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SOLID MECHANICS

CONTACT PROBLEM FOR A RIGID FLAT STAMP AND A LINEAR ELASTIC STRIP BONDED TO POROUS HALF-PLANE

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ABSTRACT: In this article, we consider the plane contact problem for indentation into a system of elastic layer deposited on a half-plane composed of a linear elastic material with voids. The Cowin-Nunziato's micro-dilatation theory is used to explain the linear elastic response of the material with voids occupying the half-plane. It is assumed that the punch is rigid with a flat base and the contact is frictionless. Using the integral Fourier transform, the problem for determining the contact pressure is reduced to a singular integral equation for the unknown contact stress, whose approximate solution is found using the collocation method. The values of the contact stresses and the deformation of the strip's free surface are analyzed by varying the thickness of the elastic strip and the material parameters for the elastic layer and the material with voids. The numerical results are presented in the form of graphs.

KEY WORDS: contact problems, indentation, porous materials, micro-dilatation theory for materials with voids.

1 INTRODUCTION

Porous materials have unique physical, mechanical, acoustic, electrical and thermal properties. Porous materials can be found in nature, e.g. geological and biological materials, or are subject to production for various applications in technics and medicine. Due to the optimal balance of weight and strength, porous materials are particularly widely used in the space industry.

The most important issue in the production and use of such materials for engineering projects or environmental risk assessment is the control and evaluation of their mechanical properties and there are different approaches in this regard. One of

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the approaches to modeling porous materials' behaviour was developed by Cowin-Nunziato [1, 2] and independently by Markov [3]. Their theory, also called micro-dilatation theory, was applied to study the behavior of materials with voids based on the linear elasticity with an additional independent kinematic variable that describes the volume fraction corresponding to the void volume. Thus, deformation and porosity are coupled fields that have a common response to the external loading.

A number of studies on contact problems employing the micro-dilatation theory for elastic materials with voids have been carried out in the recent years. In [4] and [5], contact problems in plane strain formulation were solved for a porous elastic half-plane and for a porous elastic strip, respectively. In [6–9], axisymmetric contact problems for a porous linear elastic half-space, a porous elastic layer and an elastic layer fixed on a porous half-space are considered. In [10], axisymmetric Dirichlet and Neumann problems are solved by using the technique of integral transforms to obtain analytical solutions and the case of a concentrated body load is investigated in detail. The study in [11] is devoted to the modal analysis and the eigenvalues problem for piezoelectric bodies with voids, covered with a system of electrodes, subject to a rigid flat stamp. A significant contribution to the development of the micro-dilatation theory was made in [12, 13], where static and dynamic problems for two-dimensional and three-dimensional domains were considered using the finite element method and developing user subroutines for incorporating the micro-dilatation constitutive model in a commercial finite element code. Despite the valuable research carried out within the framework of the micro-dilatation theory, this theory is still considered poorly understood from a physical point of view. Thus, in [14], the ranges of variation of the mechanical constants, involved in the constitutive relations of the micro-dilatation theory, were analysed by means of penalized cases and using a novel total energy density concept.

There are also contact problems for porous materials considered within the framework of Biot's poroelasticity. In [15], contact problems on the penetration of cylindrical and spherical indenters into a poroelastic material are considered. On the basis of an analytical approach, in [16], it was solved a contact problem of indentation of a rigid indenter into a poroelastic half-space obeying Biot's poroelasticity.

When modeling porous structures, the finite element method is the mainly used numerical approach. For example, in [17, 18] among many others, finite element simulations of the behavior of porous materials of various density under impact loading are carried out. It has to be pointed, that the analytical and semi-analytical solutions are of paramount importance for verification of the numerical simulations.

The brief overview here does not exhaust all the research in the field and due to lack of space a number of excellent results in modeling porous materials remained outside this overview. The main objective of the present work is to present the derivation of the semi-analytical solution to the plane-strain formulation of the indentation problem for a flat punch penetrating a system of elastic strip bonded to a half-plane occupied of elastic material with voids.

