

THE APPLICATION OF THE HODOGRAPH METHOD TO FREE SURFACE FLOW PROBLEM

MAY MANAL BOUNIF, ABDELKADER GASMI^{1*}

*Laboratory of Pure and Applied Mathematics,
Faculty of Mathematics and Informatics, University of M'sila, Algeria*

[Received: 29 December 2020. Accepted: 05 April 2021]

doi: <https://doi.org/10.55787/jtams.22.52.2.118>

ABSTRACT: The problem of the steady two-dimensional free-surface flow of a fluid under a sluice gate is considered. The hodograph method is used to solve this problem analytically for different values of the inclination angle of the gate wall. The obtained results agree with the experimental and numerical results given by Birkhoff & Zarantonello and Gasmi & Mekias respectively.

KEY WORDS: free-surface, zero gravity, hodograph transformation, incompressible flow.

1 INTRODUCTION

This work considers the problem of the steady free surface flow of a fluid under sluice gate. We consider the fluid region as shown in Fig 1. The fluid is assumed to be incompressible and inviscid. The flow is considered to be bidimensional, stationary, and irrotational. The effects of the superficial tension and gravity are neglected.

This problem belongs to the class of flows for which there are intersections between the free surface and a solid, for more details related on this subject see for example Birkhoff & Zarantonello [1], Batchelor [2] Bhimsen & Shivamoggi [3], Milne-Thomson [4] and Rutherford [5].

Several numerical studies have been realized on the subject; this includes the determination of the shape of the free-surface of the flow for a given solid form. Ackerberg & Liu [6] considered the flow of a fluid issuing from a slot with inclined walls using the finite difference method. Vanden-Broeck [7, 8] and Assavanat & Vanden-Broeck [9], used a series truncation technique and the boundary-integral method, to solve the problem of flow under a gate, whilst Gasmi & Mekias [10] and Gasmi [11, 12] numerically studied the jet flow and flow issuing from a Funnel. Other authors who studied similar problems include Cooker [13] Hureau & Weber [14],

*Corresponding author e-mail: abdelkader.gasmi@univ-msila.dz

Klokov & Sergeev [15], Pengp & Parker [16], Semenov & Wu [17] and Semenov et al [18, 19].

Our aim in this paper is to apply the hodograph method to solve the proposed problem analytically for several values of the inclination angle of the gate. This method proves useful to solve this problem analytically. The obtained results are compared with those calculated numerically using the series truncation method given in Gasmi & Mekias [10], and with those given in Birkhoff & Zarantonello [1], which shows that there is a good agreement between them.

The plan of this paper is as follows. In Section 2 we formulate the problem in the dimensionless variables. In Section 3 we introduce the analytical method based on the hodograph transformation to solve the considered problem. In Section 4 we present some exact solutions and their graphical representation for several values of the inclination angle, these results are also compared with previous ones given by Birkhoff & Zarantonello [1] and Gasmi & Mekias [10]. Finally, this work is concluded in Section 5.

2 FORMULATION OF PROBLEM

Let us consider the motion of a two-dimensional flow of a fluid under a gate with inclined edge, we denote the inclination angle between the edge of the gate and the horizontal by α , the opening of the gate is of width d see Fig. 1. The flow is considered to be incompressible, inviscid and steady. The effects of gravity and the surface tension are neglected.

