Journal of Theoretical and Applied Mechanics, Sofia, Vol.52 (2022) pp. 19-35
SOLID MECHANICS

ENTROPY GENERATION RATE ARISING FROM FRACTIONAL OSCILLATOR MODEL WITH NON-SINGULAR KERNEL

NORODIN A. RANGAIG*, ALVANH ALEM G. PIDO

Department of Physics, Mindanao State University-Main Campus, 9700 Marawi City, Philippines

[Received: 16 July 2020. Accepted: 05 April 2021]

doi: https://doi.org/10.55787/jtams.22.52.1.019

ABSTRACT: The developing potential applications of fractional calculus in science and engineering excites many researchers because it can describe different nonlinear phenomena. In this present work, we have studied another physical implication of the fractional operator on the spring-mass model. We derive the fractional equation of motion by considering a fractional action-like integral via the fractional operator and obtaining a fractionally-modified Euler-Lagrange equation with fractionalized Lagrangian function. We have shown that the dissipative effect due to the fractional operators generates an entropy proportional to the fractional-order ν.

KEY WORDS: entropy generation rate, fractional oscillator, fractional derivative, nonsingular kernel.

1 INTRODUCTION

These past years can be regarded as the golden era of the fractional calculus because of its tremendous applications and developments in science and engineering. There are numerous pieces of evidence on the applications of fractional calculus in real-world systems, fractional derivatives give more advantages compared to classical integer derivatives because of the convolution of a memory function and a first derivative, in which, more memory effects are accessible [1–5]. In the application of fractional calculus in modeling an oscillatory system, literature shows an increasing report in the past years. In 2001, Anchar et. al [6] investigated the motion of harmonic oscillator and give a fractional model-based Mittag-Leffler function, together with the response and damping characteristics [7, 8]. In 2015, Gomez-Aguillar [9] reported the analytic and numerical solution of a spring-mass-damper system under fractional differentiation and experimental work on the evaluation of viscous damping coefficient in fraction underdamped oscillator was conducted by Escalante et.

*Corresponding author e-mail: rangaig.norodin@msumain.edu.ph

al [10]. On the other hand, the entropy generation rate (EGR) was first established by [11] which aims to analyze the thermodynamic foundation in real engineering designs. EGR has gained attention, especially in a mechanical system since the basic concept was established from the second law of thermodynamics to analyze how the system behaves from its ideal state [12]. In 2015, Sunar et. al [13] used the concept of EGR in analyzing the damped oscillatory system and the result shows that the entropy generated is highly dependent on the damping coefficient.
It is the novel goal of this work to study the dissipative effect of the nonsingular kernel fractional derivative, namely, the Caputo-Fabrizio fractional derivative operator on the spring-mass system and to study the arising entropy generation rate. We also aim to discuss that the entropy generation is possible without the damper attached on the system (which is physically acceptable since in real-world system), and we should expect an internal friction-like term that leads on a non-conservative effect. We believe that the imposed dissipative effect due to the fractional operator implies the internal friction of a physical system. This is proven when the entropy generation rate arises on the system. Additionally, the solution of a forced oscillator will also be investigated.

2 PRELIMINARIES

2.1 CAPUTO-FABRIZIO OPERATOR

In this preliminary, we will present the definition of the imposed fractional derivatives with the non-singular kernel. The new fractional derivative with nonsingular kernel can be defined, and it is called Caputo-Fabrizio fractional derivative [14], given by the definition:

Definition 1. *Let* $f \in H^1(a,b)$, $b > a$, $\nu \in (0,1]$, *then the Caputo-Fabrizio fractional derivative is defined as*

$$ {}^{CF}_{a}D^{\nu}_{t} f(t) = \frac{C(\nu)}{1-\nu}\int_a^t f'(s)\exp\left(-\frac{\nu}{1-\nu}(t-s)\right)ds, \tag{1}$$

where $C(\nu)$ *is the normalization function such that* $C(0) = C(1) = 1$.

For more details about the property of this derivative, see [14].
However, Losada and Nieto [18] showed the explicit form of the equation (1) as

$$ {}^{CF}_{a}D^{\nu}_{t} f(t) = \frac{(2-\nu)C(\nu)}{2(1-\nu)}\int_a^t f'(s)\exp\left(-\frac{\nu}{1-\nu}(t-s)\right)ds, \tag{2}$$

where they also write the fractional integral counterpart of (2), given by the definition below.

