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SOLID MECHANICS

STUDY OF CURVATURE EFFECT ON STONELEY WAVE PROPAGATION IN DISSIPATIVE POROELASTIC CYLINDRICAL SOLIDS

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ABSTRACT: Interface between two dissimilar solids is very common in natural structures and manmade structures. Wave propagation along the interface of poroelastic cylindrical solid with a different poroelastic medium is studied in the framework of Biot's isotropic poroelasticity. Both cylindrical solid, and surrounded medium are assumed to be dissipative. Two types of interfaces namely welded contact and smooth contact are considered. Axisymmetric waves are considered to make the problem two dimensional. Frequency equation of expected Stoneley waves is obtained for pervious interface. When the angular wavenumber is zero, the frequency equation is degenerated into two parts that are pertaining to dilatational and equivoluminal cases.

KEY WORDS: poroelastic cylinder, Biot's theory, phase velocity, attenuation coefficient, wavenumber.

NOMENCLATURE

A, N, Q, R	Poroelastic constants
b	Dissipative coefficient
k	Wavenumber
Q^{-1}	Attenuation coefficient
r, θ, z	Cylindrical coordinate system
u_r, u_θ, u_z)	Solid displacements
U_r, U_θ, U_z)	Fluid displacements
t	Time
s	Fluid pressure
ρ_{11}, ρ_{12}, ρ_{22}	Mass coefficients
σ_{ij}	Stresses
ω	Frequency

1 INTRODUCTION

In general, analytical results of the wave propagation in poroelastic solids have applications in various fields such as Geophysics, Civil Engineering, and Mechanical

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Engineering. In particular, study of Stonely waves which propagate along the interface of two dissimilar media is important in Seismology and Structural Engineering. Stoneley and Rayleigh waves in thermoelastic materials with voids are studied by Singh and Tochhawng [1]. In the paper [1], Stoneley waves at the bounded and unbounded interfaces between two dissimilar half-spaces of thermoelastic materials with voids are obtained. Stoneley waves at the interface of two dissimilar poroelastic solids is studied by Tajuddin and Reddy [2], wherein, phase velocity of waves against curvature is investigated for pervious and impervious interfaces at cut-off frequency. Effect of curvature is investigated for plane waves propagating along concave and convex cylindrical surfaces [3]. Tajuddin [4, 5], studied the problems of Rayleigh waves on a flat and curved poroelstic surfaces, and Stoneley waves at interfaces. Rayleigh waves on a convex cylindrical poroelastic surface are studied by Tajuddin and Moiz [6]. In the paper [6] the propagation of Rayleigh waves on a convex cylindrical poroelastic surface is studied for pervious and impervious surfaces. Waves in poroelastic cylindrical bore filled with fluid are investigated, and this model is applied to human bones [7]. In the said papers, the dissipative nature of poroelastic medium is neglected. Dissipative structures are abundant in nature [8, 9] and cannot be neglected absolutely due to solid fluid interactions. The effect of damping on propagation of torsional waves in an initially stressed dissipative cylinder of infinite length was studied by Selim [10]. Propagation and attenuation of waves in a dissipative orthotropic medium under initial and couple stresses were studied by Selim and Ahmed [11]. Phase velocity and attenuation of both longitudinal and shear vibrations in hollow poroelastic dissipative cylinders were studied by Shah and Hussaini [12]. The effect of curvature on Stoneley waves was reported in several papers. However, the problem of curvature effect on Stoneley wave propagation in dissipative cylindrical poroelastic solids is not yet investigated. Therefore, in the present paper, the same is investigated.

The rest of paper is organized as fallows. In Section 2, formulation and solution of the problem is given. In Section 3, the boundary conditions and frequency equations are presented. In Section 4, numerical results are described. Finally, conclusion is presented in Section 5.

2 Formulation and Solution of the Problem

Consider the isotropic poroelastic solid cylinder of radius a, Material-II (say), whose axis is along the z-axis of cylindrical system $(r,\, \theta,\, z)$, which is embedded in an isotropic poroelastic medium, Material-I (say). As the interface $r = a$ is of two dissimilar solids, Stoneley waves propagate thereat. The waves here are assumed to be axisymmetric to reduce the mathematical complexity, consequently, the problem becomes plane strain. For axisymmetric waves, the displacement vectors of solid and fluid are $\vec{u}\,(u,\, v,\, 0)$ and $\vec{U}\,(U,\, V,\, 0)$, respectively. In order to solve the pertinent problem, it is convenient to introduce displacement potentials φ's and ψ's which are functions of $r,\, \theta,\, t$ as follows:

$$\begin{aligned} u &= \frac{\partial \varphi_1}{\partial r} + \frac{1}{r}\frac{\partial \psi_1}{\partial \theta}\,, \qquad v = \frac{1}{r}\frac{\partial \varphi_1}{\partial \theta} - \frac{\partial \psi_1}{\partial r}\,, \\ U &= \frac{\partial \varphi_2}{\partial r} + \frac{1}{r}\frac{\partial \psi_2}{\partial \theta}\,, \qquad V = \frac{1}{r}\frac{\partial \varphi_2}{\partial \theta} - \frac{\partial \psi_2}{\partial r}\,. \end{aligned} \tag{1}$$

