) = \frac{L_2}{L_1}\Big(\frac{m_2}{m_1 + m_2}\Big)\cos(\theta_1 - \theta_2)\,, \tag{8}$$

$$\beta_2(\theta_1, \theta_2) = \frac{L_1}{L_2}\cos(\theta_1 - \theta_2)\,, \tag{9}$$

$$F_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = -\frac{L_2}{L_1}\Big(\frac{m_2}{m_1 + m_2}\Big)\dot{\theta}_2^2\sin(\theta_1 - \theta_2) - \frac{g}{L_1}\sin\theta_1\,, \tag{10}$$

$$F_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{L_1}{L_2}\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - \frac{g}{L_2}\sin\theta_2\,. \tag{11}$$

Equations (6) and (7) are two second-order differential equations. Below we are going to convert them to four first-order differential equations.

It is interesting to note that F_1 does not depend on $\dot{\theta}_1$ and F_2 does not depend on $\dot{\theta}_2$. We can combine Eqs. (6) and (7) into a single equation using matrix notation as

$$A\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix}) = \begin{pmatrix}1 & \beta_1\\ \beta_2 & 1\end{pmatrix}\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} = \begin{pmatrix}F_1\\ F_2\end{pmatrix}\,, \tag{12}$$

where A is 2×2 square matrix depending on both θ_1 and θ_2. Its inverse can be obtained as

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix}1 & -\beta_1\\ -\beta_2 & 1\end{pmatrix} = \frac{1}{1 - \beta_1\beta_2}\begin{pmatrix}1 & -\beta_1\\ -\beta_2 & 1\end{pmatrix}\,. \tag{13}$$

From Eqs. (12) and (13), we obtain

$$\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} = A^{-1}\begin{pmatrix}F_1\\ F_2\end{pmatrix} = \frac{1}{1 - \beta_1\beta_2}\begin{pmatrix}F_1 - \beta_1 F_2\\ -\beta_2 F_1 + F_2\end{pmatrix}\,. \tag{14}$$

Finally, letting $\omega_1 = \dot{\theta}_1$ and $\omega_2 = \dot{\theta}_2$, the equations of motion of the double pendulum system may be written as a system of coupled first-order differential equations as

$$\frac{d}{dt}\begin{pmatrix}\theta_1\\ \theta_2\\ \omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}\omega_1\\ \omega_2\\ H_1(\theta_1, \theta_2, \omega_1, \omega_2)\\ H_2(\theta_1, \theta_2, \omega_1, \omega_2)\end{pmatrix}\,, \tag{15}$$

where $H_1 = \dfrac{F_1 - \beta_1 F_2}{1 - \beta_1\beta_2}$, $H_2 = \dfrac{-\beta_2 F_1 + F_2}{1 - \beta_1\beta_2}$.

The last equation can be solved numerically using a Runge-Kutta method. In the section below we are going to obtain numerical solution for this equation.

2.2 SPECIAL CASE: SMALL OSCILLATIONS

The Euler-Lagrange equations obtained in the previous section are coupled, nonlinear second order ODEs. When the angles θ_1 and θ_2 are small, the oscillations of the double pendulum system near zero equilibrium can be described by a linear system of equations. For small angles θ_1 and θ_2, one can use Maclaurin series to expand the trigonometric functions in Eq. (3) as

$$\cos\theta_1 \approx 1 - \frac{\theta_1^2}{2}\,, \quad \cos\theta_2 \approx 1 - \frac{\theta_2^2}{2}\,, \text{ and } \cos(\theta_1 - \theta_2) \approx 1 - \frac{(\theta_1-\theta_2)^2}{2} \approx 1\,.$$

Here we have taken into account that the term with $\cos(\theta_1 - \theta_2)$ contains the product of small quantities $\dot\theta_1\dot\theta_2$ and has the second order of smallness. Therefore, we can leave only the linear term in the cosine expansion.

Substituting this in the original Lagrangian (3) and considering that the potential energy is defined up to a constant, we obtain the quadratic Lagrangian for the double pendulum in the form

$$\text{(16)} \quad L = \frac{1}{2}(m_1+m_2)L_1^2\dot\theta_1^2 + \frac{1}{2}m_2L_2^2\dot\theta_2^2 + m_2L_1L_2\dot\theta_1\dot\theta_2 + \frac{1}{2}(m_1+m_2)gL_1\theta_1^2 + \frac{m_2}{2}gL_2\theta_2^2\,.$$