2 MATHEMATICAL FORMULATION

We consider, in the Cartesian coordinate system (x, y), the plane-strain problem of the interaction of a rigid flat punch with an elastic strip $0 \le y \le h$ perfectly fixed on a porous half-plane $y < 0$, Fig. 1. The micro-dilatation theory for linear elastic material with voids is used to model the behaviour of the porous material [2–5]. It is to recall that in [4, 5] contact problems were considered for a porous half-plane and for a porous strip, respectively.

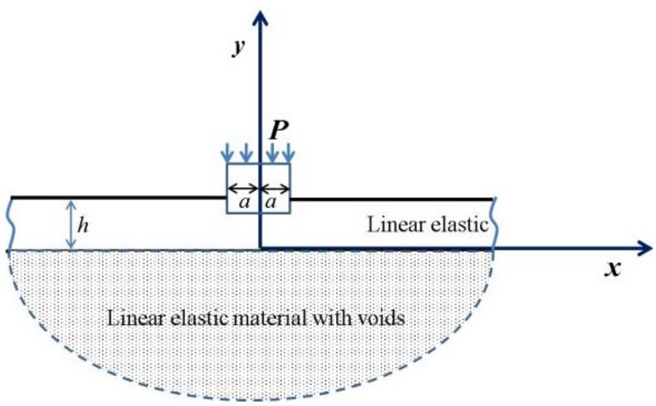

Fig. 1: Sketch of the problem with dimentions and coordinate system

The governing equations for the linear elastic material response of the strip $0 \le y \le h$ formulated for the displacement in x direction, $u_1(x, y)$, and y-direction, $w_1(x, y)$, read

$$
\begin{aligned}
&\mu_1 \Delta u_1 + (\lambda_1 + \mu_1)\Big(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial x \partial y}\Big) = 0, , \\
&\mu_1 \Delta w_1 + (\lambda_1 + \mu_1)\Big(\frac{\partial^2 u_1}{\partial x \partial y} + \frac{\partial^2 w_1}{\partial y^2}\Big) = 0 \,.
\end{aligned} \tag{1}
$$

According to the micro-dilatation theory, the governing equations for the elastic material with voids occupying the half-plane $y < 0$, are written in terms of the displacements $u_2(x, y)$, $w_2(x, y)$, and the function $\Phi(x, y)$ representing the change in the matrix volume fraction [2–5] as follows:

$$
\begin{aligned}
&\mu_2\Delta u_2 + (\lambda_2+\mu_2)\Big(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial x \partial y}\Big) + \beta\frac{\partial \Phi}{\partial x} = 0\,,\\
&\mu_2\Delta w_2 + (\lambda_2+\mu_2)\Big(\frac{\partial^2 u_2}{\partial x \partial y} + \frac{\partial^2 w_2}{\partial y^2}\Big) + \beta\frac{\partial \Phi}{\partial y} = 0\,,\\
&\alpha\Delta\Phi - \xi\Phi - \beta\Big(\frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial y}\Big) = 0\,.
\end{aligned} \tag{2}
$$

In the above equations, μ_k and λ_k, $k = 1, 2$ are the Lame constants for the linear elastic material ($k = 1$) and for the elastic material with voids ($k = 2$). The coefficients α, β, ξ represent material parameters related to the existence of voids, namely, α is the voids diffusion parameter, ξ is the "void stiffness" modulus, β is the coupling modulus. When $\beta = 0$, the first two equations in (2) describe a linear elastic deformation of the half-plane $y < 0$ and its stress state is independent of $\Phi(x, y)$ and of voids existence, respectively.