Definition 2. [18] *Let* $g(t) = {}^{CF}D_t^{\nu} f(t)$*, then the fractional integral of order* ν *is given by*

$$(3) \qquad {}^{CF}I_t^{\nu} g(t) = f(t) = \frac{2(1-\nu)}{(2-\nu)C(\nu)} g(t) + \frac{2\nu}{(2-\nu)C(\nu)} \int_a^t g(t')dt' \,,$$

where $C(\nu)$ *is shown to be* $C(\nu) = 2/(2-\nu)$*.*

Note that the classical definition of calculus is recovered if $\nu \to 1$. For more details about the Caputo-Fabrizio fractional derivative, we refer to papers [15,18,19]. We continue the rest of this work with the considerations $C(\nu) = 1$ and $a = 0$.

2.2 Entropy rate generation

The state of any dynamical physical systems in physics and engineering, at either macroscopic or microscopic scales, can be described by the laws of thermodynamics to serve as an analysis in both quantity and quality of the system [12]. A closed system's environment can be defined initially by a state at temperature $T_0(K)$ and its interactions with other system through the energy exchange and its entropy. The simple process of a system is defined between its states 1 & 2, where there is an exchange of energy and entropy. Additionally, internal energy, heat, and work are also interchangeable between the interaction of the system with its environment. Hence, the first and second law of thermodynamics for any closed system are expressed as

$$(4) \qquad \Delta U_{1\to 2} = \int_1^2 (\delta W + \delta Q) \,,$$

$$(5) \qquad \Delta S_{1\to 2} = S_{\text{gen}} + \int_1^2 \frac{\delta Q}{T} \,,$$

where S_{gen} is the entropy generated from state 1 to state 2, which also an indication that the system has irreversibility property. In an inherent process, S_{gen} must always be positive and for $S_{\text{gen}} = 0$, an ideal system is described.

Now, substituting equation (4) onto (5) through δQ and let the temperature approaches to the room temperature $T_0(K)$, we get the expression of the generated entropy

$$(6) \qquad S_{\text{gen}} = \Delta S_{1\to 2} - \frac{1}{T_0} + \int_1^2 (\delta W + \Delta U_{1\to 2}) \,.$$

Note that the external energy $\Delta U_{1\to 2}$ is an energy that varies with time. In a complete mechanical system under consideration, $\Delta U_{1\to 2}$ is entirely comprise by the kinetic energy (KE) and potential energy (PE) changes. Consider that any type of energy exchanges are ignored, such as heat transfer, heat flux and the net work produced, and done by the system is constant, we can express the entropy generation rate as

$$\dot{S}_{\text{gen}} = -\frac{1}{T_0}\left(\frac{dKE}{dt} + \frac{dPE}{dt}\right). \tag{7}$$

Equation (7) is also known as a Guoy-Stodola theorem [11], which is the direct relationship of the entropy generation rate, the over-all power and the which is a convenient way of analyzing the entropy generated by the system than the temperature.

To understand the energy exchange and the entropy generation rate of the system model, we need to determine the total energy of the system without any external force $F(t) = 0$. The total energy of such mechanical system is given by

$$U(t) = \frac{1}{2}kx^2(t) + \frac{1}{2}Mv^2(t)\,, \tag{8}$$

where the first and second term are the potential energy and kinetic energy of the system. Plugging equation (6) onto equation (7), the entropy generation rate is given by

$$\dot{S}_{\text{gen}} = -\frac{M}{T_0}\left(\omega^2 x(t) + \frac{d^2x}{dt^2}\right)v(t)\,. \tag{9}$$

For a macroscopic mechanical system, the entropy generation rate (9) best describe the energy dissipation of a damped system. From the work of Sunar et al. [13], the classical equation of motion for a damped oscillator were considered and the corresponding entropy generation were discussed. It has been proven that the entropy generation rate of a mechanical system model, such as the damped oscillator, is highly dependent to the damping coefficient [13, 22]. Zero entropy is acquired for an undamped system [22]. In the subsequent sections, we consider a classical oscillator in fractional regime using fractional derivative. As mentioned earlier, undamped oscillator gives zero entropy generation.

In the fractional scheme, entropy generation rate arises from the system, as an initial hypothesis of this work, due to the dissipative effect of the imposed fractional derivative operator [24–26].

3 Fractionally-Modified Euler-Lagrange Equation

In this section we will present the fractional action-like integral in the framework of the nonsingular fractional derivative, using the fractional integral (3) counterpart to formulate the Euler-Lagrange equation. With the help of calculus of variation, we will first define a generalized path coordinate $q(t)$ with the variational principle, and we parametrize the path coordinate as

$$q(\beta, t) = q(0, t) + \beta\eta(t), \tag{10}$$

such that $\eta(t_a) = \eta(t_b) = 0$, and $q(0, t) = q(t)$.