As a result we obtain the following two CELEs for θ_1 and θ_2, respectively:

$$\text{(17)} \quad \Big(1 + \frac{m_1}{m_2}\Big)L_1\ddot\theta_1 + L_2\ddot\theta_2 + \Big(1 + \frac{m_1}{m_2}\Big)g\theta_1 = 0\,,$$

$$\text{(18)} \quad L_2\ddot\theta_2 + L_1\ddot\theta_1 + g\theta_2 = 0\,.$$

Again as before, letting $\omega_1 = \dot\theta_1$, $\omega_2 = \dot\theta_2$ and following the procedures from Eq. (12) to Eq. (15) the equations of motion of the double pendulum system (for small oscillations) can be written as a system of coupled first-order differential equations as

$$\text{(19)} \quad \frac{d}{dt}\begin{pmatrix}\theta_1\\ \theta_2\\ \omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}\omega_1\\ \omega_2\\ h_1(\theta_1,\theta_2)\\ h_2(\theta_1,\theta_2)\end{pmatrix}\,,$$

where

(20a) $$h_1(\theta_1,\theta_2) = \frac{f_1 - \gamma_1 f_2}{1-\gamma_1\gamma_2}\ ,$$

(20b) $$h_2(\theta_1,\theta_2) = \frac{-\gamma_2 f_1 + f_2}{1-\gamma_1\gamma_2}\ , ,$$

(20c) $$\gamma_1 = \frac{L_2}{1 - L_1\left(1+\frac{m_1}{m_2}\right)}\ ,$$

(20d) $$\gamma_2 = \frac{L_1}{L_2}\ ,$$

(20e) $$f_1(\theta_1) = -\frac{g}{L_1}\theta_1\,,$$

(20f) $$f_2(\theta_2) = -\frac{g}{L_2}\theta_2\,.$$

3 SIMULATION METHOD AND NUMERICAL SOLUTION

MATLAB is a high-performance language for technical computing. It can be used for data analysis, math computations, data simulation and modeling, algorithm development and data visualization [17]. It integrates programming concepts with visualization and computation environment. MATLAB framework has user-defined procedures and built in functions that can be used to solve common problems. Examples include procedures for symbolic calculations, signal processing, and control systems. Many mathematical functions are imported in symbolic calculations, such as $\cos(x)$, $\sin(x)$, $\tan(x)$, $\ln(x)$ are evaluated by the functions `cos`, `sin`, `tan`, and `log` respectively in MATLAB [17].

In addition to mathematical built in functions; MATLAB has various graphic tools. plotting and simulating a computational data set is possible with few commands, such as `plot`, `plot3`, `pie`, `stem`, `surf`, are used to create two-dimensional plots, create three-dimensional plots, draw pie charts, draws sequential points with vertical lines and markers in the plane, and plot a spherical surface respectively [11, 13].

A sequence of MATLAB statements can be mixed segments of built in and user defined function calls, matrix creation, variable declaration, conditional statements and loops. Script and function files contains a sequence of MATLAB commands and are often called M-files. M-files can accept arguments as inputs and can produce one or more outputs when M-files are executed as shown in Fig. 2 [13].

In the next section, we solve equation (15) numerically for some specific initial conditions.

Fig. 2: Function diagram in MATLAB; it receives inputs and produce processed outputs.

4 RESULTS AND DISCUSSION

The numerical solution for equation (15) is obtained using ode113, odeset, and the if statement structure through script and function files [12,18]. Several plots have been drawn using `plot`, `plot3`, `stem3` commands with particular initial conditions. We begin by small angular displacements in Case 1, 2 and then we increase the angular displacements in Case 3, 4 to investigate the effect of increasing the initial angular displacements on the motion, while the initial angular velocity was kept constant in all cases to focus only on the effect of increasing the initial angular displacements.

The behavior of the angular displacement (i.e. θ_1 and θ_2), and the angular velocities (i.e. $\dot{\theta}_1$ and $\dot{\theta}_2$) against time has been presented in Figs. 3–6 for different initial conditions. While in Figs. 7, and 8 we show the behavior of the angular velocities (i.e. $\dot{\theta}_1$ and $\dot{\theta}_2$) versus angular displacement (i.e. θ_1 and θ_2). Figure 7 belongs to the initial condition specified in Case 1, and in Fig. 8 belongs to the initial condition specified in Case 4. On the other hand, in Figs. 9 and 10 we show the variation of θ_2 versus θ_1 for small and large initial condition angles as specified in Case 1 and Case 4 respectively.

Case 1: We consider the following initial conditions ($\theta_1 = 0.1$ rad, $\dot{\theta}_1 = 0$, $\theta_2 = 0.2$ rad, and $\dot{\theta}_2 = 0$).

In Fig. 3a, when both θ_1 and θ_2 are small (< 0.2 rad), the two masses have almost simple harmonic motion with nearly the same period, but the second mass has greater amplitude in $\dot{\theta}_2$, and this agrees with both equations (17) and (18) in which the second derivative of angular displacements is proportional to the angular displacements. The change of both $\dot{\theta}_1$, $\dot{\theta}_2$ is mostly periodic as shown in Fig. 3b with nearly same period with more regularity and higher amplitude in $\dot{\theta}_2$, and because $\dot{\theta}_1$, $\dot{\theta}_2$ are related to kinetic energy and as the amplitude of $\dot{\theta}_2$ is larger this means that the kinetic energy of the second mass is larger.