The components of the stress tensor are given below according to the linear elastic model and the linear elastic model for materials with voids

$$
\begin{aligned}
&\sigma_x^1 = \frac{\mu_1}{c_1^2}\Big[\frac{\partial u_1}{\partial x} + (1-2c_1^2)\frac{\partial w_1}{\partial y}\Big], \quad \sigma_x^2 = \frac{\mu_2}{c_2^2}\Big[\frac{\partial u_2}{\partial x} + (1-2c_2^2)\frac{\partial w_2}{\partial y} + H\Phi\Big]\\
&\sigma_y^1 = \frac{\mu_1}{c_1^2}\Big[(1-2c_1^2)\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial y}\Big], \quad \sigma_y^2 = \frac{\mu_2}{c_2^2}\Big[(1-2c_2^2)\frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial y} + H\Phi\Big],\\
&\tau_{xy}^k = \mu_k\Big(\frac{\partial u_k}{\partial y} + \frac{\partial w_k}{\partial x}\Big), \qquad k = 1, 2\,.
\end{aligned} \tag{3}
$$

In (3) the following notations for the dimensionless parameters are used [4,5]:

$$
c_k^2 = \frac{\mu_k}{\lambda_k + 2\mu_k} = \frac{1-2\nu_k}{2(1-\nu_k)}\,, \quad k = 1, 2; \qquad H = \frac{\beta}{\lambda_2 + 2\mu_2}\,,
$$

where ν_k $(k = 1,\ 2)$ are the Poisson's ratios of the elastic strip and the porous half-plane, respectively.

Taking into account (3), for the system of partial differential equations (1) and (2), we introduce the following boundary conditions at the upper surface of the strip $y = h$, where a rigid stamp with base size $2a$, subjected to a distributed force P, penetrates the system, and at the line $y = 0$, where the elastic strip and the porous half-plane are perfectly bonded

$$
\text{(4)}\quad
\begin{aligned}
&\text{for } y = h:\\
&\qquad \tau^1_{xy} = 0, \quad \frac{\partial \Phi}{\partial y} = 0, \quad \sigma^1_y = \begin{cases} q(x) & \text{for } |x| \le a\,, \\ 0 & \text{for } |x| > a\,, \end{cases}\\
&\text{for } y = 0:\\
&\qquad w_1 = w_2, \quad u_1 = u_2, \quad \tau^1_{xy} = \tau^2_{xy}, \quad \sigma^1_y = \sigma^2_y, \quad \frac{\partial \Phi}{\partial y} = 0\,.
\end{aligned}
$$

In the contact area for all $|x| \le a$ and for the applied distributed force, P, that acts on the rigid stamp in direction of negative y, the unknown contact stress, $q(x)$, is defined with: $\int_{-a}^{a} q(x)dx = P$. For $y \to -\infty$ both stresses and displacements vanish.

3 Derivation of the Contact Pressure Integral Equation

We introduce the Fourier integral transforms of the unknown function of volume fraction change $\Phi(x, y)$ and the unknown displacements u_k and w_k (k =1,2) as follows:

$$
\text{(5)}\quad
\begin{aligned}
u_k(x, y) &= \int_0^\infty U_k(\tau, y) \sin \tau x d\tau\,,\\
w_k(x, y) &= \int_0^\infty W_k(\tau, y) \cos \tau x d\tau\,, \quad k = 1, 2\,,\\
\Phi(x, y) &= \int_0^\infty F(\tau, y) \cos \tau x d\tau\,.
\end{aligned}
$$

For the unknown functions $U_k(y, \tau)$, $W_k(y, \tau)$, $F(y, \tau)$, taking into account the system of equations (1) and (2), we have to solve the following systems of ordinary differential equations (ODE) for the elastic strip (D is the notation for the operator d/dy)

$$
\text{(6)}\quad
\begin{aligned}
&c_1^2 D^2 U_1 - \tau^2 U_1 - (1 - c_1^2)\tau D W_1 = 0\,,\\
&(1 - c_1^2)\tau D U_1 + D^2 W_1 - c_1^2 \tau^2 W_1 = 0
\end{aligned}
$$

and for the porous half-plane

$$
\text{(7)}\quad
\begin{aligned}
&c_2^2 D^2 U_2 - \tau^2 U_2 - (1 - c_2^2)\tau D W_2 - H\tau F = 0\,,\\
&(1 - c_2^2)\tau D U_2 + D^2 W_2 - c_2^2 \tau^2 W_2 + HDF = 0\,,\\
&-\tau U_2 - D W_2 - \Big(\frac{l_1^2}{l_2^2} + l_1^2\Big) F + l_1^2 D^2 F = 0\,.
\end{aligned}
$$