Substitution of Eq. (1) in the equations of motion [13] gives the following:

$$\begin{aligned} P\nabla^2\varphi_1 + Q\nabla^2\varphi_2 &= (\rho_{11}\ddot{\varphi}_1 + \rho_{12}\ddot{\varphi}_2 + b(\dot{\varphi}_1 - \dot{\varphi}_2)\,, \\ Q\nabla^2\varphi_1 + R\nabla^2\varphi_2 &= (\rho_{12}\ddot{\varphi}_1 + \rho_{22}\ddot{\varphi}_2) - b(\dot{\varphi}_1 - \dot{\varphi}_2)\,, \\ N\nabla^2\psi_1 &= (\rho_{11}\ddot{\psi}_1 + \rho_{12}\ddot{\psi}_2) + b(\dot{\psi}_1 - \dot{\psi}_2)\,, \\ 0 &= (\rho_{12}\ddot{\psi}_1 + \rho_{22}\ddot{\psi}_2) - b(\dot{\psi}_1 - \dot{\psi}_2)\,, \end{aligned} \tag{2}$$

where

$$P = A + 2N, \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,.$$

In the above, A, N, Q and R are poroelastic constants, and ∇^2 is Laplace operator. b is dissipative coefficient. A 'dot' over a quality denotes partial differentiation with respect to time t. The solutions of Eq. (2) are assumed as follows:

$$\begin{aligned} \varphi_1 &= F_1(r)\, e^{i(\omega t + n\theta)}\,, \qquad \varphi_2 = F_2(r)\, e^{i(\omega t + n\theta)}\,, \\ \psi_1 &= G_1(r)\, e^{i(\omega t + n\theta)}\,, \qquad \psi_2 = G_2(r)\, e^{i(\omega t + n\theta)}\,, \end{aligned} \tag{3}$$

where θ is angular co-ordinate, n is the integer number of waves around the circumference of solid cylinder (angular wavenumber), and ω is the frequency of the waves. Substitutions of the expressions for φ_1, φ_2, ψ_1 and ψ_2 in the Eq. (3) give

$$\begin{aligned} P\Delta F_1 + Q\Delta F_2 &= -\omega^2(M_{11}F_1 + M_{12}F_2)\,, \\ Q\Delta F_1 + R\Delta F_2 &= -\omega^2(M_{12}F_1 + M_{22}F_2)\,, \\ N\Delta G_1 &= -\omega^2(M_{11}G_1 + M_{12}G_2)\,, \\ 07 &= -\omega^2(M_{12}G_1 + M_{22}G_2)\,, \end{aligned} \tag{4}$$

where $M_{11} = \rho_{11} - ib\omega^{-1}$, $M_{12} = \rho_{12} + ib\omega^{-1}$, $M_{22} = \rho_{22} - ib\omega^{-1}$, and $\Delta = \dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} - \dfrac{n^2}{r^2}$.

In order to solve Eq. (4), the first and second equations of Eq. (4) are multiplied by R and $-Q$, respectively. Addition of these resultant equations gives

(5) $$F_2 = Z_1^2 \Delta F_1 - Z_2^2 F_1 \,,$$

where Z_1^2 and Z_2^2 are $Z_1^2 = -\dfrac{(PR - Q^2)}{\omega^2(RM_{12} - QM_{22})}$, $Z_2^2 = -\dfrac{(RM_{11} - QM_{12})}{(RM_{12} - QM_{22})}$.

Substitution of Eq. (5) into the first equation of Eq. (4)

(6) $$\Delta^2 F_1 + Z_3^2 \Delta F_1 + Z_4^2 F_1 = 0 \,,$$

where $Z_3^2 = \dfrac{P - QZ_2^2 + \omega^2 Z_1^2 M_{12}}{QZ_1^2}$, $Z_4^2 = \dfrac{\omega^2(M_{11} - Z_2^2 M_{12})}{QZ_1^2}$.

Equation (6) can also be re-written as

(7) $$\Delta^2 F_1 + (\mu_1^2 + \mu_2^2)\Delta F_1 + \mu_1^2 \mu_2^2 F_1 = 0 \,,$$

where $\mu_1^2 + \mu_2^2 = Z_3^2$, $\mu_1^2 \mu_2^2 = Z_4^2$.

Equation (7) can be factored as

(8) $$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - n^2 + \mu_1^2\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - n^2 + \mu_2^2\right)F_1 = 0 \,,$$

where μ_1^2 and μ_2^2 are

(9) $$\mu_1^2 = \omega^2 V_1^{-2} \,, \qquad \mu_2^2 = \omega^2 V_2^{-2} \,.$$

In the above equation, V_1 and V_2 are the velocities of dilatational waves of first and second kind respectively. Solution of Eq. (8) together with Eq. (3), gives

(10) $$\varphi_1 = (BJ_n(\mu_1 r) + CJ_n(\mu_2 r))e^{i(\omega t + n\theta)} \,,$$

where B and C are arbitrary constants, J_n is the Bessel functions of first kind of order n. Equations (3), (5) and (10), give

(11) $$\varphi_2 = -(B\delta_1^2 J_n(\mu_1 r) + C\delta_2^2 J_n(\mu_2 r))e^{i(\omega t + n\theta)} \,.$$

In the above, δ_1^2 and δ_2^2 are

(12) $$\delta_1^2 = (RM_{12} - QM_{22})^{-1}(RM_{11} - QM_{12} - V_1^{-2}(PR - Q^2)) \,,$$

$\delta_2^2 =$ similar expression as δ_1^2 with V_1^{-2} replaced by V_2^{-2}.