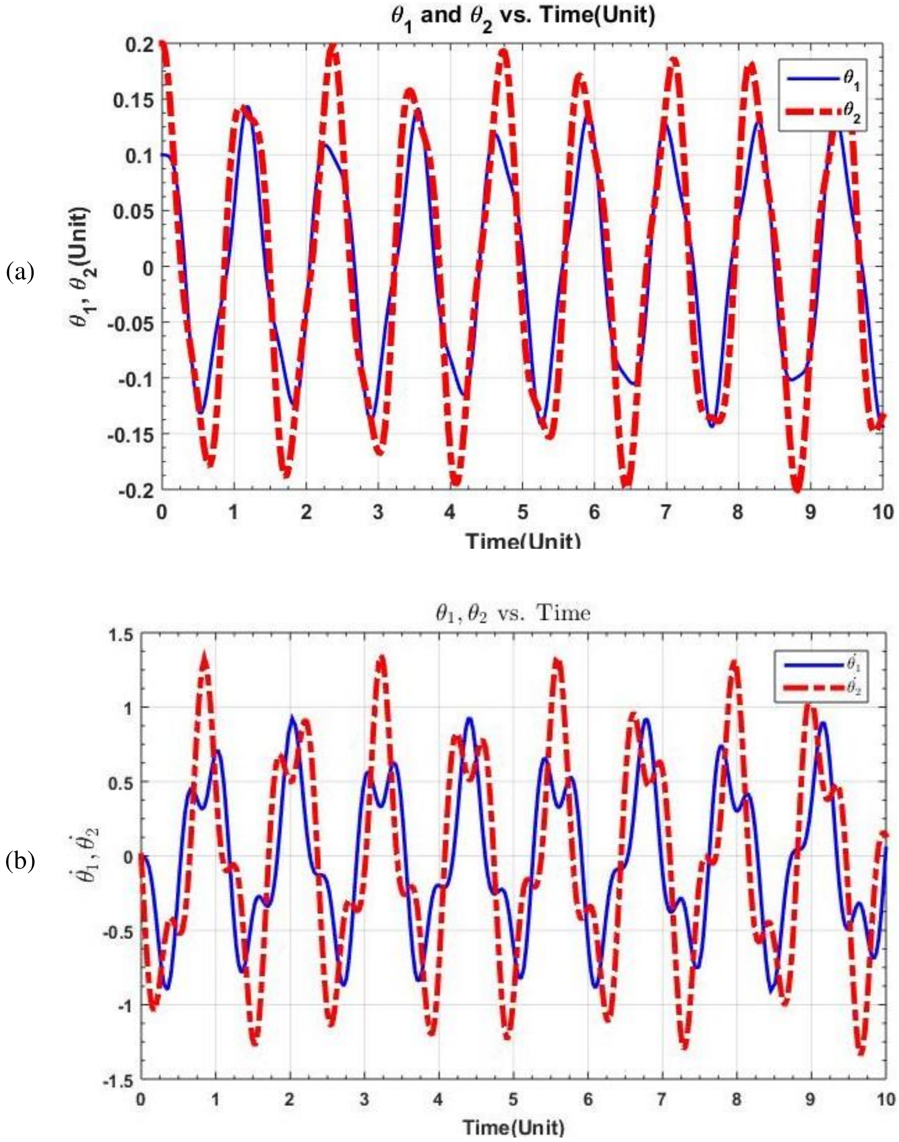

Fig. 3: (a) Shows the dynamical behavior of θ_1 and θ_2 against time, while in (b): the dynamical behavior of $\dot{\theta}_1$ and $\dot{\theta}_2$ against time (Case 1).

Case 2: The following initial conditions have been considered ($\theta_1 = 0.1$ rad, $\dot{\theta}_1 = 0$, $\theta_2 = 0.8$ rad, and $\dot{\theta}_2 = 0$).

In Case 2 as we increase θ_2 to 0.8 rad and keeping θ_1 small of 0.1 rad as the previous case. We can still notice periodic motion for both masses but with two different amplitudes for each mass as in Fig. 4a. A phase shift of nearly $\pi/2$ of the change of both $\dot{\theta}_1$, $\dot{\theta}_2$ with time giving rise of a pulsating behavior as shown in Fig. 4b

