

A METHOD FOR THE SOLUTION OF UNIFORM TORSION OF CARTESIAN ORTHOTROPIC BAR

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ABSTRACT: The object of this paper is the Saint-Venant torsion of homogeneous orthotropic bar with solid cross section. The solution of the uniform torsion of orthotropic bar with a suitable coordinate transformation is reduced to the solution of the torsion problem of an isotropic solid cross section. Both the torsion function and Prandtl's stress function formulations are used. A new method is given to determine the shape of orthotropic cross section which has the maximal torsional rigidity for a given cross sectional area.

KEY WORDS: Saint-Venant torsion, orthotropic, solid cross section, torsional rigidity.

1 INTRODUCTION

The Saint-Venant torsion of homogeneous orthotropic linearly elastic bars has been the subject of several studies from both theoretical and numerical viewpoints. There are several books [1–5] and [6] which give the detailed analysis of the problem of uniform torsion of anisotropic elastic bars. Chen and Wei studied the Saint-Venant torsion of anisotropic bars with affine transformation [7]. They proved external properties of torsional rigidity for anisotropic composite bars and they determined the shape of anisotropic homogeneous bar which has the maximum value of torsional rigidity for a given cross-sectional area [7]. The non-warping property of the twisted anisotropic cross-section is discussed in papers [7–11].

Cartesian anisotropic materials having linear elastic stress-strain relations can possess a unique type of coupling between the x , y and z directions. The Cartesian orthotropy is a lower level of the anisotropy it has weaker coupling between the x , y and z directions as in Cartesian anisotropy. The material anisotropy in the case of Saint-Venant torsion (uniform torsion) has at least one plane of elastic symmetry

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which is normal to the axis of the bar [1–7, 9–11]. In this case the torsional and bending deformations are separated [2].

The solution of the Saint-Venant torsion problem of orthotropic homogeneous bar is reduced with a suitable coordinate transformation to the problem of Saint-Venant torsion of an isotropic bar. The torsion function and Prandtl's stress function formulation are also presented. By using the results of [12], which says that for given cross-sectional area the optimum shape of the solid cross-section is circle having the maximum value of torsional rigidity, we determine the optimum shape of the orthotropic cross-section by the use of the coordinate transformation introduced in this paper. We note, the optimum shape of Cartesian anisotropic cross section at first was derived by Banichuk [13, 14].

It must be mentioned that the affine transformation of the Cartesian coordinates x and y to reduce the anisotropic Saint-Venant torsion problem

$$(1) \quad k_{44} \frac{\partial^2 P}{\partial x^2} - 2k_{45} \frac{\partial^2 P}{\partial x \partial y} + k_{55} \frac{\partial^2 P}{\partial y^2} = -2 \quad (x, y) \in A_{xy},$$

$$(2) \quad P(x, y) = 0 \quad (x, y) \in \partial A_{xy}$$

to an isotropic one has been used by [5], and [2].

In Eqs. (1) and (2) $P = P(x, y)$ is the Prandtl's stress function, k_{44} , k_{55} and $k_{45} = k_{55}$ are the shear flexibility coefficients and A_{xy} is the cross section and ∂A_{xy} is the boundary curve of A_{xy} . [5] considered the next affine changes of variables x and y to ξ and η

$$(3) \quad \xi = \alpha x + \beta y, \quad y = \gamma x + \delta y, \quad \varepsilon = \alpha\delta - \beta\gamma \neq 0.$$

Introducing the new variables ξ and η into Eqs. (1) and (2) gives ([5])

$$(4) \quad \frac{1}{k} \left(\frac{\partial^2 p}{\partial \xi^2} + \frac{\partial^2 p}{\partial \eta^2} \right) = -2 \quad (\xi, \eta) \in A_{\xi\eta},$$

$$(5) \quad p(\xi, \eta) = 0 \quad (\xi, \eta) \in \partial A_{\xi\eta}.$$

Here

$$(6) \quad p(\xi, \eta) = P \left(\frac{\delta\xi - \beta\eta}{\varepsilon}, \frac{-\gamma\xi + \alpha\eta}{\varepsilon} \right),$$

and k is the shear modulus of the isotropic beam. We have [5]

$$(7) \quad k_{44}\alpha^2 - 2k_{45}\alpha\beta + k_{55}\beta^2 = k_{44}\gamma^2 - 2k_{45}\gamma\delta + k_{55}\delta^2 = 1/k,$$

$$(8) \quad k_{44}\alpha\gamma - k_{45}(\alpha\delta + \beta\delta) + k_{55}\beta\delta = 0.$$

The affine transformation Eq. (3) is area holding mapping if $\varepsilon = 1$.