Proposition 1. *If q is an extremizable path, then applying the fractional derivative operator (2) to the parametrized path coordinate (10), we have*

$$^{CF}D_t^{\nu} q(\beta, t) = \, ^{CF}D_t^{\nu} q(0, t) + \beta \, ^{CF}D_t^{\nu}\eta(t). \tag{11}$$

We also recall the definition of fundamental theorem of the calculus of variation, which is very important in the development of this work.

Definition 3. *For any arbitrary continuous function $\eta(t)$ up to its second derivative, and if*

$$\int_{t_a}^{t_b} G(\tau)\eta(\tau)d\tau = 0, \tag{12}$$

then $G(\tau)$ must identically vanish in the interval $[t_a, t_b]$.

Now, we can formulate the fractional action-like integral in the variational problem in the subsequent definitions.

Definition 4. *Suppose we have a continuous manifold T and let the L be a classical Lagrangian function $L = L(q(t), \dot{q}(t); t) : \Re^{3d} \to \Re$, $d \geq 1$. So, for any smooth path $q : [t_a, t_b] \to T$ with a fixed boundary condition. We define the fractional action integral under the nonsingular kernel scheme as:*

$$^{CF}S^{\nu}[q] = \, ^{CF}I^{\nu} \left(L(q(t), \dot{q}(t); t)\right). \tag{13}$$

Definition 5. *Let the fractional Lagrangian function be $L = L(q(t), \, ^{*}D^{\nu}q(t); t) : \Re^{3d} \to \Re$, then we can write the fractional action integral under nonsingular kernel scheme as with fractional Lagrangian function as*

$$^{CF}S^{\nu}[q] = \, ^{CF}I^{\nu} \left(L(q(t), \, ^{CF}D^{\nu}q(t); t)\right). \tag{14}$$

According to the definitions presented above, it shows that the Lagrangian can be formulated, classically or fractionally (10-11). The presented definitions (6) and (7) shows the action-like integrals for the classical and fractionalized Lagrangian functions, respectively. However, in the case of the classical formulation of the Lagrangian function, the resulting Euler-Lagrange equation is in classical form despite applying the fractional integral [23]. We present the next theorem for the fractionally-modified Euler-Lagrange equation.

Theorem 3.1 (Fractional Euler-Lagrange equation). *If $q : [t_a, t_b] \to T$ is the extremizable path coordinate of the fractional action-like integral in the fractional space with fractional Lagrangian function $L_\nu = L_\nu(q(t), {}^{CF}Dq(t); t) : \Re^{3d} \to \Re$, then the Euler-Lagrange equation is fractionalized with the form*

$$\frac{\partial L}{\partial q} - \left[{}^{CF}D_t^\nu \left(\frac{\partial L}{\partial \, {}^{CF}D_t^\nu q} \right) \right] = 0. \tag{15}$$

Proof: We can express the fractional action-like integral with fractionalized Lagrangian function as

(16)

$${}^{CF}S^\alpha[q] = (1-\alpha)\left(L_\alpha(q(t), {}^{CF}D_t^\alpha q(t); t)\right) + \alpha \int_0^t \left(L_\alpha(q(s), {}^{CF}D_s^\alpha q(s); s)\right) ds,$$

using Definition 5. The second term on the right-hand side of (16) can be simplified using the integral-by-parts method under fractional calculus and it can absorb the first term since $L_\alpha(q(t), {}^{CF}D_t^\alpha q(t); t)$ is not explicitly dependent to time and by applying the variational principle, we can get (16) under Proposition 1. For more details, please see [23]. ∎

The obtained fractionally-modified Euler-Lagrange equation generalizes the classical form with the fractional formulation of Lagrangian function. As a goal of this study, we will consider a fractional Lagrangian function of a harmonic oscillator of the form

$$L_\nu = \frac{1}{2} m \left({}^{CF}D^\nu x(t) \right)^2 - \frac{1}{2} k x^2, \quad \text{where } m > 0 \;\&\; k > 0. \tag{17}$$

The derivation of the fractional equation of motion will be discussed in the next section, together with the presentation of the reduction to ordinary differential equation of the obtained fractional equation of motion and its corresponding entropy generation rate.

4 Results and Discussions

In previous sections, we presented the fractional Euler-Lagrange equation under the Caputo-Fabrizio fractional derivative with fractionalized Lagrangian describing a harmonic oscillator. The established notion of the consequences in utilizing Caputo-Fabrizio is widely studied in [9,10,25,26], which the kernel imposed leads to a dissipative effect on the system. In this section, we will also study the known dissipative effect of the fractional derivative with non-singular kernel imposed on a fractional oscillator and applying Theorem 3.1 and considering the fractional Lagrangian function (17). We shall discuss the implication of the imposed memory kernel by solving the resulting fractional differential equation, that describe fractional oscillator, via reduction to ordinary differential equation and using the concept of entropy generation rate. In addition, the solution of the forced fractional oscillator is also studied. We present the main result of this work in the subsequent discussion.