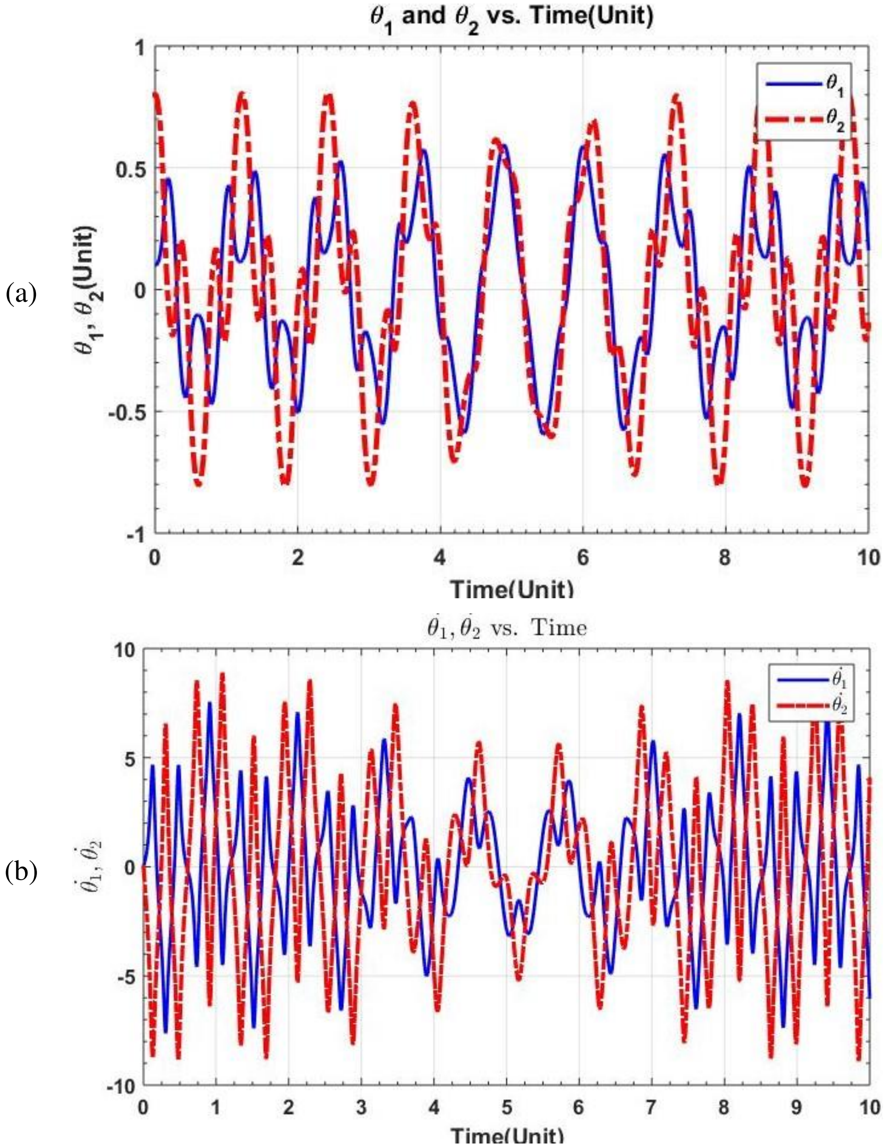

Fig. 4: (a) The dynamical behavior of θ_1 and θ_2 against time, while in (b): the dynamical behavior of $\dot{\theta}_1$ and $\dot{\theta}_2$ against time (Case 2).

Case 3: The initial conditions are ($\theta_1 = 1.5$ rad, $\dot{\theta}_1 = 0$, $\theta_2 = 1.5$ rad, and $\dot{\theta}_2 = 0$).

By increasing both θ_1 and θ_2 up to 1.5 rad for both in Case 3 the motion is getting more complicated. It looks like each mass has a large amplitude periodic motion as an envelope and small periodic oscillations inside the main cycle as shown in Fig. 5a. This behavior is being repeated in Case 4 (Fig. 6a) after the increasing of θ_2 to 3.1 rad keeping $\theta_1 = 1.5$ rad but with more small oscillations inside the main virtual envelop in comparison with Case 3 shown in Fig. 5a.