A matrix representation of the affine transformation Eq. (3) is used by [7] to analyse the Saint-Venant torsion of anisotropic beam. In this paper the Saint-Venant's torsion of Cartesian orthotropic beam is reduced to the Saint-Venant's torsion of isotropic beam. The shear modulus of the isotropic beam is the geometrical mean of the principle shear moduli of the considered anisotropic beam.

2 GOVERNING EQUATIONS FOR ORTHOTROPIC BAR

We consider the Saint-Venant torsion of orthotropic homogeneous linearly elastic bar with a solid cross section (see Fig. 1). The unit vectors of the Cartesian coordinate system Oxy are \mathbf{e}_x and \mathbf{e}_y . The cross section of the bar is denoted by A and the boundary curve of A is ∂A . Arc-length defined on ∂A is denoted by s , $\mathbf{t} = \mathbf{t}(s)$ and $\mathbf{n} = \mathbf{n}(s)$ represent the tangential and normal vectors to boundary curve ∂A . The equation of the boundary curve ∂A is

$$(9) \quad F(x, y) = 0 \quad (x, y) \in \partial A$$

From Eq. (9) it follows that

$$(10) \quad dF(x, y) = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = \left(\frac{\partial F}{\partial x} \mathbf{e}_x + \frac{\partial F}{\partial y} \mathbf{e}_y \right) \cdot d\mathbf{r} \\ = \left(\frac{\partial F}{\partial x} \mathbf{e}_x + \frac{\partial F}{\partial y} \mathbf{e}_y \right) \cdot \left(\frac{dx}{ds} \mathbf{e}_x + \frac{dy}{ds} \mathbf{e}_y \right) ds = 0 \quad (x, y) \in \partial A.$$

In Eq. (10) the dot between two vectors denotes their scalar product. It is evident the unit tangential vector $\hat{\mathbf{t}}$ (Fig. 1) is

$$(11) \quad \hat{\mathbf{t}} = \frac{dx}{ds} \mathbf{e}_x + \frac{dy}{ds} \mathbf{e}_y, \quad \overrightarrow{OP} = x(s) \mathbf{e}_x + y(s) \mathbf{e}_y$$

and the normal vector \mathbf{n} can be expressed in terms of $F = F(x, y)$ as

$$(12) \quad \mathbf{n} = n_x \mathbf{e}_x + n_y \mathbf{e}_y = \frac{\partial F}{\partial x} \mathbf{e}_x + \frac{\partial F}{\partial y} \mathbf{e}_y.$$

The shear moduli of orthotropic material in xz and yz directions are denoted by G_x G_y .

2.1 TORSION AND PRANDTL'S STRESS FUNCTION FORMULATION FOR ORTHOTROPIC BAR

Let $\phi = \phi(x, y)$ be the torsion function of the cross section shown in Fig. 1 and the rate of twist is represented by ϑ and the applied torque is T . It is known that

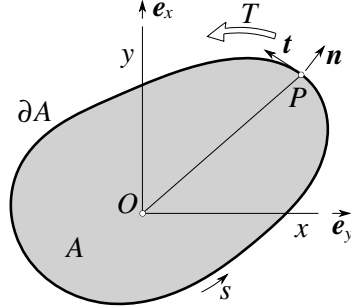


Fig. 1: Solid orthotropic cross section.

the solution of the Saint-Venant's torsion of Cartesian orthotropic elastic bar can be obtained from the next Neumann' type boundary value problem [2,3,6]

$$(13) \quad G_x \frac{\partial^2 \phi}{\partial x^2} + G_y \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (x, y) \in A,$$

$$(14) \quad G_x \left(\frac{\partial \phi}{\partial x} - y \right) n_x + G_y \left(\frac{\partial \phi}{\partial y} + x \right) n_y = 0 \quad (x, y) \in \partial A,$$

where

$$(15) \quad n_x = \frac{\partial F}{\partial x}, \quad n_y = \frac{\partial F}{\partial y}.$$