In the above equations the following notation with dimension of length are used [4,5]:

$$l_1^2 = \alpha/\beta\,,; \qquad l_2^2 = \alpha/\xi\,.$$

From (4) we obtain that U_k and W_k, $k = 1, 2$, have to satisfy the following boundary conditions imposed on lines $y = h$ and $y = 0$:

$$\text{(8)} \quad \text{for } y = h: \quad DU_1 - \tau W_1 = 0\,, \qquad (1 - 2c_1^2)\tau U_1 + DW_1 = \frac{c_1^2}{\mu_1} Q(\tau)$$

$$\text{(9)} \quad \text{for } y = 0: \quad U_1 = U_2, \quad W_1 = W_2, \quad DF = 0$$

$$(1 - 2c_1^2)\tau U_1 + DW_1 = \mu \frac{c_1^1}{c_2^2}[(1 - 2c_2^2)\tau U_2 + DW_2 + HF]$$

$$DU_1 - \tau W_1 = \mu(DU_2 - \tau W_2)\,,$$

where $\mu = \mu_2/\mu_1$. For the unknown contact stress $q(x)$ at the line $h = y$, we have

$$\text{(10)} \qquad q(x) = \int_0^\infty Q(\tau)\cos\tau x d\tau\,, \qquad Q(\tau) = \frac{2}{\pi}\int_0^\infty q(x)\cos\tau x dx$$

The general solution of the ODE system (7) reads

$$\text{(11)} \qquad \begin{aligned} U_1(\tau, y) &= (a_1 + a_2 y)e^{\tau y} + (a_3 + a_4 y)e^{-\tau y}\,, \\ W_1(\tau, y) &= (b_1 + b_2 y)e^{\tau y} + (b_3 + b_4 y)e^{-\tau y}\,, \end{aligned}$$

where

$$\text{(12)} \qquad \begin{aligned} b_1 &= -a_1 + a_2 \frac{1 + c_1^2}{\tau(1 - c_1^2)}\,, \quad b_2 = -a_2\,, \\ b_3 &= a_3 + a_4 \frac{1 + c_1^2}{\tau(1 - c_1^2)}\,, \quad b_4 = a_4 \end{aligned}$$

The general solution of the ODE system (7) is given by

$$\text{(13)} \qquad \begin{aligned} U_2(\tau, y) &= (d_1 + d_2 y)e^{\tau y} + d_3 e^{my}\,, \\ W_2(\tau, y) &= (s_1 + s_2 y)e^{\tau y} + s_3 e^{my}\,, \\ F(\tau, y) &= (t_1 + t_2 y)e^{\tau y} + t_3 e^{my}\,, \end{aligned}$$

where

$$
\begin{aligned}
&s_1 = d_1 + d_2\frac{c_2^2 - N + 1}{\tau(c_2^2 + N - 1)}\,, \quad && s_2 = -d_2\,, \quad && s_3 = -\frac{md_3}{\tau}\,, \\
&t_1 = \frac{2d_2c_2^2l_2^2}{l_1^2(c_2^2 + N - 1)}\,, && t_2 = 0\,, && t_3 = \frac{d_3l_2^2(m^2 - \tau^2)}{Nl_1^2\tau}\,, \\
&m = \frac{\sqrt{l_2^2\tau - N + 1}}{l_2}\,, && && N = \frac{l_2^2}{l_1^2}H = \frac{\beta^2}{\xi(\lambda_2 + 2\mu_2)}\,.
\end{aligned}
\tag{14}
$$

It is shown in [4] that $0 \le N < 1 - c_2^2$, N – dimensionless parameter.