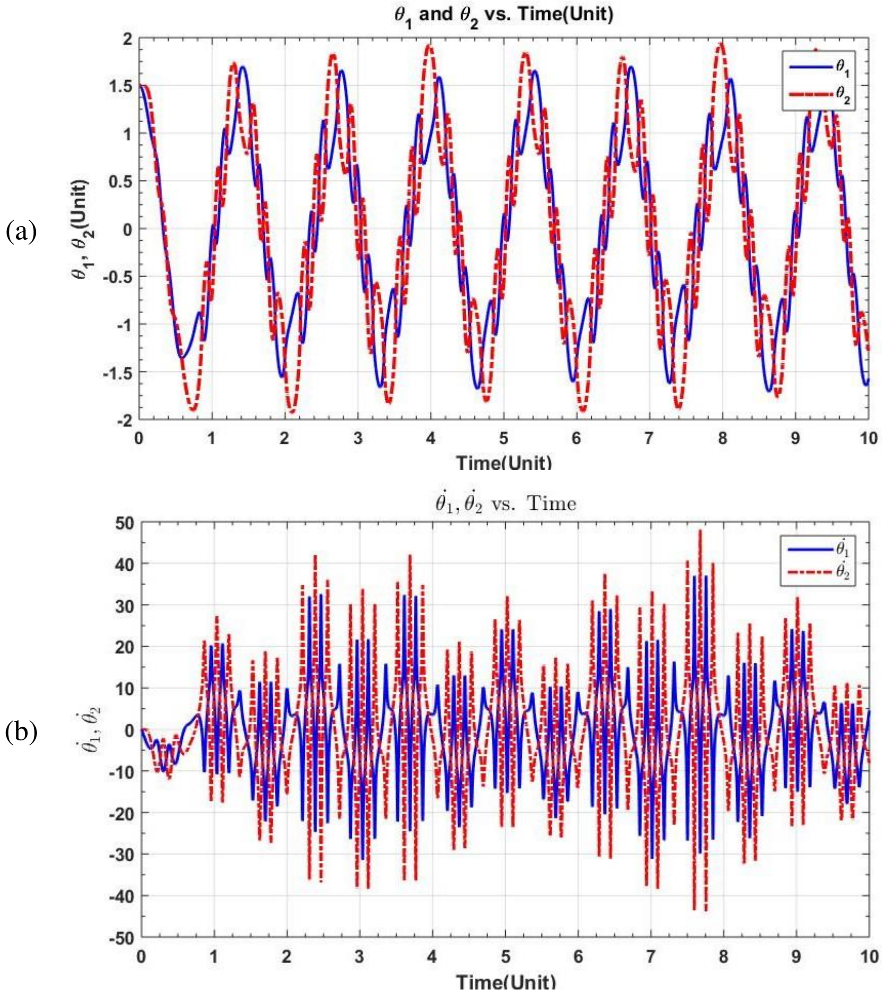

Fig. 5: (a) The dynamical behavior of θ_1 and θ_2 against time, while in (b): the dynamical behavior of $\dot{\theta}_1$ and $\dot{\theta}_2$ against time (Case 3).

Case 4: The initial condition for this case are ($\theta_1 = 1.5$ rad, $\dot{\theta}_1 = 0$, $\theta_2 = 3.1$ rad and $\dot{\theta}_2 = 0$).

The change of $\dot{\theta}_1$, $\dot{\theta}_2$ with time in both Cases 3 and 4 is still out of phase of nearly $\pi/2$ and with pulsating behavior but with sudden increase in pulse amplitude in Case 4.

The variation of both $\dot{\theta}_1$, $\dot{\theta}_2$ with the angular displacements θ_1 and θ_2 is being illustrated for both small and big angles initial conditions in Figs. 7 and 8.