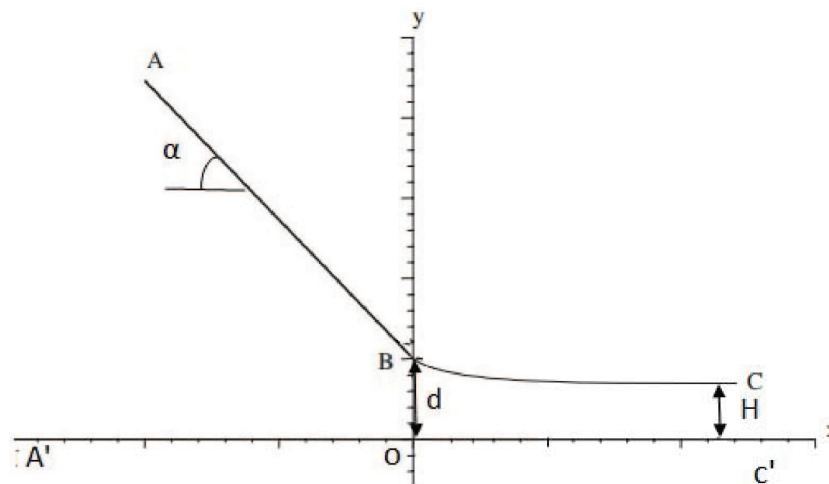


Fig. 1: Sketch of the flow and of the coordinates. The width of the opening is d , the depth of the flow at infinity is H and the inclination angle of the gate is α .

We introduce the cartesian coordinates with the Bottom $A'OC'$ on the x axis and the y axis is vertically through the point B . Far upstream the fluid is at rest, and far downstream, we assume that the velocity approaches a constant U and the depth of the fluid tends to a constant H .

The dimensionless variables are defined by choosing U as the unit velocity and d as the unit length. We introduce the potential function φ and the stream function ψ . Without loss of generality we choose $\varphi = 0$ at $(x, y) = (0, 1)$ and $\psi = 1$ on the streamline ABC . It follows from the choice of the dimensionless variables that $\psi = 0$ on the streamline $A'OC'$. In order to use the conformal mapping techniques, we consider the flow in the complex plane $z = x + iy$ and the complex potential function

$$(1) \quad f(z) = \varphi(z) + i\psi(z).$$

By this the mathematical problem is to determine the variable ψ which verifies the following conditions:

$$(2) \quad \Delta\psi = 0,$$

in the interior of the flow field.

$$(3) \quad \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = 1,$$

on the free surface BC .

$$(4) \quad \frac{\partial\psi}{\partial x} = 0,$$

on the horizontal wall $A'OC'$.

$$(5) \quad \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} \tan \alpha,$$

on the inclined wall AB .

Next we introduce the complex velocity

$$(6) \quad \eta(z) = C \frac{df}{dz} = u - iv = qe^{-i\theta}.$$

Here $C = H/d$ is the contraction coefficient which is defined as the ratio of the thickness H of the flow as $x \rightarrow \infty$ to length of the opening of the gate d , $q = \sqrt{u^2 + v^2}$ and θ is the velocity direction relative to the real axis.

The two components of the potential function (the potential velocity and the stream function) satisfies the Cauchy-Riemann conditions, then

$$(7) \quad \frac{\partial \varphi}{\partial q} = -\frac{1}{q} \frac{\partial \psi}{\partial \theta}, \quad \frac{\partial \psi}{\partial q} = \frac{1}{q} \frac{\partial \varphi}{\partial \theta}.$$

The transformation (6) maps the flow field of z plane to a sector of the disk such as the free surface is mapped into a part of the circumference of the circle in the hodograph plane (see Fig. 2).

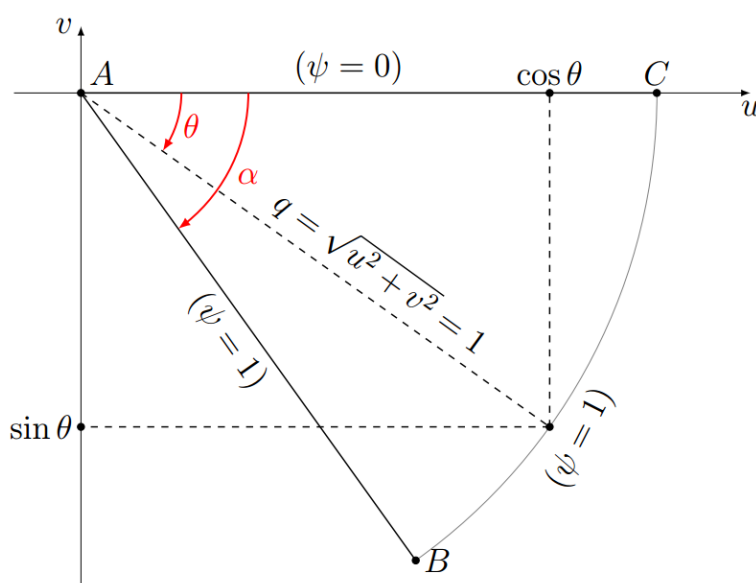


Fig. 2: Hodograph plane (u, v) .