4.1 EGR AND SOLUTION OF FRACTIONAL OSCILLATOR

Considering an unforced oscillator, we aim to utilize the presented fractional Euler-Lagrange equation to study further the fractional oscillator and the arising EGR. Starting from fractional Lagrangian function (17) and Theorem 3.1, we get the fractional equation of motion

$$^{CF}D_t^{2\nu}x(t) + \omega_n^2 x(t) = 0, \tag{18}$$

where $\omega_n = \sqrt{k/m}$ is the natural frequency of the oscillator. Obviously, the classical model for harmonic oscillator is obtained for $\nu \to 1$. Now, in order to solve this differential equation, we use the reduction to ODE method by applying the fractional integral, twice, in equation (18) to yield the integral equation,

$$x(t) + \omega_n^2 \, {}^{CF}I_t^{2\nu}x(t) = 0, \tag{19}$$

we can further write (19) as

$$x(t) + \omega_n^2 \Big\{ (1-\nu)^2 x(t) + 2\nu(1-\nu)\int_0^t x(s)ds + \nu^2 \int_0^t ds \int_0^s x(u)du \Big\} = 0 \tag{20}$$

provided that $x(0)$ is constant and that the integrals exists. Performing a direct application of classical derivative, the fractional differential equation (18) can be expressed as a second order ODE of the form

$$\ddot{x}(t) + \frac{2\omega_n^2\nu(1-\nu)}{1+(1-\nu)^2}\dot{x}(t) + \frac{\omega_n^2\nu^2}{1+(1-\nu)^2}x(t) = 0. \tag{21}$$

We have reduced the fractional differential equation (18) to (21) where the direct consequence of the kernel of the derivative is shown in the second term of (21). Looking at (21), the damping-like term emerges that causes the dissipative effect on the dynamical system. In the best of our knowledge, this explicit form of the dissipative effect has not been reported previously on the fractional oscillator. Salti et al. [20] presented the investigation of a differential equation with order $\beta = 1 + \nu$ and they also obtained a second-order ODE, but not similar to our result (21). The results presented in this work describes a more accurate physical system because of the illustrated fractional Euler-Lagrange equation, which (18) was drawn.

Now, since the derived auxiliary equation of motion (21) have the same form with those in damped oscillation, we present the solution of the form

$$x(t) = X \exp\left(-\frac{\omega_n^2 \nu(1-\nu)}{1+(1-\nu)^2} t\right) \cos\left(\omega_n' t + \psi\right), \tag{22}$$

where X is the amplitude and

$$\omega_n' = \frac{\omega_n \nu}{1+(1-\nu)^2} \sqrt{1+(1+\omega_n^2)(1-\nu)^2}\,. \tag{23}$$

Clearly, for $\nu \to 1$, the solution (22) reduces to the classical solution. In this formulation, we can not get the critical and over-damped type of damping. The only resulting allowed damping from this formulation is underdamping case. To test this, we present the figure below. For any varied value of the fractional order ν,we can see the dissipative effect and it is noticeable that the oscillation becomes classical oscillation and lowering the ν makes the oscillation more damped.

Next, we present the corresponding entropy generation rate of the system presented above. From equation (9), the entropy generation rate is written as

$$\dot{S}(t) = -\frac{m}{T_0}\left(\omega_n^2 x(t) + \ddot{x}(t)\right)\dot{x}(t). \tag{24}$$

Substituting equation (21) to (24), we have

$$\dot{S}(t) = \frac{k}{T_0}\left(\frac{2\nu(1-\nu)}{1+(1-\nu)^2}\dot{x}(t) + \left(\frac{\nu^2}{1+(1-\nu)^2} - 1\right)x(t)\right)\dot{x}(t). \tag{25}$$

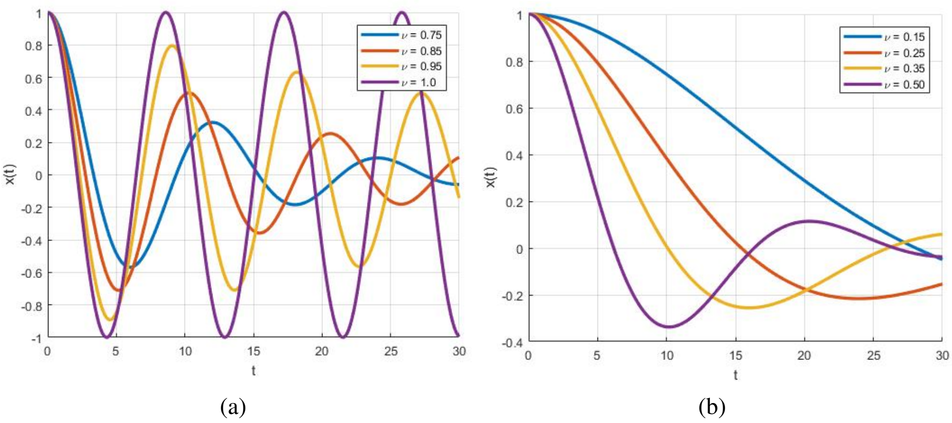


Fig. 1: Displacement of fractional oscillator under nonsingular fractional derivative operator.