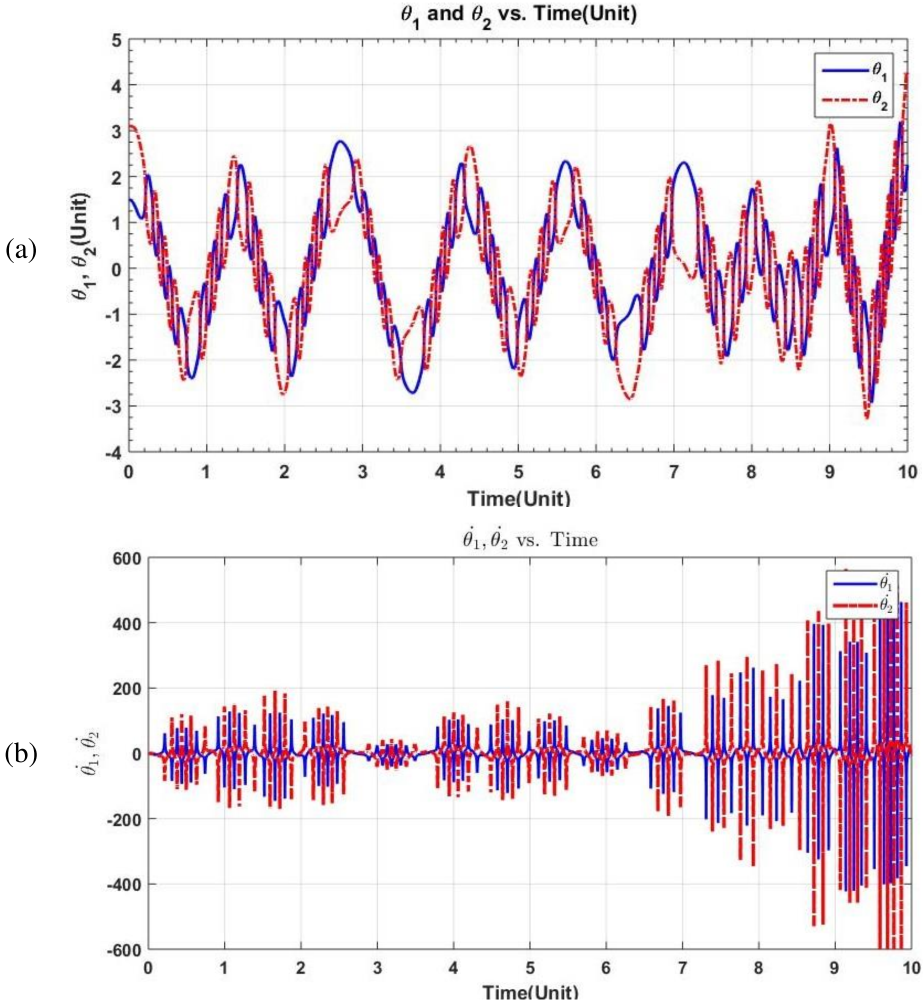

Fig. 6: (a) Shows the behavior of both θ_1 and θ_2 against time, while (b) shows the behavior of $\dot{\theta}_1$ and $\dot{\theta}_2$ against time (Case 4).

In Fig. 7 for small angles of both masses (Case 1) one can notice from the first quadrant that as both $\dot{\theta}_1$, $\dot{\theta}_2$ increase the angles θ_1, θ_2 decrease which ensure simple harmonic motion for both masses as simple pendulum (as the kinetic energy increases the potential energy decreases and vice versa). Whereas when the two angles are

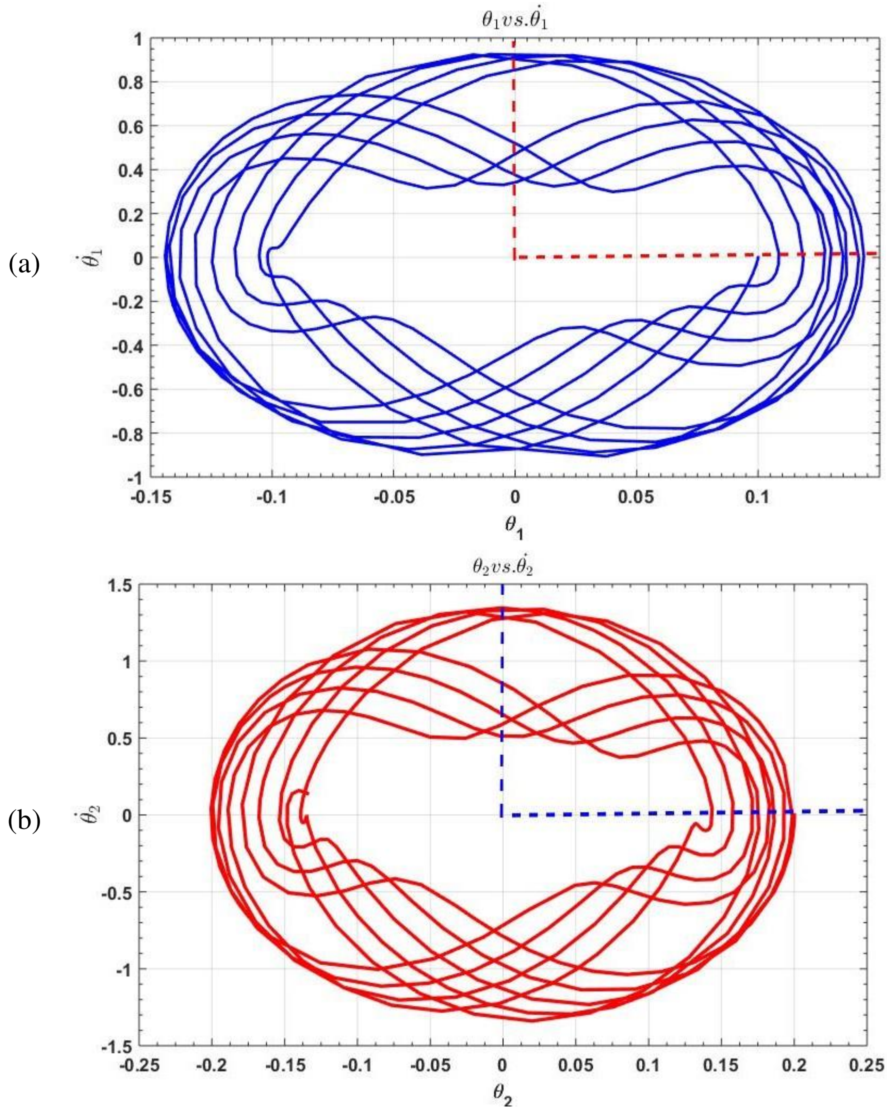

Fig. 7: (a) The behavior of $\dot{\theta}_1$ against θ_1, while in (b): The behavior of $\dot{\theta}_2$ against θ_2 for Case 1.

increased to become $\theta_1 = 1.50$ rad, and $\theta_2 = 3.1$ rad (Case 4) the previous periodic behavior is no more noticed and a chaotic behavior dominates on the motion of the double pendulum system, as it is clear in Fig. 8.