Then the mathematical problem is to determine the stream function ψ as a function of the two variables q and θ which verifies the following equation:

$$(8) \quad \frac{\partial^2 \psi}{\partial q^2} + \frac{1}{q} \frac{\partial \psi}{\partial q} + \frac{1}{q^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0,$$

in the flow field.

3 ANALYTICAL SOLUTION

In the hodograph plane the conditions (3), (4) and (5) are easily transformed into the conditions

$$(9) \quad \psi(1, \theta) = \psi(q, -\alpha) = 1; \quad -\alpha \leq \theta \leq 0, \quad 0 < q < 1 \text{ on } BC \text{ and } AB$$

and

$$(10) \quad \psi(q, 0) = 0; \quad 0 \leq q \leq 1, \quad \text{on } A'OC'.$$

Then to solve the above considered problem defined by Eq. (8) and the conditions (9) and (10), we pose that

$$(11) \quad \psi(q, \theta) = \xi(q, \theta) - \frac{\theta}{\alpha}.$$

As ξ satisfies the following problem

$$(12) \quad \begin{cases} \frac{\partial^2 \xi}{\partial q^2} + \frac{1}{q} \frac{\partial \xi}{\partial q} + \frac{1}{q^2} \frac{\partial^2 \xi}{\partial \theta^2} = 0, \\ \xi(q, 0) = \xi(q, -\alpha) = 0, \quad 0 \leq q \leq 1, \\ \xi(1, \theta) = (1 + \theta/\alpha), \quad -\alpha \leq \theta \leq 0. \end{cases}$$

By using the separation variable method, firstly we seek the solution of the problem (12) in the form

$$(13) \quad \xi(q, \theta) = h(q)g(\theta),$$

by substituting the product solution form (13) into (12), we converted a partial differential equation into two ordinary differential equations. Each differential equation involves only one of the independent variables q or θ , we obtain

$$(14) \quad \begin{cases} g''(\theta) + \lambda^2 g(\theta) = 0, \\ g(0) = g(-\alpha) = 0, \end{cases}$$

where λ is constant.

The solution of (14) is given in the form:

$$(15) \quad g(\theta) = A \cos(\lambda\theta) + B \sin(\lambda\theta),$$

and after using the conditions of (12) we find that:

$$(16) \quad \lambda = \{\lambda_n, n \in N^* \quad \text{where} \quad \lambda_n = -\frac{n\pi}{\alpha}\},$$

and by the associated eigenfunctions λ_n the equation (15) is given as:

$$(17) \quad g_n(\theta) = B_n \sin\left(-\frac{n\pi}{\alpha}\theta\right), \quad n = 1, 2, \dots,$$

then, the solution of the problem (12) is given by

$$(18) \quad \xi(q, \theta) = B_n(q) \sin\left(-\frac{n\pi}{\alpha}\theta\right), \quad n = 1, 2, \dots$$

The general solution of the problem can be obtained by the superposition of solution of the homogeneous problem (12) of the form

$$(19) \quad \xi(q, \theta) = \sum_{n=1}^{\infty} B_n(q) \sin\left(-\frac{n\pi}{\alpha}\theta\right).$$

By replacing (19) in (12) and after using Euler's method, we get:

$$(20) \quad \xi(q, \theta) = -\frac{2}{\alpha} \sum_{n=1}^{\infty} \frac{\alpha}{n\pi} q^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\theta\right),$$