Clearly, the entropy generation rate is highly dependent to the fractional order. The acquired equation (25) is one of the important effect on the dynamical system under the fractional regime with nonsingular kernel. Simulation were carried out considering a unit mass and a constant natural frequency $\omega_n = 0.73$ Hz with an initial displacement and velocity $x(0) = 1$, $v(0) = 0$, respectively. In Figs. (1a) and (1b), displacement of the fractional oscillator under the nonsingular kernel derivative is shown.

The validation of the said speculation of the fractional oscillator is shown via Fig. (2), where we plot the phase diagram of the system. Clearly, the analysis is constant for $\nu \to 1$, which we can get the classical model. The effect of the fractional operator on the system allows it to manifest a dissipative behavior, like an external friction in the case of fractional oscillator. This is actually an already established notion as one of the advantages of fractional derivative operator. However, its implications on the dynamical system need more attention, such as the reversibility of the system. To give further analysis, we present the theory of entropy generation rate of a mechanical system (25). The dynamic response of the fractional oscillator will be studied, which arises due to the fractional operator.

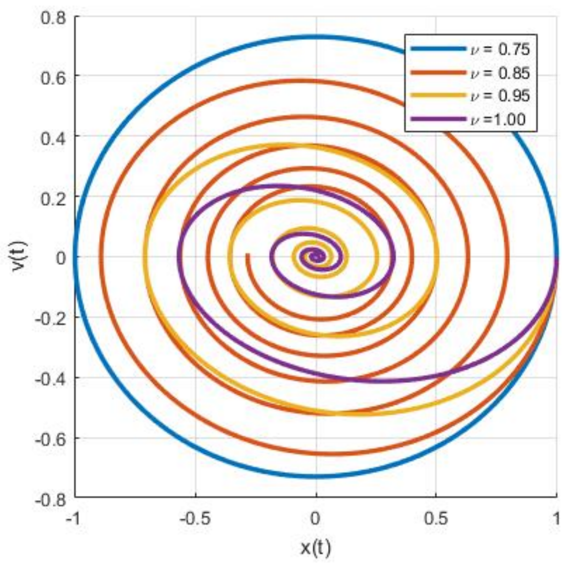


Fig. 2: Phase diagram of the fractional oscillator model.

The entropy generation rate of the fractional oscillator model system was determined by carrying out the total energy of the system from its dynamical response

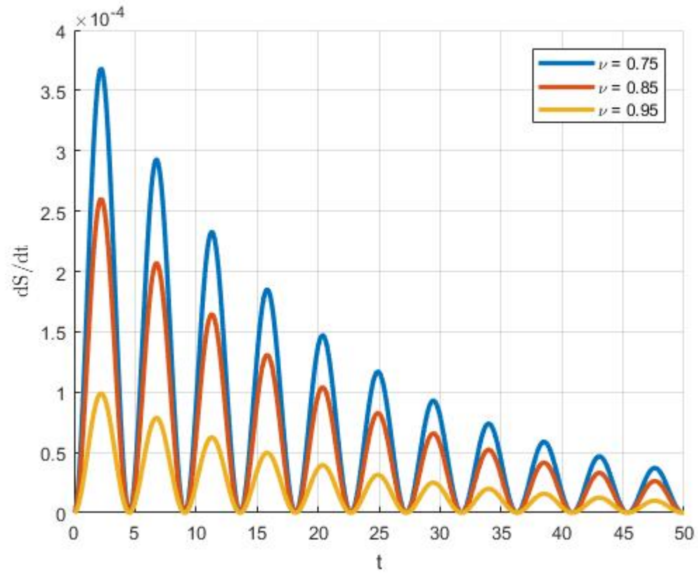


Fig. 3: Entropy rate generation of the fractional oscillator with the given values of ν.

under the non-singular fractional derivative. The physical consequence of applying the fractional derivative operator is manifested by the underdamping-like effect on the system which generates an entropy. In Fig. (3), we presented the sample simulation of the entropy rate generation of the fractional oscillator. Clearly, the plot shows an interesting point, where decreasing the fractional order ν increases the entropy generation rate, while for increasing ν, the entropy generation rate decreases. Notice that equation (21) suggests a larger irreversible signature when ν decreases, which supports the mentioned statement in the previous sentence.