On the other hand, small irregular oscillations are clear within the main semi-periodic motion as in Fig. 5 in which θ_1, θ_2 are related with time.

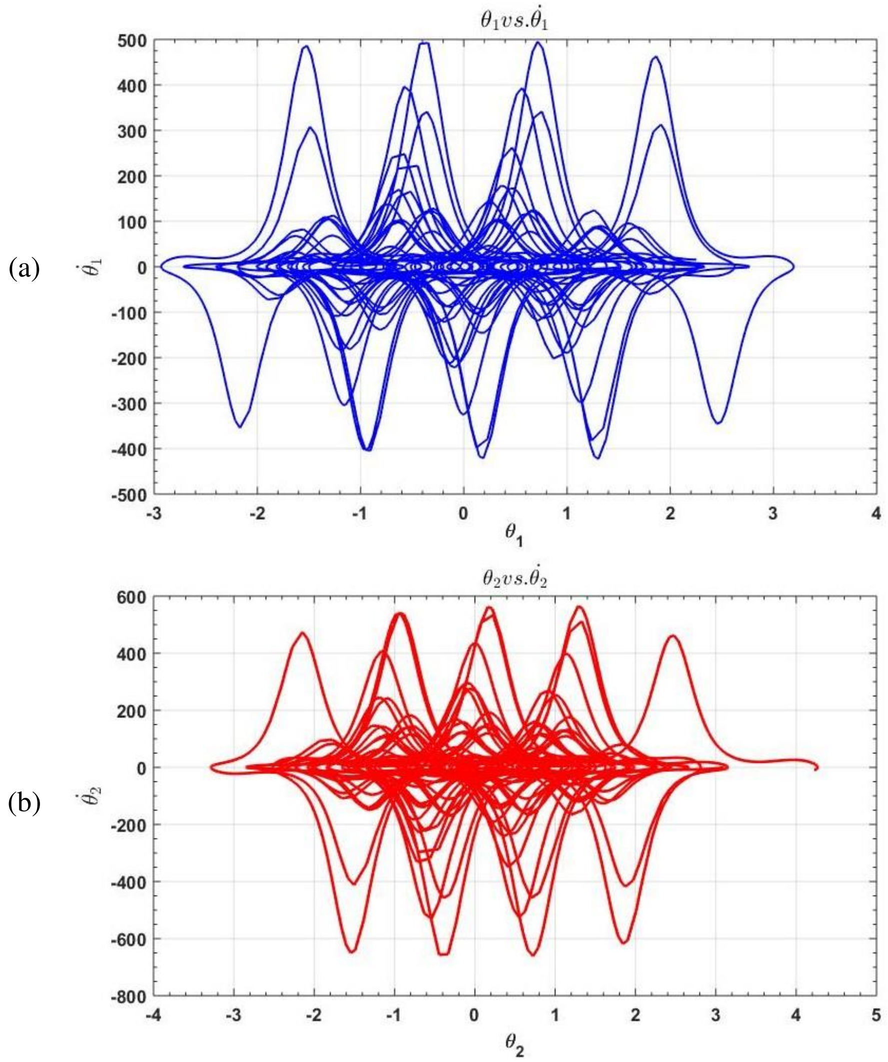

Fig. 8: (a) The behavior of $\dot{\theta}_1$ against θ_1, while in (b): shows the behavior of $\dot{\theta}_2$ against θ_2 for Case 4.

Finally, the variation of the angular displacement of the second mass θ_2 with the angular displacement of the first mass θ_1 is shown in Fig. 9 for the initial conditions specified in Case 1 and in Fig. 10 for the initial conditions given in Case 4. In Fig. 9

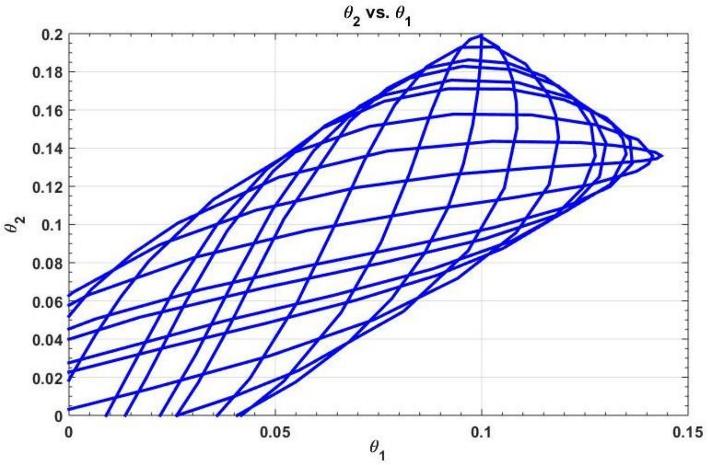

Fig. 9: The dynamical behavior of θ_2 against θ_1 for the initial conditions specified in Case 1.