The stream function ψ is given by:

$$(21) \quad \psi(q, \theta) = -\frac{2}{\alpha} \sum_{n=1}^{\infty} \frac{\alpha}{n\pi} q^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\theta\right) - \frac{1}{\alpha}\theta,$$

for (7) and (21), we obtains the velocity potential φ

$$(22) \quad \varphi(q, \theta) = \frac{2}{\alpha} \sum_{n=1}^{\infty} \frac{\alpha}{n\pi} q^{\frac{n\pi}{\alpha}} \cos\left(\frac{n\pi}{\alpha}\theta\right) + \frac{1}{\alpha} \ln(q).$$

Thus, the potential function is:

$$(23) \quad f(q, \theta) = \frac{2}{\alpha} \sum_{n=1}^{\infty} \frac{\alpha}{n\pi} \eta^{\frac{n\pi}{\alpha}} + \frac{1}{\alpha} \ln(\eta),$$

Using the relation (6), we obtain:

$$(24) \quad dz = \frac{2C}{\alpha} \left(\frac{\eta^{\frac{\pi}{\alpha}-2}}{1 - \eta^{\frac{\pi}{\alpha}}} \right) d\eta + \frac{C}{\alpha\eta^2} d\eta$$

4 FREE STREAMLINE FORMS FOR VARIOUS VALUE OF α

- For $\alpha = \pi/4$,

substitute this value in equation (24) we get:

$$(25) \quad dz = \frac{8C}{\pi} \left(\frac{\eta^2}{1 - \eta^4} \right) d\eta + \frac{4C}{\pi\eta^2} d\eta,$$

by integrating both sides of this equation, we get:

$$(26) \quad z = \frac{4C}{\pi} \left(\operatorname{arctanh}(\eta) - \arctan(\eta) - \frac{1}{\eta} \right) + cst1,$$

On the free streamline we have:

$$(27) \quad \eta = e^{-i\theta}, \quad \text{with} \quad -\frac{\pi}{4} \leq \theta \leq 0.$$

Then we have:

$$(28) \quad cst1 = -\frac{2C}{\pi} \left[\operatorname{arctanh}\left(\frac{\sqrt{2}}{2}\right) - \sqrt{2} - \frac{\pi}{2} \right] \\ + i \left[1 - \frac{2C}{\pi} \left(-\operatorname{arctanh}\left(\frac{\sqrt{2}}{2}\right) + \sqrt{2} - \frac{\pi}{2} \right) \right].$$

By separating the real parts and the imaginary parts, we obtain:

$$(29) \quad x = \frac{2C}{\pi} \left(\operatorname{arctanh}(\cos \theta) - \operatorname{arctanh}\frac{\sqrt{2}}{2} - 2 \cos \theta + \sqrt{2} \right), \\ y = 1 - \frac{2C}{\pi} \left(\operatorname{arctanh}[\cos(\theta + \pi/2)] - \operatorname{arctanh}\frac{\sqrt{2}}{2} + 2 \sin(\theta) + \sqrt{2} \right).$$

The amplitude of the flow at the infinity is calculated as:

$$(30) \quad C = \frac{H}{d} = \lim_{\theta \rightarrow 0} y(\theta) = 1 - \frac{2C}{\pi} \left(-\operatorname{arctanh}\left(\frac{\sqrt{2}}{2}\right) + \sqrt{2} \right) \\ \Rightarrow C = \frac{\pi}{\left[\pi - 2 \operatorname{arctanh}\left(\frac{\sqrt{2}}{2}\right) + 2\sqrt{2} \right]}.$$

Figure 3 presents the comparison of the exact form of the free streamlines with the numerical solution calculated by the series truncation method for $\alpha = \pi/4$.