In terms of Prandtl's stress function $U = U(x, y)$ the solution of the torsion problem of the orthotropic homogeneous bar is obtained from the following Dirichlet-type boundary value problem

$$(16) \quad \frac{1}{G_y} \frac{\partial^2 U}{\partial x^2} + \frac{1}{G_x} \frac{\partial^2 U}{\partial y^2} = -2 \quad (x, y) \in A,$$

$$(17) \quad U = 0 \quad (x, y) \in \partial A.$$

Knowing the solutions of boundary value problem Eqs. (13), (14) and Eqs. (16), (17) the stress field τ_{xz} and τ_{yz} can be computed by the next formulae [1-6]

$$(18) \quad \tau_{xz} = \vartheta G_x \left(\frac{\partial \phi}{\partial x} - y \right) = \vartheta \frac{\partial U}{\partial y},$$

$$(19) \quad \tau_{yz} = \vartheta G_y \left(\frac{\partial \phi}{\partial y} + x \right) = -\vartheta \frac{\partial U}{\partial x},$$

where

$$(20) \quad \vartheta = \frac{T}{R}.$$

In Eq. (20) T is the applied torque and R denotes the torsional rigidity of orthotropic solid cross section A , it can be computed in terms of $\phi = \phi(x, y)$ or in terms of $U = U(x, y)$ as

$$(21) \quad R = \int_A \left[x G_y \left(\frac{\partial \phi}{\partial y} + x \right) - y G_x \left(\frac{\partial \phi}{\partial x} - y \right) \right] dA$$

$$= \int_A (G_y x^2 + G_x y^2) dA - \int_A \left[G_x \left(\frac{\partial \phi}{\partial x} \right)^2 + G_y \left(\frac{\partial \phi}{\partial y} \right)^2 \right] dA,$$

$$(22) \quad R = 2 \int_A U dA = \int_A \left[\frac{1}{G_y} \left(\frac{\partial U}{\partial x} \right)^2 + \frac{1}{G_x} \left(\frac{\partial U}{\partial y} \right)^2 \right] dA.$$

2.2 TORSION AND PRANDTL'S STRESS FUNCTION FORMULATIONS FOR ISOTROPIC BAR

In the case of isotropic bar

$$(23) \quad G_x = G_y = G.$$

Substitution of Eq. (23) into the equations of orthotropic beam we get the boundary value problems of Saint-Venant torsion for isotropic material. The cross section a of the isotropic bar is a simply connected plane domain in the plane OXY (see Fig. 2). The boundary curve of a is denoted by ∂a , whose equation is

$$(24) \quad f(X, Y) = 0 \quad (X, Y) \in \partial a.$$

The normal vector to the boundary curve ∂a can be represented as (Fig. 2)

$$(25) \quad \mathbf{N} = N_x \mathbf{e}_x + N_y \mathbf{e}_y = \frac{\partial f}{\partial X} \mathbf{e}_x + \frac{\partial f}{\partial Y} \mathbf{e}_y.$$

For isotropic bar $\varphi = \varphi(X, Y)$ denotes the torsion function and $u = u(X, Y)$ denotes the Prandtl's stress functions. From Eqs. (13-14) and (16-17) it follows that

$$(26) \quad \frac{\partial^2 \varphi}{\partial X^2} + \frac{\partial^2 \varphi}{\partial Y^2} = 0 \quad (X, Y) \in a,$$

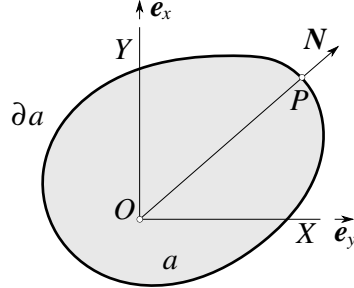


Fig. 2: Solid isotropic cross section.