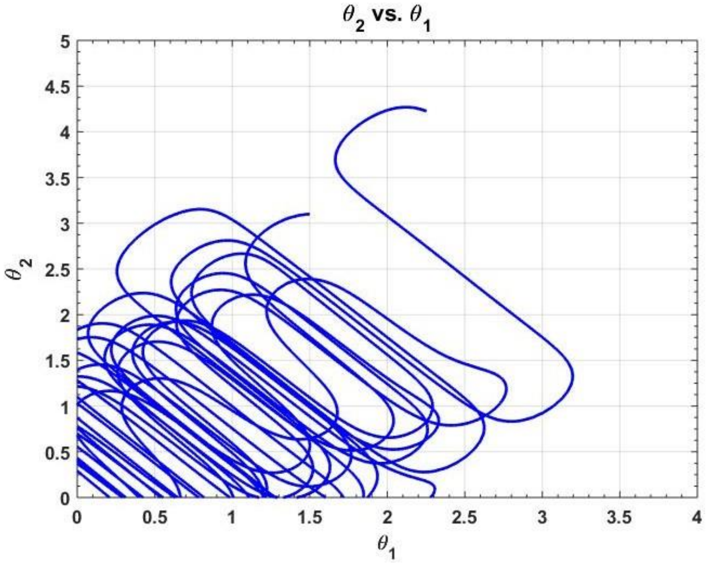

Fig. 10: The dynamical behavior of θ_2 against θ_1 for the initial conditions specified in Case 4.

a regular direct proportionality is clear and this emphasizes again simple harmonic motion and agrees with the previous findings from Figs. 3 and 7 and equations (17) and (18). Whereas a reverse and irregular proportionality is a dominant behavior in Fig. 10, which emphasizes again the chaotic behavior of the double pendulum as mentioned. These two behaviors were expected and this is what makes a double pendulum system of so interest.

5 CONCLUSION

A system consisting of two simple pendulums attached together in such a way that they form a double pendulum is investigated. The classical Lagrangian describing the system been obtained, from which we derived the equations of motion of the system. The dynamical behavior of the two masses depends on the initial conditions. The motion of the system is nearly simple harmonic with small initial angles displacement, and a chaotic motion is being noticed for large initial angles displacement.

REFERENCES

[1] L.D. LANDAU, E.M. LIFSHITZ (1976) "Mechanics: V. 1" (Course of Theoretical Physics S) 3rd ed. Butterworth-Heinemann.

[2] L.N. HAND, J.D. FINCH (1998) "Analytical Mechanics". Cambridge University Press.

[3] J.B. MARION, S.T. THORNTON (1988) "Classical Dynamics of Particles and Systems‘', 3rd ed. Harcourt Brace Jovanovich.

[4] R.G. FOWLES, G.L. CASSIDAY (2005) "Analytical Mechanics", 7th ed. Thomson Brooks/Cole.

[5] H. GOLDSTEIN, CH.P. POOLE JR., J.L. SAFKO (2001) "Classical Mechanics", 3rd ed. Pearson.

[6] H. PATRICK (2013) "A Student's Guide to Lagrangians and Hamiltonians" (Student's Guides) 1st ed. Cambridge University Press.

[7] J.L. BRENNER (2013) "Problems in Differential Equations". Dover Publication.

[8] V.I. ARNOLD (1989) "Ordinary Differential Equations", 1st ed. The MIT Press, Cambridge, England.

[9] W. TRENCH (1999) "Elementary Differential Equations", 1st ed. Brooks Cole.

[10] D.G. ZILL (2000) "A First Course in Differential Equations: The Classic", 5th ed. Cengage Learning.

[11] http://matlab.izmiran.ru/help/techdoc/demo_example.html. Last access 22.01.2020.

[12] P. HOWARD (2007) "Solving ODE in MATLAB". University of Maryland.

[13] A. GILAT (2011) "MATLAB: an Introduction with Applications", 4th ed. Ohio University.

[14] A.C. CALVÃO, T.J.P. PENNA (2015) The double pendulum: a numerical study. *European Journal of Physics* **36** 045018. DOI: https://doi.org/10.1088/0143-0807/36/4/045018.

[15] K. MARKOV (1995) "Guidence for Analytical Mechanics". Sofia University Press, Bulgaria.

[16] F.J. VESELY (2013) Of pendulums, polymers, and robots: Computational mechanics with Constraints. *American Journal of Physics* **81** 537-544.

[17] D. HOUCQUE (2005) "Introduction to MATLAB for Engineering Students", Version 1.2.

[18] S. ATTAWAY (2009) "MATLAB: A Practical Introduction to Programming and Problem Solving". College of Engineering, Boston University.