Proceeding on similar lines, potential functions ψ_1 and ψ_2 are obtained, which are given below:

$$\psi_1 = DJ_n(\mu_3 r)e^{i(\omega t+n\theta)}\,, \qquad \psi_2 = -M_{12}M_{22}^{-1}\psi_1\,, \tag{13}$$

where D is arbitrary constant, and μ_3 is

$$\mu_3^2 = \omega^2 V_3^{-2}\,. \tag{14}$$

In Eq. (14) V_3 is the shear wave velocity. Equations (10), (13) and (1) give the following displacement components:

$$\begin{aligned} u &= (BJ_n'(\mu_1 r) + CJ_n'(\mu_2 r) + inr^{-1}DJ_n(\mu_3 r))e^{i(\omega t+n\theta)}, \\ v &= ir^{-1}(BnJ_n(\mu_1 r) + CnJ_n(\mu_2 r) + irDJ_n'(\mu_3 r))e^{i(\omega t+n\theta)}. \end{aligned} \tag{15}$$

Substitutions of there in stress-displacement relations [13] gives are following stresses:

$$\begin{aligned} \sigma_{rr} &= (BS_{11} + CS_{12} + iDS_{13})\, e^{i(\omega t+n\theta)}\,, \\ \sigma_{r\theta} &= i\,(BS_{21} + CS_{22} - iDS_{23})\, e^{i(\omega t+n\theta)}\,, \\ -\,s &= (BS_{31} + CS_{32})\, e^{i(\omega t+n\theta)}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} S_{11} &= 2NJ_n''(\mu_1 r) - (P-2N)\,P^2 J_n'(\mu_1 r) + QP^2\delta_1^2 J_n(\mu_1 r)\,, \\ S_{21} &= 2Nr^{-1}(J_n'(\mu_1 r) - r^{-1}J_n(\mu_1 r))\,, \\ S_{23} &= N(J_n''(\mu_3 r) - r^{-1}J_n'(\mu_3 r) + n^2 r^{-2}J_n(\mu_3 r))\,, \\ S_{31} &= P^2(Q - R\delta_1^2)J_n(\mu_1 r)\,, \end{aligned}$$

S_{12}, S_{32} = similar expressions as S_{11}, S_{31} with μ_1, δ_1^2 replaced by μ_2, δ_2^2, respectively,

S_{22}, S_{13} = similar expressions as S_{21} with μ_1 replaced by μ_2, μ_3, respectively.

3 Boundary Conditions and Frequency Equations

Intact at the interface of two solids depends on roughness of the interface, and broadly there will be two types of interfaces namely welded contact and smooth contact. Two cases are considered separately in the following:

3.1 WELDED CONTACT

The boundary conditions at $r = a$ in this case are

$$
\begin{aligned}
&(\sigma_{rr})_{M_1} = (\sigma_{rr})_{M_2}\,, \\
&(\sigma_{r\theta})_{M_1} = (\sigma_{r\theta})_{M_2}\,, \\
&(u)_{M_1} = (u)_{M_2}\,, \\
&(v)_{M_1} = (v)_{M_2}\,, \\
&(s)_{M_1} = 0\,, \\
&(s)_{M_2} = 0\,.
\end{aligned}
\tag{17}
$$

The above boundary conditions physically state that solid stresses and displacements at the interface are continuous, and no fluid pressure thereat. These boundary conditions lead to the following frequency equation:

$$
|A_{ij}| = 0\,, \quad (i,\, j = 1, 2, \ldots, 6)\,. \tag{18}
$$

where $A_{11} = S_{11}$, $A_{13} = iS_{13}$, $A_{31} = J_n'(\mu_1 a)$, $A_{33} = ina^{-1}J_n'(\mu_3 a)$, $A_{41} = nJ_n(\mu_1 a)$, $A_{43} = iaJ_n'(\mu_3 a)$, $A_{53} = A_{54} = A_{55} = A_{56} = A_{61} = A_{62} = A_{63} = A_{66} = 0$;

A_{12} A_{21}, A_{22}, A_{51}, A_{52} = similar expressions as A_{11} with S_{11} replaced by S_{12}, S_{21}, S_{22}, S_{51}, S_{52} respectively.

A_{23} = similar expression as A_{13} with S_{13} replaced by S_{23}.

A_{32}, A_{42} = similar expressions as A_{31}, A_{41} with μ_1 replaced by μ_2 respectively.