Satisfying the conditions (8) and (9), we can obtain the unknown coefficients a_1, a_2, a_3, a_4, d_1, d_2, d_3. We are not giving here the expressions for these parameters due to their cumbersomeness. For the purpose of determining the contact stress $q(x)$ and analysing the surface settlement, we present explicitly only the expression for $W_1(\tau, h)$

$$
W_1(\tau, h) = \frac{Q(\tau)l_2}{2\mu_1(1 - c_1^2)}L(u)\,, \tag{15}
$$

$$
\begin{aligned}
&L(u) = \frac{L_{10} + L_{11}\mu + L_{12}\mu^2}{L_{20} + L_{21}\mu + L_{22}\mu^2} \\
&L_{10} = 2(1 - c_1^2)(\sinh(2ud) + 2ud)(c_2^2m_2 - m_1)\,, \\
&L_{11} = -4m_1\cosh(2ud) + c_1^2c_2^2m_2\sinh(2ud)\,, \\
&L_{12} = -[(1 + c_1^2)\sinh(2ud) + 2du(c_1^2 - 1)](c_2^2m_2 + m_1)\,, \\
&L_{20} = 4u(1 - c_1^2)^2(\cosh(2ud) - 2u^2d^2 - 1)(c_2^2m_2 - m_1)\,, \\
&L_{21} = 8u(1 - c_1^2)[c_1^2c_2^2m_2\cosh(2ud) - m_1\sinh(2ud) - c_2^2m_2(2u^2d^2(c_1^2 - 1) + c_1^2)]\,, \\
&L_{22} = 4u(c_2^2m_2 + m_1)[(c_1^4 - 1)\cosh(2ud) - 2u^2d^2(1 - c_1^2)^2 - c_1^4 - 1]\,.
\end{aligned}
$$

In the above relations the following notations are used:

$$
h = dl_2\,, \quad u = \tau l_2\,, \quad T = \sqrt{u^2 - N + 1}\,,
$$
$$
m_1 = T(T + u)(1 - N)\,, \quad m_2 = 2Nu^2 - T - T^2\,.
$$

Combining (5) and (15), we end with the following expression for the vertical displacement along the line $y = h$:

$$
w_1(tl_2, h) = \frac{1 - \nu_1}{\mu_1}\int_0^\infty Q\Big(\frac{u}{l_2}\Big)L(u)\cos utdu\,,
$$

where we use the dimensionless $t = x/l_2$. Employing the notation $w(t) = w_1(tl_2, h)l_2^{-1}$ for the dimensionless vertical displacement and taking into account equation (10), we have

$$w(t) = \frac{1}{\pi}\int_{-b}^{b} p(\tau)k(t-\tau)d\tau \quad \text{with} \quad k(\vartheta) = \int_{0}^{\infty} L(u)\cos u\vartheta du\,, \tag{16}$$

where for readability we used the notations $p(\tau) = \dfrac{1-\nu_1}{\mu_1}q(\tau l_2)$ and $b = a/l_2$ (both dimensionless). It can be shown that $\lim\limits_{u\to\infty} uL(u) = 1$, then, based on the representation

$$M(u) = L(u) - 1/u\,, \tag{17}$$

we find that, [4]

$$\frac{d}{dt}w(t) = \frac{1}{\pi}\int_{-b}^{b} p(\tau)K(t-\tau)d\tau \tag{18}$$

$$K(\vartheta) = \frac{1}{\vartheta} + \int_{0}^{\infty} uM(u)\sin u\vartheta du \tag{19}$$

The integral in equation (19) converges because $\lim\limits_{u\to\infty} uM(u) = 0$ and $\lim\limits_{u\to 0} uM(u) =$ const.