4.2 Solution for forced fractional oscillator

Next, we consider the forced oscillator of the form

$$ {}^{CF}D_t^{2\nu} x(t) + \omega_n^2 x(t) = F(t), \tag{26} $$

where $F(t)$ is dependent to the driving force. We can get the solution of fractional forced oscillation by using the reduction to ODE method via the application of fractional integral counterpart of the operator (2). We can express (26) as

$$ x(t) + \omega_n^2 I_t^{2\nu} x(t) = I_t^{2\nu} F(t), \tag{27} $$

which can be simplified as

$$\ddot{x}(t) + \frac{2\omega_n^2\nu(1-\nu)}{1+(1-\nu)^2}\dot{x}(t) + \frac{\omega_n^2\nu^2}{1+(1-\nu)^2}x(t) = f(t), \tag{28}$$

where

$$f(t) = \frac{(1-\nu)^2}{1+(1-\nu)^2}\frac{d^2F(t)}{dt^2} + \frac{2\nu(1-\nu)}{1+(1-\nu)^2}\frac{dF(t)}{dt} + \frac{\nu^2}{1+(1-\nu)^2}F(t). \tag{29}$$

Assuming that $F(t)$ is continuous up to its second derivative. Clearly, the forced fractional oscillator can be expressed as a forced damped vibration which the solution can be easily shown. However, we show the analysis for the effect of the fractional order ν. Before we show the solution(s), we must evaluate the discriminant of the auxiliary equation of motion

$$M(\nu, \omega_n) = \frac{4\omega_n^2\nu^2}{1+(1-\nu)^2}\left[1 - \frac{\omega_n^2(1-\nu)^2}{1+(1-\nu)^2}\right] \tag{30}$$

and different solution may be obtained, depending on the sign of M. For simplicity, we let

$$\lambda_1 = \frac{2\omega_n^2\nu(1-\nu)}{1+(1-\nu)^2}, \quad \lambda_2 = \frac{\omega_n^2\nu^2}{1+(1-\nu)^2},$$

so that we can write (28) as

$$\ddot{x}(t) + \lambda_1\dot{x}(t) + \lambda_2 x(t) = f(t). \tag{31}$$

Now, consider the discriminant $M = 0$, then we can derive the corresponding solution with natural frequencies $\omega_n = 0$ and $\omega_n = \sqrt{\frac{1+(1-\nu)^2}{(1-\nu)^2}}$. Obviously, the general solution of auxiliary equation of motion can be written as

$$x(t) = e^{\lambda_1 t}\left\{z_1 - \int_0^t \tau f(\tau)e^{-\lambda_1\tau/2}d\tau + t\left(z_2 + \int_0^t f(\tau)e^{-\lambda_1\tau/2}d\tau\right)\right\}, \tag{32}$$

where z_1 and z_2 are constants which can be obtained by imposing an initial condition. Clearly, for the case of $\omega_n = 0$, the solution yields

$$x(t) = z_1 - \int_0^t \tau f(\tau)d\tau + t\left(z_2 + \int_0^t f(\tau)d\tau\right), \tag{33}$$

which is the solution of the Newton's second law of motion considering that $f(t)$ is continuous and has a physical meaning. In addition, for the case of $\omega_n = \sqrt{\frac{1+(1-\nu)^2}{(1-\nu)^2}}$, the general solution yields

$$x(t) = e^{\frac{2\nu}{1-\nu}t}\left\{z_1 - \int_0^t \tau f(\tau)e^{-\frac{\nu}{1-\nu}\tau}d\tau + t\left(z_2 + \int_0^t f(\tau)e^{-\frac{\nu}{1-\nu}\tau}d\tau\right)\right\}. \tag{34}$$

Looking at this different analytical solution, the fractional order has a huge impact on the system since it can give a whole new point of view. However, it is an interesting notion since the natural frequency is found to be connected with the fractional order, which gives a clearer physical of the effect of nonsingular kernel derivative. For further generalization of the solution, we assume $M > 0$ so that it yields

$$(35)\quad x(t) = e^{\lambda_1 t/2}\left\{y_1 e^{\sqrt{M}t/2} + y_2 e^{-\sqrt{M}t/2}\right\} + e^{\lambda_1 t/2}\Big\{\frac{e^{\sqrt{M}t/2}}{\sqrt{M}}\int_0^t f(\tau)e^{-\frac{(\lambda_1+\sqrt{M})\tau}{2}} - \frac{e^{-\sqrt{M}t/2}}{\sqrt{M}}\int_0^t f(\tau)e^{-\frac{(\lambda_1-\sqrt{M})\tau}{2}}\Big\},$$

where y_1 and y_2 are constants. We have successfully written the analytical solution of the forced fractional oscillator. In the different cases of the values of M, we are certain that the solution does exist. Next, we give an example of forced fractional oscillator and we present the amplitude analysis with respect to the fractional order ν.