A_{14}, A_{15}, A_{16}, A_{24}, A_{25}, A_{26}, A_{34}, A_{35}, A_{36}, A_{44}, A_{45}, A_{46}, A_{64}, A_{65} = similar expressions as A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}, A_{31}, A_{32}, A_{33}, A_{41}, A_{42}, A_{43}, A_{51}, A_{52} with N, P, Q, R, J_n replaced by N_1, P_1, Q_1, R_1, Y_n respectively.

Because of dissipative nature of the medium $b \neq 0$ and all the above equations involve complex terms. As a result, Eq. (18) cannot be investigated as it is. Hence, the frequency equation is investigated when the angular wavenumber (n) approaches zero. In this case, the frequency equation Eq. (18) results in two equations, and are given by

$$
\begin{vmatrix} A_{23} & A_{26} \\ A_{43} & A_{46} \end{vmatrix} = 0 \tag{19}
$$

and

$$
\begin{vmatrix} A_{11} & A_{12} & A_{14} & A_{15} \\ A_{31} & A_{32} & A_{34} & A_{35} \\ A_{51} & A_{52} & 0 & 0 \\ 0 & 0 & A_{64} & A_{65} \end{vmatrix} = 0\,. \tag{20}
$$

From Eq. (19), it is clear that the displacement is azimuthal and equivoluminal, while from the Eq. (20), it is clear the displacement is purely radial and dilatational as that of the paper [14]. Equation (20) is still complex valued and implicit in nature. Therefore, for numerical process, the following approximations [15] are used. These approximations hold good when the argument is large, that is when the curvature $(1/ka)$ is small gives

$$\begin{aligned} J_0(x) &\approx \sqrt{\frac{2}{\pi x}}\cos(x-\frac{\pi}{4})\,, \\ Y_0(x) &\approx \sqrt{\frac{2}{\pi x}}\sin(x-\frac{\pi}{4})\,. \end{aligned} \tag{21}$$

From Eqs. (20) and (21), the following equation is obtained:

$$|A_{jk}|=0, \qquad (j,k=1,2,3,4)\,, \tag{22}$$

where $A_{jk}=a_{jk}+ia'_{jk}$, $(j,k=1,2,3,4)$, $i=\sqrt{-1}$, a_{jk} and a'_{jk} are given in the Appendix.

3.2 Smooth contact

Boundary conditions in this case are

$$\begin{aligned} (\sigma_{rr})_{M_1} &= (\sigma_{rr})_{M_2}\,, \\ (\sigma_{r\theta})_{M_1} &= 0, \\ (\sigma_{r\theta})_{M_2} &= 0, \\ (u)_{M_1} &= (u)_{M_2}, \\ (s)_{M_1} &= 0, \\ (s)_{M_2} &= 0. \end{aligned} \tag{23}$$

These boundary conditions physically state that normal stress and displacement component are continuous, and there are no shear stresses and fluid pressure. These boundary conditions result in the following frequency equation:

$$|B_{ij}|=0\,, \qquad (i,j=1,2,\dots,6)\,, \tag{24}$$

where $B_{11}=A_{11}$, $B_{21}=J'_n(\mu_1 a)$, $B_{23}=ina^{-1}J_n(\mu_3 a)$, $B_{34}=B_{35}=B_{36}=0$, $B_{41}=B_{42}=B_{43}=B_{53}=B_{54}=B_{55}=B_{56}=B_{61}=B_{62}=B_{63}=B_{66}=0$, B_{12}, B_{13}, B_{31}, B_{32}, B_{33}, B_{51}, B_{52} = similar expressions as B_{11} with A_{11} replaced by A_{12}, A_{13}, A_{21}, A_{22}, A_{23}, A_{51}, A_{52} respectively.

B_{22} = similar expression as B_{21} with μ_1 replaced by μ_2.

B_{14}, B_{15}, B_{16}, B_{24}, B_{25}, B_{26}, B_{44}, B_{45}, B_{46}, B_{64}, B_{65} = similar expressions as B_{11}, B_{12}, B_{13}, B_{21}, B_{22}, B_{23}, B_{31}, B_{32}, B_{33}, B_{51}, B_{52} with N, P, Q, R, J_n replaced by N_1, P_1, Q_1, R_1, Y_n respectively.

The frequency equation (24) results in two equations, when angular wavenumber approaches zero as in earlier case. The first equation is

$$B_{23}B_{46} = 0\,, \tag{25}$$

while the second equation is the same as that of Eq. (20). From this, it is clear that dilatational waves are independent of nature of contact, while shear waves are not independent of nature of contact.

4 Numerical Results

In the presence of dissipation (b), frequency equations (19), (22) and (25) are implicit and complex valued and give phase velocity and attenuation coefficient. For numerical process, two cases are considered. In the case-I, cylindrical core solid is sandstone saturated with kerosene [16] (Material-I say), and is surrounded by sandstone saturated with water (Material-II say). In the case-II, cylindrical core solid is sandstone saturated with water [17], and is surrounded by sandstone saturated with kerosene. The parameter values of above two poroelastic materials are given in the Table 1. The roots of Eq. (19) (pertaining to welded contact) along with that of its counterpart of purely elastic solid (i.e., when ρ_{12} and ρ_{22} are taken to be zero) are computed, and presented in Table 2. From these results, it is clear that, the purely elastic solid values are greater than that of poroelstic solid. The roots of Eq. (25) (the case of smooth contact) is presented along with its counterpart of purely elastic solid in Table 3. From Table 3, it is clear that, the values of poroelastic solid are, in general, greater than that of purely elastic solid unlike earlier case.