It had been shown in [19] that any unbalanced load applied to the upper line, $y = h$, results in an infinite logarithmic increase in displacements. However, we can calculate the relative vertical displacement at the upper surface, the dimensionless w^*, defined as

$$w^*(t) = \pi[w(t) - w(0)] = \int_{-b}^{b} p(\tau)k^*(t,\tau)d\tau\,, \tag{20}$$

$$k^*(t,\tau) = 2\int_{0}^{\infty} L(u)\sin\frac{u(t-2\tau)}{2}\sin\frac{ut}{2}du\,. \tag{21}$$

The integral in (21) converges according to the behaviour of function $L(u)$ at zero and at infinity.

4 Determination of the Contact Pressure

Based on (18), for the case of a rigid flat stamp, penetrating the system and subjected to a constant distributed load P, we have a singular integral equation with Cauchy type kernel for determination of the dimensionless contact stress $p(t)$ for $|t| \le b$, [4]

$$\int_{-b}^{b} p(\tau)K(t-\tau)d\tau = 0 \quad \text{for} \quad |t| \le b\,. \tag{22}$$

To solve the integral equation (22), the direct colocation method is applied [4, 5, 19]. The interval $[-b, b]$ is divided into n sub-intervals $[-b+\varepsilon(j-1), -b+\varepsilon j]$, $(j = 1, \dots, n)$, $\varepsilon = 2b/n$ and the set of central points is defined as: $\tau_j = -b + \varepsilon(j-1/2)$ $(j = 1, 2, \dots, n)$, while the collocation points are taken at the ends of the sub-intervals: $t_i = -b + \varepsilon i$ $(i = 1, 2, \dots, n-1)$. Thus, the integral equation (22) is converted to a system of n algebraic equations with respect to $p_j = p(\tau_j)$, $j = 1, \dots, n$ [4, 5, 20]

$$\begin{aligned} &\sum_{j=1}^{n} a_{ij}p_j = 0\,, \quad \text{with} \quad a_{ij} = K(t_i - \tau_j), \quad i = 1, \dots, n-1\,, \\ &\varepsilon \sum_{j=1}^{n} p_j = \tilde{P}\,, \quad \text{with} \quad \tilde{P} = \frac{1-\nu_1}{\mu_1 l_2} P = \int_{-b}^{b} p(\tau)d\tau\,. \end{aligned} \tag{23}$$

It is straightforward to show that

$$\begin{aligned} a_{ij} &= a_{1,\,j-i+1} \quad (2 \le i \le n-1,\ i \le j \le n)\,, \\ a_{ij} &= a_{1,\,i-j+2} \quad (2 \le i \le n-1,\ 1 \le j \le i-1) \end{aligned} \tag{24}$$

The relations (24) significantly reduce the computational efforts. According to (24), it is sufficient to calculate the elements of only the first row of the matrix in (23), and the remaining elements are readily expressed through them.

Using the solution of (24), the relative deformation of the upper surface can be approximated based on (20) leading to the following expression:

$$w^*(t) = \varepsilon \sum_{j=1}^{n} p_j k^*(t, \tau_j)\,. \tag{25}$$

5 Numerical Results and Discussion

Hereafter, we analyse the considered contact problem using the approximate solutions for the contact pressure and the relative vertical displacement of the upper surface of the strip. The main attention is paid to the influence on the contact pressure and the relative vertical displacements of the material parameters $N$and μ, and the relative thickness of the elastic strip $d = h/l_2$. For the numerical calculations it is assumed that $b = a/l_2 = 1$, $\nu_1 = \nu_2 = 0.3$, $\tilde{P} = 1$.