- For $\alpha = \pi/2$,

it becomes that the values of the angle θ are given by $-\pi/2 \leq \theta \leq 0$, and from the equation (24), we get:

$$(31) \quad dz = \frac{4C}{\pi} \left(\frac{1}{1-\eta^2} \right) d\eta + \frac{2C}{\pi\eta^2} d\eta,$$

by integrating both sides of this Equation, we get:

$$z = \frac{2C}{\pi} \left(\ln \left(\frac{\eta+1}{1-\eta} \right) - \frac{1}{\eta} \right) + cst2.$$

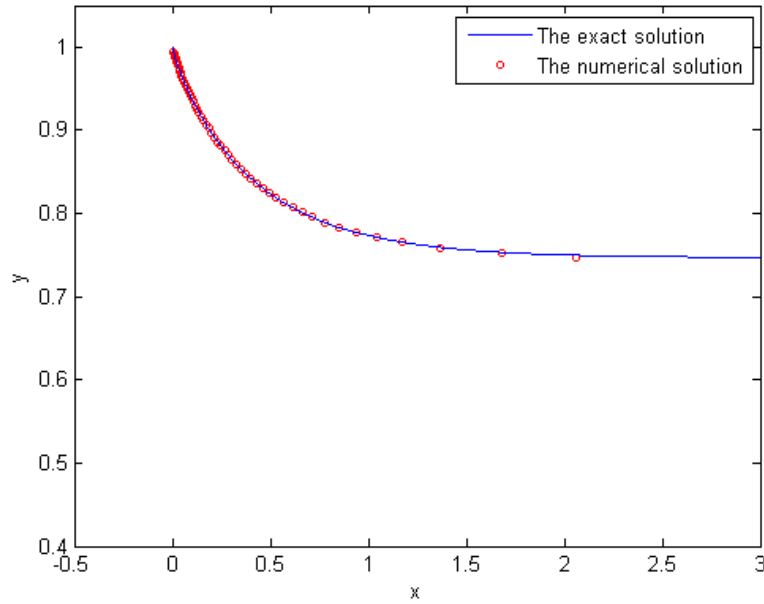


Fig. 3: Comparison of the analytical solution and the numerical solutions for $\alpha = \frac{\pi}{4}$

On the free streamline we have:

$$(32) \quad \eta = e^{-i\theta}, \quad \text{with} \quad -\frac{\pi}{2} \leq \theta \leq 0.$$

On the other hand, by using the conditions at point B ($\theta = -\pi/2$ and $z = i$) we find the free streamlines:

$$(33) \quad \begin{aligned} x &= \frac{2C}{\pi}(\operatorname{arctanh}(\cos \theta) - \cos \theta). \\ y &= 1 - \frac{2C}{\pi}(\sin \theta + 1). \end{aligned}$$

The amplitude of the flow at the infinity is calculated as:

$$(34) \quad C = \frac{H}{d} = \lim_{\theta \rightarrow 0} y(\theta) = 1 - \frac{2C}{\pi} \Rightarrow C = \frac{\pi}{(\pi + 2)}.$$

This solution is also obtained in [2] by the free streamline theory methods and the comparison of this result with the numerical solution obtained by [10] is shown in Fig. 4.

The free surface shapes for different values of the inclination angle α is presented in Fig. 5