$$(27) \quad \left(\frac{\partial \varphi}{\partial X} - Y \right) N_x + \left(\frac{\partial \varphi}{\partial Y} + X \right) N_y = 0 \quad (X, Y) \in \partial a$$

and

$$(28) \quad \frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2} = -2G \quad (X, Y) \in a,$$

$$(29) \quad u = 0 \quad (X, Y) \in \partial a.$$

Shearing stresses in isotropic bar are denoted by t_{xz} and t_{yz} and we have [1–6]

$$(30) \quad t_{xz} = \nu G \left(\frac{\partial \varphi}{\partial X} - Y \right) = \nu \frac{\partial u}{\partial Y},$$

$$(31) \quad t_{yz} = \nu G \left(\frac{\partial \varphi}{\partial Y} + X \right) = -\nu \frac{\partial u}{\partial X}.$$

The torsional rigidity of isotropic solid cross section is indicated by r . We have

$$(32) \quad r = G \int_a \left[X \left(\frac{\partial \varphi}{\partial Y} + X \right) - Y \left(\frac{\partial \varphi}{\partial X} - Y \right) \right] da \\ = G \int_a (X^2 + Y^2) da - G \int_a \left[\left(\frac{\partial \varphi}{\partial X} \right)^2 + \left(\frac{\partial \varphi}{\partial Y} \right)^2 \right] da.$$

and

$$(33) \quad r = 2 \int_a u(X, Y) da = \frac{1}{G} \int_a \left[\left(\frac{\partial u}{\partial X} \right)^2 + \left(\frac{\partial u}{\partial Y} \right)^2 \right] da.$$

3 SOLUTION OF TORSION PROBLEM OF ORTHOTROPIC CROSS SECTION WITH COORDINATE TRANSFORMATION

We consider the next coordinate transformation

$$(34) \quad x \rightarrow X = x \sqrt[4]{\frac{G_y}{G_x}}, \quad y \rightarrow Y = y \sqrt[4]{\frac{G_x}{G_y}} \quad (x, y) \in A \cup \partial A.$$

The inverse of transformation given by Eq. (34)

$$(35) \quad X \rightarrow x = X \sqrt[4]{\frac{G_x}{G_y}}, \quad Y \rightarrow y = Y \sqrt[4]{\frac{G_y}{G_x}} \quad (X, Y) \in a \cup \partial a.$$

The one-to-one map Eq. (34) preserving the area, this fact follows from next equation

$$(36) \quad dA = dx dy = \left| \frac{\partial(x, y)}{\partial(X, Y)} \right| dX dY = \begin{vmatrix} \sqrt[4]{\frac{G_x}{G_y}} & 0 \\ 0 & \sqrt[4]{\frac{G_y}{G_x}} \end{vmatrix} da = da$$

that is we have

$$(37) \quad A = \int_A dA = \int_a \left| \frac{\partial(x, y)}{\partial(X, Y)} \right| da = \int_a da = a.$$

Let $\varphi = \varphi(X, Y)$ be defined as

$$(38) \quad \varphi(X, Y) = \phi \left(X \sqrt[4]{\frac{G_x}{G_y}}, Y \sqrt[4]{\frac{G_y}{G_x}} \right).$$

It is very easy to show the validity of Eqs. (39) and (40)

$$(39) \quad \frac{\partial \phi}{\partial x} = \frac{\partial \varphi}{\partial X} \sqrt[4]{\frac{G_y}{G_x}}, \quad \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial X^2} \sqrt{\frac{G_y}{G_x}},$$

$$(40) \quad \frac{\partial \phi}{\partial y} = \frac{\partial \varphi}{\partial Y} \sqrt[4]{\frac{G_x}{G_y}}, \quad \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \varphi}{\partial Y^2} \sqrt{\frac{G_x}{G_y}}.$$