For the verification of the applied here solution strategy let us consider the particular case when $N = 0$, $\mu = 1$. In this case, the problem is reduced to the interaction of a flat stamp and linear elastic half-plane, whose exact solution is known, [19, 21] and according to this solution the contact stress $p(t)$ is given as

(26) $$p(t) = \frac{\tilde{P}}{\pi\sqrt{b^2 - t^2}} .$$

At the same time, we calculated the contact stress $p(t)$ using (23) accounting that in this case $L(u) = u^{-1}$. Table 1 presents the comparison of the contact stress calculated using (26) with the contact stress, calculated using the procedure in (23) for $n = 100$ and $n = 500$. In Table 1, in the first raw, are given the coordinates of the points below the stamp, that is the value of t. As can be seen from Table 1, for $n = 500$, the results practically coincide with the exact solution in the inner region of the contact; the largest error is observed in the vicinity of the edge of the stamp.

Table 1

	t						
	0.01	0.21	0.41	0.61	0.81	0.95	0.99
$n = 100$	0.320	0.327	0.351	0.405	0.551	1.073	2.832
$n = 500$	0.318	0.326	0.349	0.402	0.544	1.030	2.372
(26)	0.318	0.326	0.349	0.402	0.543	1.019	2.256

Further, for the original problem, the results of the calculations of contact stresses and relative deformation of the surface for some values of the parameters N, μ, d are shown and analyzed in the form of graphs. Figures 2 and 3 show the graphs of the reduced contact pressure $p(t)$ for $d = 1$ and $d = 0.5$, and for various values of the parameter $\mu = \mu_2/\mu_1$. Figure 4 depicts the contact pressure $p(t)$ at various values of the parameter $N = \dfrac{\beta^2}{\xi(\lambda_2 + 2\mu_2)} = \dfrac{\beta^2(1 - 2\nu_2)}{2\xi\mu_2(1 - \nu_2)}$. Figures 5 and 6 present the distribution of the relative vertical displacement $w^*(t)$ of the upper strip surface outside the stamp and allow a comparison of $w^*(t)$ for different values of the parameters d, N and μ.

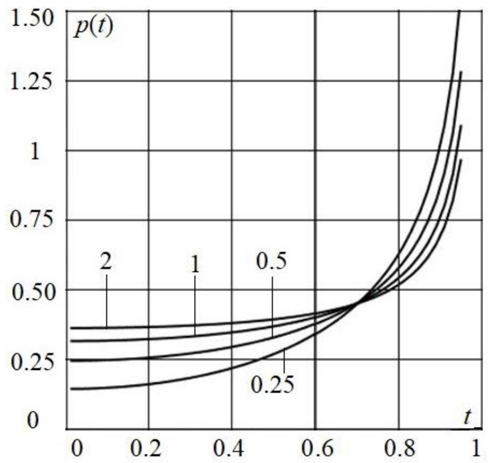

Fig. 2: Contact pressure $p(t)$ for $d = 1$, $N = 0.5$ and $\mu = 0.25$; 0.5; 1; 2.

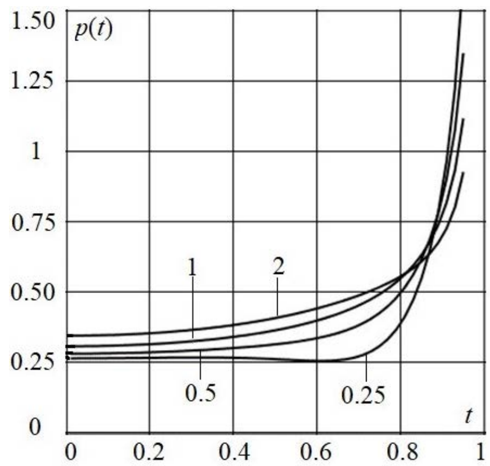

Fig. 3: Contact pressure $p(t)$ for $d = 0.25$, $N = 0.5$ and $\mu = 0.25$; 0.5; 1; 2.