Table 1: Values of poroelastic material parameters

Material parameter	Material-I	Material parameter	Material-II
N [N/m^2]	0.2765×10^{10}	N_1 [N/m^2]	0.922×10^{10}
P [N/m^2]	0.5825×10^{10}	P_1 [N/m^2]	2.2876×10^{10}
Q [N/m^2]	0.7635×10^{10}	Q_1 [N/m^2]	0.013×10^{10}
R [N/m^2]	0.0326×10^{10}	R_1 [N/m^2]	0.0637×10^{10}
ρ_{11} [kg/m^3]	1.9261×10^3	$(\rho_{11})_1$ [kg/m^3]	1.90302×10^3
ρ_{12} [kg/m^3]	-0.002137×10^3	$(\rho_{12})_1$ [kg/m^3]	0
ρ_{22} [(kg/m^3]	0.21537×10^3	$(\rho_{22})_1$ [kg/m^3]	0.268×10^3

Table 2: Phase velocities in the case of equivoluminal waves and welded contact

Poroelastic solid	Purely elastic solid
35.7734	78.4453
23.6953	52.4141
17.5859	39.4297
13.8672	31.6641
11.3359	26.4922
9.4766	22.8047
8.0547	20.0391
6.9453	17.8828

Table 3: Phase velocities in the case of dilatational waves and smooth contact

Poroelastic solid	Purely elastic solid
77.4297	55.2109
38.7109	36.8047
25.8047	27.6016
38.7109	22.0859
30.9766	18.3984
12.8984	15.7734
22.1172	13.8047
19.3516	12.2734

Equation (20) corresponds to dilatational modes, and is same for both type of boundaries. For both the boundaries when angular wavenumber is infinite, the phase velocity and attenuation coefficient against curvature $(1/ka)$ are computed. The attenuation coefficient (Q^{-1}) is given by [18] $Q^{-1} = 2Im(\omega)/Re(\omega)$. $Im(\omega)$ is the frequency of imaginary part while $Re(\omega)$ is the frequency of the real part. Wave characteristics are computed using the numerical process bisection method implemented in MATLAB, and the results are presented in Figs. 1–3. Figures 1 and 2 depict the variations of phase velocity, and attenuation coefficient against curvature in the case of welded contact and smooth contact. From these figures, it is clear that, phase velocity values are, greater than that of attenuation coefficient in the both

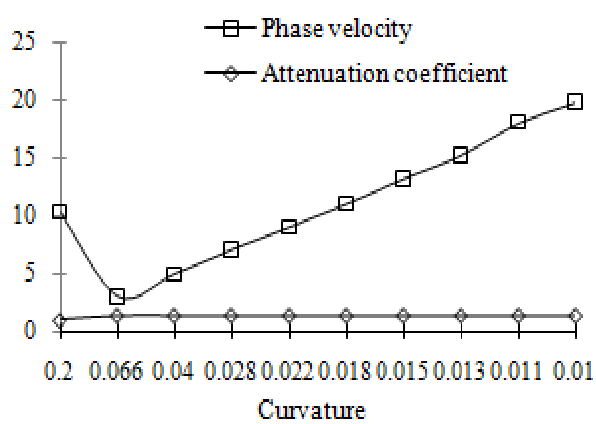

Fig. 1: Variation of phase velocity and attenuation coefficient against curvature in the case-I.

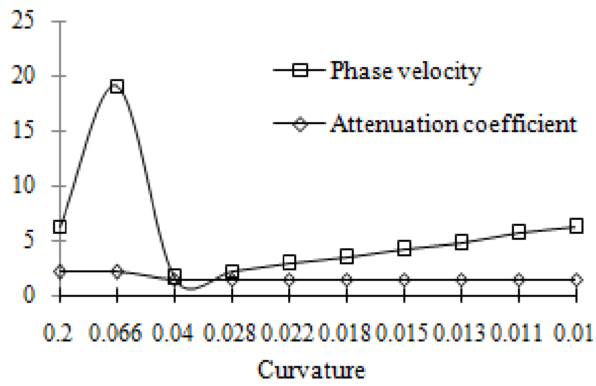

Fig. 2: Variation of phase velocity and attenuation coefficient against curvature in the case-II.