Substitution of Eqs. (39), (40) into Eqs. (13-14) gives

$$(41) \quad \frac{\partial^2 \varphi}{\partial X^2} + \frac{\partial^2 \varphi}{\partial Y^2} = 0,$$

$$(X, Y) \in a = \left\{ X, Y \mid X = x \sqrt[4]{\frac{G_y}{G_x}}, Y = y \sqrt[4]{\frac{G_x}{G_y}}, (x, y) \in A \right\},$$

$$(42) \quad \sqrt{G_x G_y} \left[\left(\frac{\partial \varphi}{\partial X} - Y \right) \frac{\partial f}{\partial X} + \left(\frac{\partial \varphi}{\partial Y} + X \right) \frac{\partial f}{\partial Y} \right] = 0$$

$$(X, Y) \in \partial a = \left\{ X, Y \mid X = x \sqrt[4]{\frac{G_y}{G_x}}, Y = y \sqrt[4]{\frac{G_x}{G_y}}, (x, y) \in \partial A \right\}.$$

Torsional rigidity of the isotropic cross section in terms of $\varphi = \varphi(X, Y)$ can be computed from the next formula

$$(43) \quad r = \sqrt{G_x G_y} \int_a \left[X \left(\frac{\partial \varphi}{\partial Y} + X \right) - Y \left(\frac{\partial \varphi}{\partial X} - Y \right) \right] da$$

$$= \sqrt{G_x G_y} \int_a (X^2 + Y^2) da - \sqrt{G_x G_y} \int_a \left[\left(\frac{\partial \varphi}{\partial X} \right)^2 + \left(\frac{\partial \varphi}{\partial Y} \right)^2 \right] da.$$

Let $u = u(X, Y)$ be defined as

$$(44) \quad u(X, Y) = U \left(X \sqrt[4]{\frac{G_x}{G_y}}, Y \sqrt[4]{\frac{G_y}{G_x}} \right).$$

From Eq. (38) it follows that

$$(45) \quad \frac{\partial U}{\partial x} = \frac{\partial u}{\partial X} \sqrt[4]{\frac{G_y}{G_x}}, \quad \frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 u}{\partial X^2} \sqrt{\frac{G_y}{G_x}},$$

$$(46) \quad \frac{\partial U}{\partial y} = \frac{\partial u}{\partial Y} \sqrt[4]{\frac{G_x}{G_y}}, \quad \frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 u}{\partial Y^2} \sqrt{\frac{G_x}{G_y}},$$

The correctness of Eqs. (45) and (46) follows from the rule of differentiation of the composite functions. Substitution of Eqs. (45) and (46) into Eq. (16), (17) gives

$$(47) \quad \frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2} = -2\sqrt{G_x G_y}$$

$$(X, Y) \in a = \left\{ X, Y \mid X = x \sqrt[4]{\frac{G_y}{G_x}}, Y = y \sqrt[4]{\frac{G_x}{G_y}}, (x, y) \in A \right\},$$

$$(48) \quad u(X, Y) = 0$$

$$(X, Y) \in \partial a = \left\{ X, Y \mid X = x \sqrt[4]{\frac{G_y}{G_x}}, Y = y \sqrt[4]{\frac{G_x}{G_y}}, (x, y) \in \partial A \right\}.$$

Torsional rigidity of the isotropic cross section in terms of $u = u(X, Y)$ can be obtained as

$$(49) \quad r = 2 \int_a u(X, Y) da = \sqrt{G_x G_y} \int_a \left[\left(\frac{\partial u}{\partial X} \right)^2 + \left(\frac{\partial u}{\partial Y} \right)^2 \right] dA.$$

It is very easy to show that the torsional rigidity of orthotropic and isotropic cross section is the same. The validity of this statement follows from next equation

$$(50) \quad R = 2 \int_A U(x, y) dx dy \\ = 2 \int_a U \left(X \sqrt[4]{\frac{G_x}{G_y}}, Y \sqrt[4]{\frac{G_x}{G_y}} \right) \left| \frac{\partial(x, y)}{\partial(X, Y)} \right| dX dY = 2 \int_a u(X, Y) da.$$

It is evident, knowing the solution of Saint-Venant torsion for isotropic cross section then the solution for the orthotropic cross section can be represented as

$$(51) \quad \phi(x, y) = \varphi \left(x \sqrt[4]{\frac{G_y}{G_x}}, y \sqrt[4]{\frac{G_x}{G_y}} \right), \quad U(x, y) = u \left(x \sqrt[4]{\frac{G_y}{G_x}}, y \sqrt[4]{\frac{G_x}{G_y}} \right).$$

The validity of Eq. (51) follows from the definition of $u = u(X, Y)$ which is given by Eq. (44).