Example: One of the common driving force of a mechanical damped system is given by $F(t) = F_0\cos(\omega t)$. Then starting from (26) and using (28), we have the term

$$(36)\qquad f(t) = F_0\left\{(-a\omega^2 + c)\cos(\omega t) - b\omega\sin(\omega t)\right\},$$

where

$$a = \frac{(1-\nu)^2}{1+(1-\nu)^2}, \quad b = \frac{2\nu(1-\nu)}{1+(1-\nu)^2}, \quad c = \frac{\nu^2}{1+(1-\nu)^2}$$

and the resulting auxiliary equation of motion follows the expression:

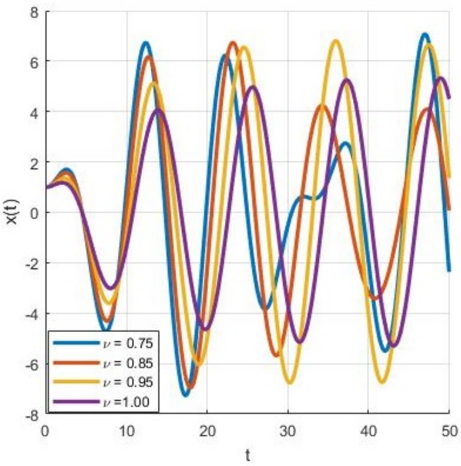
$$(37)\qquad \ddot{x}(t) + \lambda_1\dot{x}(t) + \lambda_2 x(t) = F_0\left\{(-a\omega^2 + c)\cos(\omega t) - b\omega\sin(\omega t)\right\}.$$

Now, applying the method shown in (35) and conducting a computer simulation on the displacement of a fractional oscillator, we present the figure below.

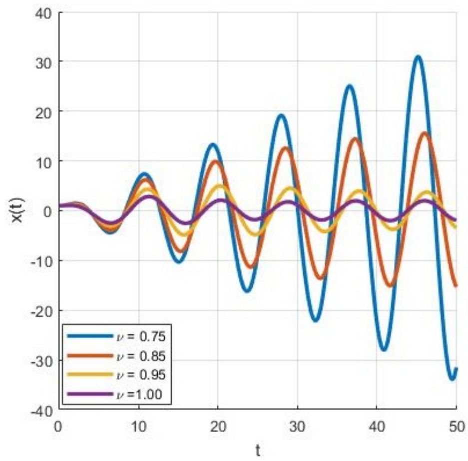
According to the presented simulation, the fractional-order gives a flexible dynamics of the consider fractional oscillator. We simulated for different cases of the driving frequency and natural frequency. In figure (4a), the case for $\omega < \omega_n$ is shown transient oscillation of the amplitude response is depicted relative to the fractional-order ν and the driving force. For the case $\omega = \omega_n$, the oscillation is unstable for lower ν and $\omega > \omega_n$, the higher amplitude of oscillation is observed.

Now, for further analysis, we shall give a brief discussion of the amplitude analysis. Consider the particular solution,

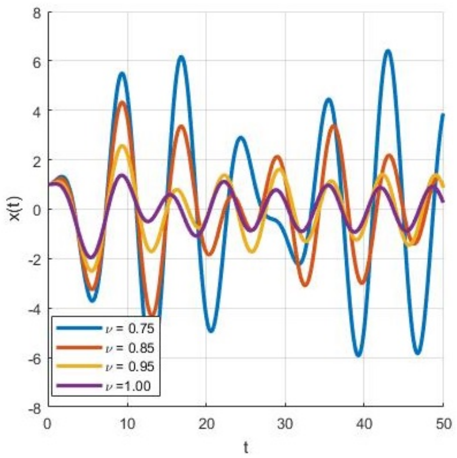
$$(38)\qquad x_p(t) = A\cos(\omega t) + B\sin(\omega t),$$



(a)



(b)



(c)

Fig. 4: Displacement of the forced fractional oscillator with nonsingular kernel and (a) $\omega < \omega_n$, (b) $\omega = \omega_n$, (c) $\omega > \omega_n$.

or the form,

(39) $$x_p(t) = X\cos(\omega t - \phi),$$

where $A = X\cos(\phi)$, $B = X\sin(\phi)$, and $\phi = \tan^{-1}\left(\frac{B}{A}\right)$. From the auxiliary equation of motion (31), we get