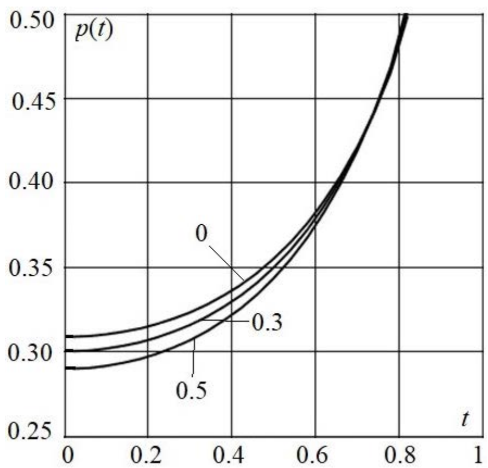

Fig. 4: Contact pressure $p(t)$ for $d = 0.1$, $\mu = 0.5$ and $N = 0$; 0.3; 0.5.

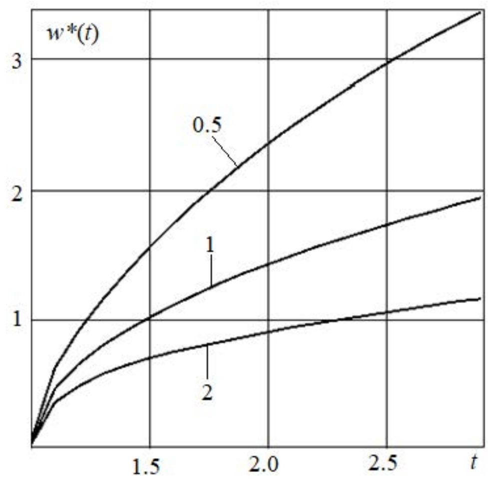

Fig. 5: Relative vertical displacements of the upper surface for $d = 0.5$, $N = 0.5$ and for $\mu = 0.5$; 1; 2.

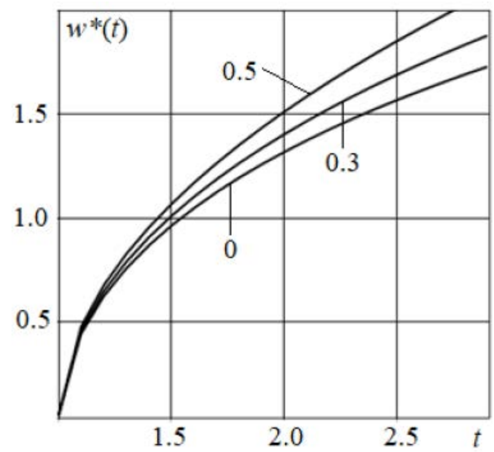

Fig. 6: Relative vertical displacements of the upper surface for $d = 0.05$, $\mu = 1$ and $N = 0$; 0.3; 0.5.

6 Discussion and Conclusions

The parametric analysis of the obtained in this study solution of the contact problem for a flat stamp penetrating a system of a linear elastic strip bonded to a half-plane composed of a material with voids allows to draw the following conclusions. Figures 2 and 3 show that at a constant load acting on the rigid stamp and constant values of the other model parameters, the contact pressure in the inner contact region is decreasing with decreasing μ, while at the same time it is increasing in the region close to the edge of the contact region. The same behaviour is observed with the increase in the initial porosity of the half-space, i.e. for larger coupling numbers $N$the contact pressure in the inner contact region is smaller (Fig. 4). Comparing the results in Figs. 2, 3, 4 for $\mu = 0.5$ (stiffer strip) and $N = 0.5$ it can be concluded that with the increase in the relative thickness d of the elastic strip, the contact stress increases.

Furthermore, the relative deformation of the upper surface outside the contact zone increases with the decrease in the parameter μ (Fig. 5).The same holds with the increase in the parameter N (Fig. 6) and, according to the results in Figs. 5 and 6 for $\mu = 0.5$ and $N = 0.5$, with the decrease in the relative thickness d of the elastic strip at fixed values of the rest of the considered parameters.

The obtained solution in plane-strain formulation of the indentation problem for a flat punch penetrating a system of elastic strip perfectly bonded to a half-plane occupied of elastic material with voids can be used to verify numerical algorithms for problems employing the micro-dilatation theory for elastic materials with voids.

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