4 AN EXAMPLE FOR THE APPLICATION OF COORDINATE TRANSFORMATION

Let us consider an isotropic circular cross section with a circular cut (see Fig. 3). The use of the solution of Saint-Venant torsion for this cross section we can obtain the solution of Saint-Venant torsion problem for an elliptical cross section weakened by an elliptical cut as shown in Fig. 4. We introduce the following designation

$$(52) \quad \alpha = \sqrt[4]{\frac{G_y}{G_x}}, \quad \beta = \sqrt[4]{\frac{G_x}{G_y}}.$$

According to Section 2 of this paper we have the connection of cross section a and A (Fig. 3 and Fig. 4)

$$(53) \quad a_1 = \frac{C}{\alpha}, \quad b_1 = \frac{C}{\beta}, \quad a_2 = \frac{B}{\alpha}, \quad b_2 = \frac{B}{\beta}.$$

The solution of the Saint-Venant torsion problem for isotropic cross section in the present example is as follows [1]

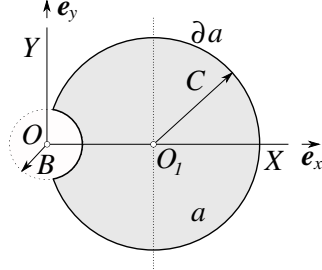


Fig. 3: Isotropic circular cross section with a circular cut (Weber's problem).

$$(54) \quad u(X, Y) = \frac{G}{2} \left(B^2 - \frac{2CB^2X}{X^2 + Y^2} + 2CX - X^2 - Y^2 \right) \quad (X, Y) \in a \cup \partial a,$$

$$(55) \quad \varphi(X, Y) = -aY \left(\frac{b^2}{X^2 + Y^2} + 1 \right) \quad (X, Y) \in a \cup \partial a,$$

$$(56) \quad r = \frac{G}{24} \frac{B}{C} \left[(21CB^2 + 6C^3) \sqrt{4C^2 - B^2} + \left(24 \frac{C^5}{B} - 12BC^3 - 48BC^3 \right) \arctan \frac{\sqrt{4C^2 - B^2}}{B} \right].$$

Substitution of Eqs. (55) and (56) into Eqs. (51)_{1,2} gives the next results for the torsion and Prandtl's stress function of orthotropic elliptical cross section weakened by an elliptical cut (Fig. 4).

$$(57) \quad \phi(x, y) = -C \left(\frac{B^2 \beta y}{\alpha^2 x^2 + \beta^2 y^2} + \beta y \right) \quad (x, y) \in A \cup \partial A$$

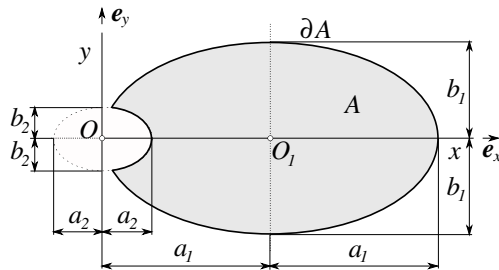


Fig. 4: Orthotropic elliptical cross section with an elliptical cut.

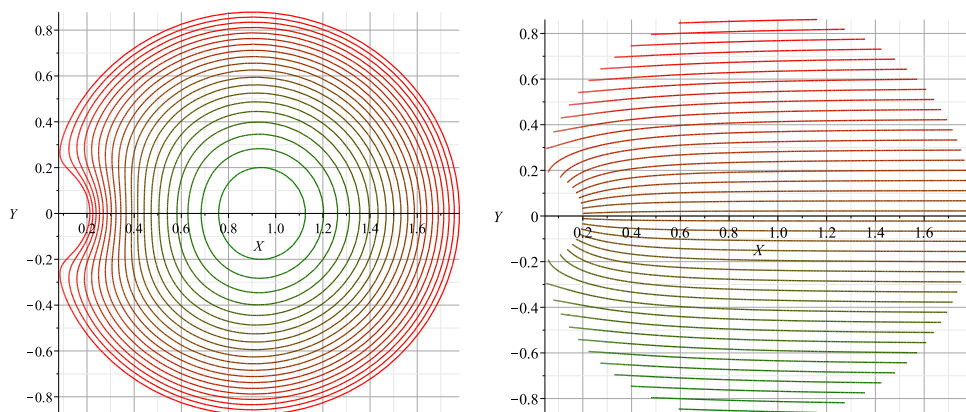


Fig. 5: Contour curves of Prandtl’s stress function $u = u(X, Y)$ and torsion function $\varphi = \varphi(X, Y)$.