(40) $$A = F_0\left(\frac{(c\omega_n^2 - \omega^2)(c - a\omega^2) + b\omega_n^2\omega^2}{(c\omega_n^2 - \omega^2)^2 + b^2\omega_n^4\omega^2}\right),$$

(41) $$B = F_0 \left(\frac{(b\omega_n^2\omega)(c - a\omega^2) - b\omega(c\omega_n^2 - \omega^2)}{(c\omega_n^2 - \omega^2)^2 + b^2\omega_n^4\omega^2} \right).$$

Then we obtain the amplitude as

(42) $$X = F_0 \frac{1}{\sqrt{(c\omega_n^2 - \omega^2)^2 + b^2\omega_n^4\omega^2}}.$$

Such that the phase angle ϕ is given by

(43) $$\phi = \frac{(b\omega_n^2\omega)(c - a\omega^2) - b\omega(c\omega_n^2 - \omega^2)}{(c\omega_n^2 - \omega^2)(c - a\omega^2) + b\omega_n^2\omega^2}.$$

From Fig. (5), we have the amplitude plot versus the frequency of the driving force. We can observe three regions in the peaks of the amplitude relative to the imposed value of the fractional-order and the driving frequency. In the case of $\omega < \omega_n$, a lower damping regime can be observed for decreasing ν since the peak increases. However, increasing ν decreases the amplitude peak and gain another increase in amplitude at around $\omega \approx 0.6\omega_n$, amplitude peak gradually increases as $\omega \to \omega_n$ until it reaches an asymptotic value at $\nu = 1$ where there is no damping. Hence, classical observation of amplitude analysis occurs at around $\omega \geq \omega_n$. In general, around $\omega < 0.6\omega_n$, lower damping oscillation dominates the system because the excitation due to the driving force overcomes the damping imposed by the fractional-order ν.

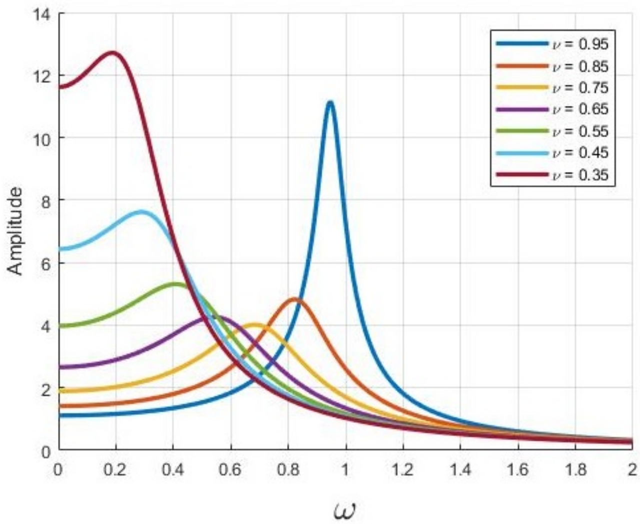


Fig. 5: Amplitude vs. frequency of the driving force.

Meanwhile, at case $1 > \omega > 0.6\omega_n$, the peak has a gradual increase making the system less damped. Lastly, for larger driving frequency, fractional scheme, and classical analysis converges.

5 Conclusion

This work presents the detailed derivation on the fractional equation of motion from the fractionally-modified Euler-Lagrange equation and we show that the fractional equation of motion can be reduced to the form of a classical ordinary differential equation, which the analytical solution is already established. We have shown the explicit form of the arising damping term due to the imposed memory kernel of the fractional derivative. The obtained damping term can significantly contribute to the non-idealness of the obtained fractional oscillator model, as shown from the simulated EGR. In general, the fractional-order ν can make the state of the system to be non-ideal and a dissipative effect can be observed. Also, the forced fractional oscillator was analyzed and the general solution was presented. The acquired general solution is consistent with the classical solution if $\nu \to 1$. Interestingly, different amplitude peaks can be observed at the lower driving frequency as presented in Fig. (5). Lastly, this paper gives additional insights into the effect of the fractional derivative operator on dynamical a systems.

Acknowledgment

The authors are highly grateful to the Department of Physics, Mindanao State University-Main Campus, Marawi City for the support shown during the process of conducting this work. N. A. Rangaig would like to acknowledge the Department of Science and Technology-Science Education Institute (DOST-SEI) for the scholarship grant that serves as a financial support throughout these past years. N. A. Rangaig would also like to express his warmth appreciation to his wife whose love and support are unfathomable. Lastly, the Authors would like to acknowledge the reviewers whose valuable suggestions and comments improved the final version of this manuscript.

Declaration

The authors declare that this work has no competing interest nor any party is affected prior to submission for publication.

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