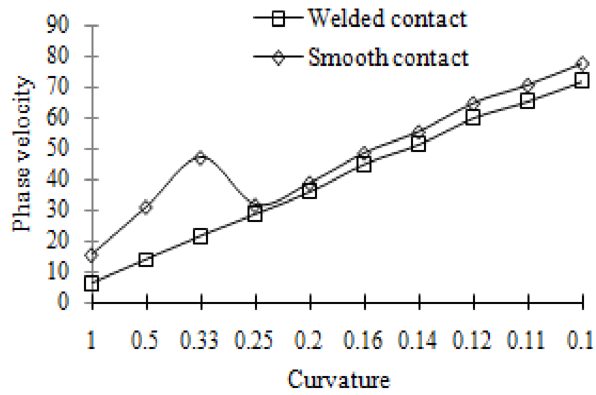

Fig. 3: Variation of phase velocity against curvature in the cases of welded and smooth contacts.

cases. Moreover, attenuation coefficients in both the cases are almost linear. Figure 3 depicts the variations of phase velocity against curvature in the cases of welded and smooth contacts. From Fig. 3, it is clear that, phase velocity in the case of welded contact is less than that of smooth contact. As curvature increases, phase velocity increases for both the cases.

5 Conclusion

The curvature effect on Stoneley wave propagation in poroelastic dissipative cylindrical solid surrounded by different poroelastic medium is investigated using the Biot's theory. Phase velocity and attenuation coefficient against curvature are computed in two types of boundaries. From the numerical results, it is seen that the results of the phase velocity are greater than that of attenuation coefficient. Also it is clear that the phase velocity for welded contact is less than that of smooth contact.

APPENDIX

$$
\begin{aligned}
a_{11} =&(QP^2L_3 - (P-2N)P^2 - 2N)m_{11} + QP^2L_4m_{12}\\
&+ \frac{2N}{\omega Z_1^{-1}a}(m_{21}\cos(T_1) + m_{22}\sin(T_1));\\
a_{31} =&\sqrt{\frac{2}{\pi}}(\omega Z_1^{-1}a)^{\frac{-5}{2}}\Bigg[(QP^2L_3 - 2N)(\omega Z_1^{-1}a)^2 - \frac{(P-2N)P^2}{2}\cos(T_1)(\omega Z_1^{-1}a)\\
&+ \Big(N + \frac{(P-2N)P^2}{2}\cos(2T_1)\Big)\cos\frac{\pi}{8} + \Bigg(\frac{(P-2N)P^2}{2}\sin(T_1)(\omega Z_1^{-1}a)\\
&- QP^2L_4(\omega Z_1^{-1}a)^2 - \Big(\frac{N}{2} + \frac{(P-2N)P^2}{2}\Big)\sin\frac{\pi}{8}\Bigg)\sin(X_1)\cosh(X_2)\\
&+ \Bigg((QP^2L_3 - 2N)(\omega Z_1^{-1}a)^2 - \frac{(P-2N)P^2}{2}\cos(T_1)(\omega Z_1^{-1}a)\\
&- \Big(\frac{3N}{2} - \frac{(P-2N)P^2}{2}\Big)\cos(2T_1)\Bigg)\sin\frac{\pi}{8} + (QP^2L_4(\omega Z_1^{-1}a)^2\\
&- \Bigg(\frac{(P-2N)P^2}{2}\sin(T_1)(\omega Z_1^{-1}a) - \Big(\frac{3N}{2} - \frac{(P-2N)P^2}{2}\Big)\cos\frac{\pi}{8}\Bigg)\cos(X_1)\sinh(X_2)\\
&- \Big(2N + \frac{(P-2N)P^2}{2}\Big)\Big(\cos\frac{\pi}{8}\cos(T_1) + \sin\frac{\pi}{8}\sin(T_1)\Big)\cos(X_1)\cosh(X_2)\\
&- \Big(\sin\frac{\pi}{8}\cos(T_1) - \cos\frac{\pi}{8}\sin(T_1)\Big)\sin(X_1)\sinh(X_2)\Bigg];\\
a_{51} =&P^2((Q - RL_3)m_{11} + RL_4m_{12});\\
a'_{51} =&P^2((Q - RL_3)m_{12} - RL_4m_{11});\\
a_{54} =&a_{55} = a_{61} = a_{62} = a'_{54} = a'_{55} = a'_{61} = a'_{62} = 0;\\
a'_{11} =&(QP^2L_3 - (P-2N)P^2 - 2N)m_{12} - QP^2L_4m_{11}\\
&+ \frac{2N}{\omega Z_1^{-1}a}(m_{22}\cos(T_1) + m_{21}\sin(T_1));
\end{aligned}
$$