$$(58) \quad U(x, y) = -\frac{\sqrt{G_x G_y}}{2} \left(B^2 - \frac{2CB^2\alpha x}{\alpha^2 x^2 + \beta^2 y^2} + 2C\alpha x - \alpha^2 x^2 - \beta^2 y^2 \right)$$

$(x, y) \in A \cup \partial A.$

Figs. 5 and 6-7 illustrate the contour lines of Prandtl’s stress function and torsion function for isotropic and orthotropic cross section. The following numerical data are used to draw the contour lines:

$$(59) \quad \begin{aligned} B &= 0.2 \text{ m} & C &= 0.9 \text{ m} & G_x &= 6 \times 10^8 \text{ Pa} \\ G_y &= 3 \times 10^8 \text{ Pa} & G &= \sqrt{G_x G_y} = 4.24266407 \times 10^8 \text{ Pa.} \end{aligned}$$

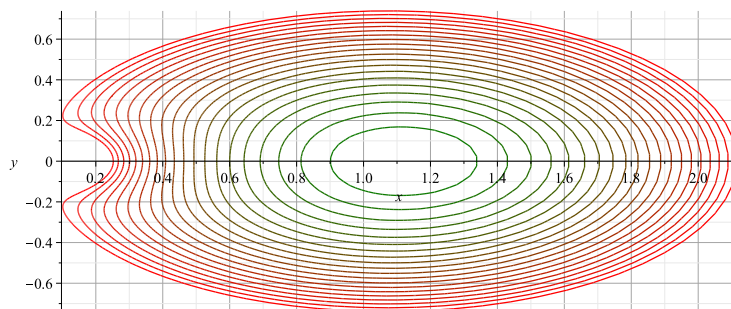
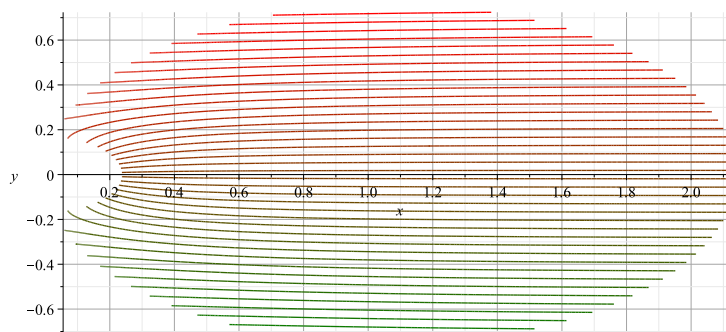


Fig. 6: Contour curves of Prandtl’s stress function $U = U(x, y)$.

Fig. 7: Contour curves of torsion function $\phi = \phi(x, y)$.

In the present numerical problems for the torsional rigidity we obtain

$$(60) \quad R = r = 4.01684287 \times 10^8 \text{ Nm}^2.$$

5 OPTIMAL SHAPE OF TWISTED ORTHOTROPIC SOLID CROSS SECTION

Pólya and Szegő proved that an extreme property of solid isotropic circular cross section which says that, the shape of isotropic homogeneous cross section which has the maximum value of torsional rigidity for a given cross sectional area is a circular cross section [12]. From this statement a very simple upper bound follows for the torsional rigidity of an arbitrary solid isotropic cross section

$$(61) \quad r \leq G \frac{a^2}{2\pi}.$$

Equality in Eq. (61) is reached only for isotropic solid cross section whose area is a .