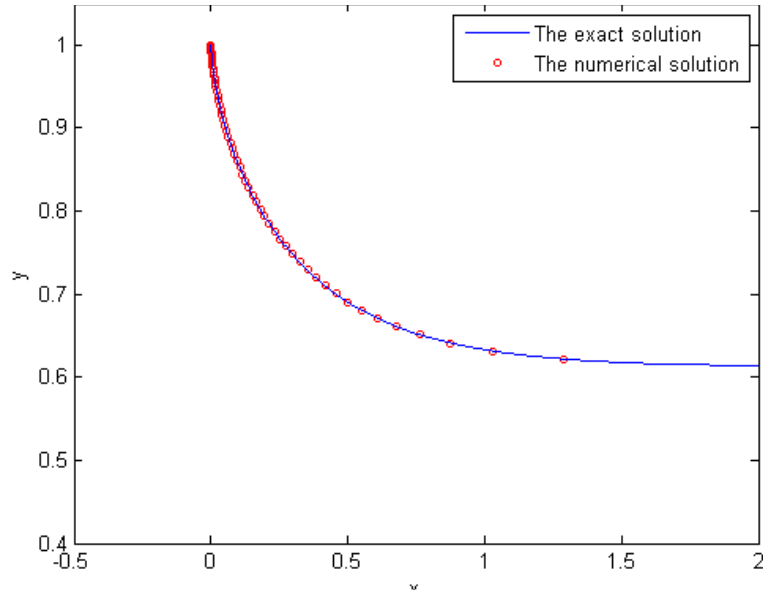


Fig. 4: Comparison of the analytical solution and the numerical solutions for $\alpha = \frac{\pi}{2}$

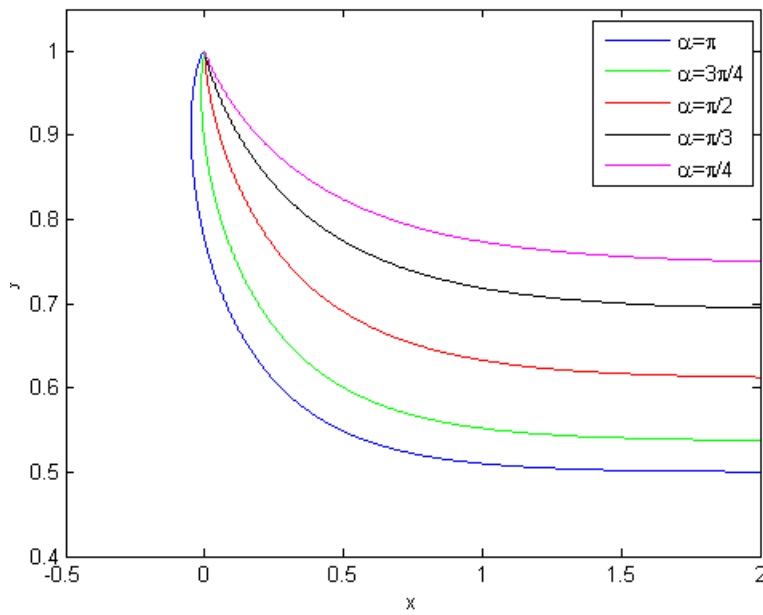


Fig. 5: The variation of the free surface with respect to α

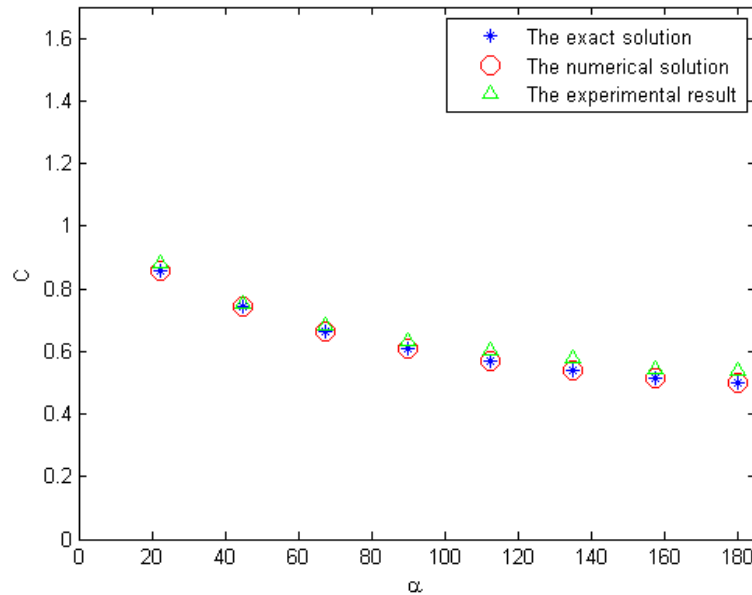


Fig. 6: Comparison of the analytical values, experimental data and the numerical results of the contraction coefficient C for several value of the inclination angle α

The comparison of the calculated coefficient of contraction of the flow for different values of the inclination angle α with the experimental results given in [1] and the numerical solutions given in [10] is shown in Fig. 6.