$$
\begin{aligned}
a_{31}' =& \sqrt{\frac{2}{\pi}}(\omega Z_1^{-1}a)^{\frac{-5}{2}}\Bigg[((QP^2L_3 - 2N)(\omega Z_1^{-1}a)^2 - \frac{(P-2N)P^2}{2}\cos(T_1)(\omega Z_1^{-1}a) \\
&+ \Big(\frac{3N}{2} - \frac{(P-2N)P^2}{2}\cos(2T_1)\Big)\sin\frac{\pi}{8} + \Big(-\frac{(P-2N)P^2}{2}\sin(T_1)(\omega Z_1^{-1}a) \\
&+ QP^2L_4(\omega Z_1^{-1}a)^2 + \frac{3N}{2} + \frac{P-2N)P^2}{2}\Big)\sin(2T_1)\cos\frac{\pi}{8}\Bigg)\sin(X_1)\cosh(X_2) \\
&+ ((QP^2L_4 - 2N)(\omega Z_1^{-1}a)^2 - \frac{(P-2N)P^2}{2}\sin(T_1)(\omega z_1^{-1}a) \\
&+ (\frac{3N}{2} + \frac{(P-2N)P^2}{2})\cos(2T_1))\cos\frac{\pi}{8} + (QP^2L_4(\omega Z_1^{-1}a)^2 \\
&- \Big(\frac{(P-2N)P^2}{2}\sin(T_1)(\omega Z_1^{-1}a) + \Big(\frac{3N}{2} + \frac{(P-2N)P^2}{2}\Big)\sin\frac{\pi}{8}\Big)\cos(X_1)\sinh(X_2) \\
&+ ((2N + \frac{(P-2N)P^2}{2})(\sin\frac{\pi}{8}\cos(T_1) - \cos\frac{\pi}{8}\sin(T_1))\sin(X_1)\sinh(X_2) \\
&- (\cos\frac{\pi}{8}\sin(T_1) + \sin\frac{\pi}{8}\cos(T_1))\cos(X_1)\cosh(X_2)\Bigg]; \\
T_{11} =& \cos(\frac{\omega}{Z_1}a\cos(T_1))\cosh(\frac{\omega}{Z_1}a\sin(T_1)) + \sin(\frac{\omega}{Z_1}a\cos(T_1))\cosh(\frac{\omega}{Z_1}a\sin(T_1)); \\
m_{11} =& (\pi\omega Z_1^{-1}a)^{\frac{-1}{2}}(\cos\frac{\pi}{8}T_{11} - \sin\frac{\pi}{8}T_{12}); \\
T_{12} =& \sin(\frac{\omega}{Z_1}a\cos(T_1))\sinh(\frac{\omega}{Z_1}a\sin(T_1)) - \cos(\frac{\omega}{Z_1}a\cos(T_1))\sinh(\frac{\omega}{Z_1}a\sin(T_1)); \\
m_{12} =& (\pi\omega Z_1^{-1}a)^{\frac{-1}{2}}(\cos\frac{\pi}{8}T_{12} + \sin\frac{\pi}{8}T_{11}); \\
m_{21} =& (\pi\omega Z_1^{-1}a)^{\frac{-1}{2}}(\cos\frac{\pi}{8}T_{21} + \sin\frac{\pi}{8}T_{22}); \\
T_{21} =& \sin(\frac{\omega}{Z_1}a\cos(T_1))\cosh(\frac{\omega}{Z_1}a\sin(T_1)) - \cos(\frac{\omega}{Z_1}a\cos(T_1))\cosh(\frac{\omega}{Z_1}a\sin(T_1)); \\
T_{22} =& \cos(\frac{\omega}{Z_1}a\cos(T_1))\sinh(\frac{\omega}{Z_1}a\sin(T_1)) + \sin(\frac{\omega}{Z_1}a\cos(T_1))\sinh(\frac{\omega}{Z_1}a\sin(T_1)); \\
m_{22} =& (\pi\omega Z_1^{-1}a)^{\frac{-1}{2}}(\sin\frac{\pi}{8}T_{21} - \cos\frac{\pi}{8}T_{22}); \\
X_1 =& P\rho_{22} - 2Q\rho_{12} + R\rho_{11} + P^2\rho_{22}^2 + 4Q^2\rho_{12}^2 + R^2\rho_{11}^2 - 4PQ\rho_{12}\rho_{22} - 4QR\rho_{11}\rho_{22} \\
&+ 2PR\rho_{11}\rho_{22} - b^2\omega^{-2}(P^2 + 4Q^2 + R^2 + 4PQ + 4QR + 2PR) \\
&- 4(PR - Q^2)(\rho_{11}\rho_{22} - \rho_{12}^2); \\
X_2 =& b\omega^{-1}(4(PQ + 2Q^2 + QR))\rho_{12} - (P + 2Q + R) - (PR + 2QR + R^2)2\rho_{11} \\
&- 2(P^2 + 2PQ + PR) + 4(\rho_{11} + \rho_{22} + 2\rho_{12});
\end{aligned}
$$