Let us consider an isotropic solid circular cross section which is shown in Fig. 8. The solution of the Saint-Venant's torsion problem for this cross section is ($G = \sqrt{G_x G_y}$)

$$(62) \quad u(X, Y) = \frac{\sqrt{G_x G_y}}{2\rho^2} \left(1 - \left(\frac{X}{\rho}\right)^2 - \left(\frac{Y}{\rho}\right)^2 \right) \quad (X, Y) \in a \cup \partial a$$

$$(63) \quad \varphi(X, Y) = 0 \quad (X, Y) \in a \cup \partial a$$

$$(64) \quad r = \sqrt{G_x G_y} \frac{\rho^4 \pi}{2}.$$

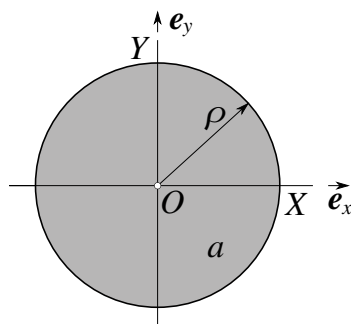


Fig. 8: Isotropic circular cross section.

The coordinate transformation introduced in Section 2 of this paper gives a solution for the corresponding orthotropic solid cross section in the next form:

$$\begin{aligned}
 (65) \quad U(x, y) &= \frac{\sqrt{G_x G_y}}{2} \left(\rho^2 - x^2 \sqrt{\frac{G_y}{G_x}} - y^2 \sqrt{\frac{G_x}{G_y}} \right) \\
 &= \rho^2 \frac{\sqrt{G_x G_y}}{2} \left(1 - \frac{x^2}{\left(\rho^4 \sqrt{\frac{G_x}{G_y}} \right)^2} - \frac{y^2}{\left(\rho^4 \sqrt{\frac{G_y}{G_x}} \right)^2} \right) \quad (x, y) \in A \cup \partial A.
 \end{aligned}$$

On the boundary contour ∂A the Prandtl's stress function $U = U(x, y)$ vanishes i.e.

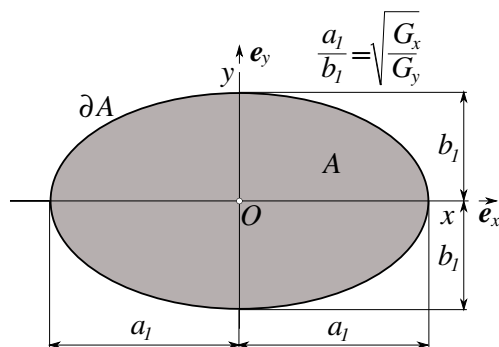


Fig. 9: Optimal shape of orthotropic cross section.

we have the equation of the boundary contour is an ellipse whose semi-axes are

$$(66) \quad a_1 = \rho^4 \sqrt{\frac{G_x}{G_y}}, \quad b_1 = \rho^4 \sqrt{\frac{G_y}{G_x}}.$$

It is evident

$$(67) \quad \phi(x, y) \equiv 0 \quad (x, y) \in A \cup \partial A.$$

The torsional rigidity of this orthotropic cross section is

$$(68) \quad R_0 = \sqrt{G_x G_y} \frac{a^2}{2\pi} = \sqrt{G_x G_y} \frac{A^2}{2\pi}$$

and we have for any orthotropic cross section whose cross-sectional area $A = \rho^2 \pi$

$$(69) \quad R \leq R_0.$$

In (69) equality reached only if A an elliptical cross section whose semi-axes are given by Eq. (66). Since the isotropic circular cross section does not warp the optimal orthotropic cross section does also not warp according to [7].

Here, we note that, the Prandtl stress function of orthotropic cross section having zero warping satisfies the Banichuk's condition which determines the shape of optimal orthotropic cross section [13].

6 CONCLUSIONS

This paper deals with the Saint-Venant torsion of homogeneous orthotropic bar with solid cross section. By a suitable coordinate transformation the solution of torsion problem of orthotropic bar is reduced to the solution of the torsion problem of isotropic bar. Knowing the Prandtl stress function and torsion function of isotropic bar with a simple calculation we get the solution of the torsion problem for an orthotropic bar. By the use of the coordinate transformation defined in this paper a new method is given to obtain the optimal shape of orthotropic cross section. An example illustrates the developed method for an orthotropic elliptical cross section having an elliptical cut.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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