Finally through all the presented results we observe that the obtained results using the hodograph method agree with those calculated numerically by series truncation method [10].

5 CONCLUSION

In this paper, hodograph transformation method has been successfully applied to solve two-dimensional steady flow problem of a flow under sluice gate, this method was used for free nonlinear boundary problem. The obtained analytically results agree with those calculated numerically by Gasmi and Mekias [10], using the series-truncation technique. Also this method is very simple and straightforward, which reduces the propped problem to a one-dimensional problem.

REFERENCES

- [1] G. BIRKHOFF, E.H. ZARANTONELLO E.H (1957) "Jet, Wakes and Cavities". Academic Press, New York.
- [2] G.K. BATCHELOR (1967) "An Introduction to Fluid Dynamics". Cambridge: Cambridge University Press.
- [3] BHIMSEN K. SHIVAMOGGI (1998) "Theoretical Fluid Dynamics". 2nd Ed., Wiley-Interscience.
- [4] L.M. MILNE-THOMSON (1968) "Theoretical Hydrodynamics". 5th Ed., Macmillan, New York.
- [5] D.E. RUTHERFORD (1959) "Fluid Dynamics". Oliver & Boyd.
- [6] R.C. ACKERBERG, TA-JO LIU (1987) The effects of capillarity on the contraction coefficient of a jet emanating from a slot. *The Physics of Fluids* **30** 289296.
- [7] J-M. VANDEN-BROECK (1986) Flow under a gate. *The Physics of Fluids* **29** 3148-3151.
- [8] J-M. VANDEN-BROECK (1997) Numerical calculations of the free-surface flow under a sluice gate. *Journal of Fluid Mechanics* **330** 339-347.
- [9] J. ASAVANAT, J-M. VANDEN-BROECK (1996) Nonlinear free-surface flow emerging from vessels and flows under sluice gate. *The ANZIAM Journal* **38** 63-86.
- [10] A. GASMI, H. MEKIAS (2003) The effect of surface tension on the contraction coefficient of a jet. *Journal of Physics A: Mathematical and General* **363** 851-862.
- [11] A. GASMI (2014) Two-dimensional cavitating flow past an oblique plate in a channel. *Journal of Computational and Applied Mathematics* **259** 851-862.
- [12] A. GASMI (2014) Numerical Study of Two-Dimensional Jet Flow Issuing from a Funnel. In: A. Ansari (eds) "Advances in Applied Mathematics. Springer Proceedings in Mathematics & Statistics", vol 87. Springer, Cham., pp. 161-169.
- [13] M.J. COOKER (1999) Potential flow for a steady jet in a wedge. *The Quarterly Journal of Mechanics and Applied Mathematics* **52** 127-140.
- [14] J. HUREAU, R. WEBER (1998) Impinging free jets of ideal fluid. *Journal of Fluid Mechanics* **372** 357-374.
- [15] V.V. KLOKOV, D.E. SERGEEV (2012) Simulation of the stationary electrochemical surface treatment by two asymmetric cathode plates. *Journal of Applied Mechanics and Technical Physics* **536** 961-965.
- [16] W. PENG, D.F. PARKER (1997) An ideal fluid jet impinging on an uneven wall. *Journal of Fluid Mechanics* **333** 231-255.
- [17] Y.A. SEMENOV, G.X. WU (2013) Asymmetric impact between liquid and solid wedges. *Proceedings of the Royal Society A* **4691** 20120203.
- [18] Y.A. SEMENOV, G.X. WU, A.A. KOROBKIN (2015) Impact of liquids with different densities. *Journal of Fluid Mechanics* **766** 5-27.
- [19] Y.A. SEMENOV , G.X. WU, J.M. OLIVER (2013) Splash jet generated by collision of two liquid wedges. *Journal of Fluid Mechanics* **737** 132-145.