$$
\begin{aligned}
X_3 =&\frac{2X_1(PR-Q^2)}{(X_1^2+X_2^2)};\\
X_4 =&\frac{2X_2(Q^2-PR)}{(X_1^2+X_2^2)};\\
Z_1 =&(X_3^2+X_4^2)^{\frac{1}{4}};\\
T_1 =&\frac{1}{2}\tan^{-1}(\frac{X_4}{X_3});\\
X_5 =&P_1\rho_{22}-2Q\rho_{12}+R\rho_{11}-P^2\rho_{22}^2-4Q^2\rho_{12}^2-R^2\rho_{11}^2+4PQ\rho_{12}\rho_{22}\\
&+4QR\rho_{11}\rho_{22}-2PR\rho_{11}\rho_{22}+b^2\omega^{-2}(P^2+4Q^2+R^2+4PQ+4QR+2PR);\\
X_6 =&b\omega^{-1}((PQ+R^2+2QR)2\rho_{11}-(PQ+2Q^2+QR)4\rho_{12}\\
&+(PR+2PQ+P^2)2\rho_{22}-(P+2Q+R)-4(PR-Q^2)(\rho_{11}+\rho_{22}+2\rho_{12});\\
L_1 =&R\rho_{11}-Q\rho_{12}-Z_1^2(PR-Q^2)\cos(2T_1);\\
L_2 =&R+Q+2Z_1^2(PR-Q^2)\cos(T_1)\sin(T_1);\\
L_3 =&\frac{L_1(R\rho_{12}-Q\rho_{22})-b^2\omega^{-2}L_2(R+Q)}{R^2(\rho_{12}+b^2\omega^{-2})+Q^2(\rho_{22}+b^2\omega^{-2})-2RQ(\rho_{12}\rho_{22}-b^2\omega^{-2})};\\
L_4 =&\frac{b\omega^{-1}(L_2(Q\rho_{22}-R\rho_{12})-L_1(R+Q))}{R^2(\rho_{12}+b^2\omega^{-2})+Q^2(\rho_{22}+b^2\omega^{-2})-2RQ(\rho_{12}\rho_{22}-b^2\omega^{-2})};
\end{aligned}
$$

a_{12}, a'_{12} = similar expressions as a_{11}, a'_{11} with L_3, L_4, m_{11}, m_{12}, m_{21}, m_{22}, T_1, Z_1 replaced by L_7, L_8, m_{13}, m_{14}, m_{23}, m_{24}, T_2, Z_2 respectively,

a_{14}, a_{15}, a'_{14}, a'_{15} = similar expressions as a_{11}, a_{12}, a'_{11}, a'_{12} with N, P, Q, L, T, Z, m_{11}, m_{12}, m_{13}, m_{14}, m_{21}, m_{22}, m_{23}, m_{24} replaced by N_1, P_1, Q_1, l, t, z, m_{31}, m_{32}, m_{33}, m_{34}, m_{41}, m_{42}, m_{43}, m_{44} respectively,

a_{32}, a'_{32} = similar expressions as a_{31}, a'_{31} with L_3, L_4, T_1, Z_1, X_1, X_2 replaced by L_7, L_8, T_2, Z_2, X_3, X_4 respectively,

a_{34}, a_{35}, a'_{34}, a'_{35} = similar expressions as a_{31}, a_{32}, a'_{31}, a'_{32} with N, P, Q, L, T, z, X replaced by N_1, P_1, Q_1, l, t, Z, x respectively,

a_{52}, a'_{52} = similar expressions as a_{51}, a'_{51} with L_3, L_4, m_{11}, m_{12} replaced by L_7, L_8, m_{13}, m_{14} respectively,

a_{64}, a_{65}, a'_{64}, a'_{65} = similar expressions as a_{51}, a_{52}, a'_{51}, a'_{52} with P, Q, R, L, m replaced by P_1, Q_1, R_1, l, m_1 respectively,

m_{13}, m_{14}, m_{23}, m_{24} = similar expressions as m_{11}, m_{12}, m_{21}, m_{22} with Z_1, T_{11}, T_{12}, T_{21}, T_{22} replaced by Z_2, T_{13}, T_{14}, T_{23}, T_{24} respectively,

m_{31}, m_{32}, m_{33}, m_{34}, m_{41}, m_{42}, m_{43}, m_{44} = similar expressions as m_{11}, m_{12}, m_{13}, m_{14}, m_{21}, m_{22}, m_{23}, m_{24} with Z, T_{11}, T_{12}, T_{13}, T_{14}, T_{21}, T_{22}, T_{23}, T_{24} replaced by z, t_{31}, t_{32}, t_{33}, t_{34}, t_{41}, t_{42}, t_{43}, t_{44} respectively,

T_{13}, T_{14}, T_{23}, T_{24} = similar expressions as T_{11}, T_{12}, T_{21}, T_{22} with Z_1, T_1 replaced by Z_2, T_2 respectively,

t_{31}, t_{32}, t_{33}, t_{34}, t_{41}, t_{42}, t_{43}, t_{44} = similar expressions as T_{11}, T_{12}, T_{13}, T_{14}, T_{21}, T_{22}, T_{23}, T_{24} with Z, T replaced by z, t respectively,

Z_2, T_2 = similar expressions as Z_1, T_1 with X_3, X_4 replaced by X_7, X_8 respectively,

z_1, t_1, z_2, t_2 = similar expressions as Z_1, T_1, Z_2, T_2 with X replaced by x respectively,

X_7, X_8 = similar expressions as X_3, X_4 with X_1, X_4 replaced by X_5, X_6 respectively,

L_5, L_6, L_7, L_8 = similar expressions as L_1, L_2, L_3, L_4 with z_1, T_1, L_1, L_2 replaced by z_2, T_2, L_5, L_6 respectively,

l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 = similar expressions as L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8 with N, P, Q, R, Z, ρ_{11}, ρ_{12}, ρ_{22} replaced by N_1, P_1, Q_1, R_1, z, $(\rho_{11})_1$, $(\rho_{12})_1$, $(\rho_{22})_1$ respectively.

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