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SPATIAL BEHAVIOUR IN MICROSTRETCH ELASTICITY WITH MASS AND THERMAL DIFFUSION

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ABSTRACT: We analyse the spatial behaviour of diffusive microstretch thermoelasticity with microconcentrations and microtemperatures in the linear, anisotropic and non-centro-symmetric case. We define a function with a weight depending on time that characterizes the principal characteristics of the model, then study its properties and derive a differential inequality. Then we obtain an exponential spatial decay that is independent of time and is valid for unbounded bodies.

KEY WORDS: diffusion, microconcentrations, microstretch thermoelasticity, microtemperatures.

NOTATIONS

u_i	displacement vector field	φ	function of microdilatation
φ_i	vector of microrotation	t_{ij}	stress tensor
ρ	mass density of reference	f_i	body force
h_j	vector of microstretch	g	internal body force
l	external microstretch body load	m_{ij}	tensor of couple stress
g_i	body couple density		
C	concentration	η_i	flux vector of mass diffusion
S	microentropy	T_0	absolute temperature in the reference configuration
q_i	vector of heat flux	s	supply of heat per unit mass
δ_{ij}	Kronecker's delta	q_{ij}	first heat flux moment tensor
Q_i	microheat flux average	G_i	first heat supply moment vector
T_i	microtemperatures	$\tilde{\sigma}_i$	micromass diffusion flux average
T	absolute temperature	P	chemical potential of the particle
C_i	microconcentrations	η_{ij}	tensor of first mass diffusion flux moment
ε_{ijk}	alternating symbol	ε_i	first moment of energy vector

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1 Introduction

The field of mechanics of generalized continua is of great interest nowadays, many books having been published recently, see for instance [1–3]. It is important for the characterization of media with microstructure and one of its pioneers is Eringen (see, for instance, [4]). Other results about bodies with microstructure were deduced in [5–10]. A topic of interest in this domain is microstretch elasticity, which was proposed by Eringen in [4]. Various properties of microstretch elasticity and its extensions were studied recently, see [4, 11–15]. This mathematical model may be applied in various areas of science, engineering and even biomechanics. For instance, microstretch elasticity can be used to model human iliopsoas, a myofascial kinetic continuum, or functionally related muscles [16, 17]. But it is also useful for the description of asphalt, of composite materials which are reinforced with chopped elastic fibers and other granular materials [4].

Very important from a practical point of view is the description of the process of diffusion of hydrogen in metals. According to [3], it is useful in the creation of safe technologies and storage systems. For instance, there are systems for transport of gas and, in particular, for the storage of hydrogen, in which hydrogen is contained in nanostructures, metals or composites [3]. In fact, ”hydrogen is always present in metals and semiconductors”, with a concentration ranging from ”0.2 ppm (for aluminum alloys and high strength steels) up to 80 ppm (for titanium alloys)” [3, p. 38]. The same article [3] explains that the mechanical stresses in crystal matrix affect not only the temperature and concentration gradient, but also the diffusion of atoms in solids. Another article [18] emphasizes that the strength of metals is affected in a significant way by hydrogen, which is important in structural design. For example, the internal hydrogen which is accumulated and redistributed in the material may create and propagate voids.

Therefore, the system of partial differential equations introduced by Aouadi et al. in [19] and numerically analysed by Bazarra et al. in [20] for thermoelasticity with microconcentrations and microtemperatures provides a good model for studying such diffusion processes. This theory has appplications in ”many engineering applications such as satellite problems, aircraft landing on water, the manufacturing of integrated circuits or the oil extraction”, see [20, p. 31]. The concept of microconcentrations is a relatively recent one and was analysed in different contexts. For example, Bitsadze derived a solution in exact form for the Dirichlet boundary value problem for an isotropic circle in [21] and employed elementary functions in order to build the fundamental and singular matrices of solutions in [22]. Moreover, Giorgashvili et al. employed the potential method to study existence of classical solutions in [23]. In [24], Kansal derived the fundamental solutions for a micromorphic medium.

Now, we continue the analysis of the concept of microconcentrations and microtemperatures in the context of diffusive microstretch thermoelasticity based on the mathematical model from [11], where we studied wave propagation. This model characterizes materials for which the micro-particles may rotate, dilate or contract (this is the microstretch effect) and are equipped with microtemperatures and microconcentrations. The intended practical applications of this model are mentioned above.

In this study, the goal is to describe the spatial behaviour in the linear, anisotropic and non-centro-symmetric case, We define a function with a weight depending on time which describes the main characteristics of the model, then study its properties and deduce a differential inequality. Then we obtain an exponential spatial decay that is independent of time and is valid for unbounded bodies. The spatial behaviour of the thermoelastodynamic processes for microstretch solids was already studied in [15] in a simplified case. Many other problems in mechanics of generalized continua were approached from the point of view of the spatial behaviour, see for instance [25–27].

The article has four sections. In the second section we give some preliminaries, then we investigate the spatial behaviour and in the final section we draw the conclusions.

2 Preliminaries

We consider the three-dimensional euclidean space and a fixed system of rectangular axes Ox_i, $i = 1, 2, 3$. In this space, we assume that at a fixed moment in time we have a bounded or unbounded regular region Ω, whose boundary surface is denoted by $\partial\Omega$. As a usual convention, we employ the Einstein summation over repeated indices. The derivatives are represented by a superposed dot if they are with respect to the time variable and by a comma followed by a subscript if they are with respect to a spatial variable. The functions appearing in the mathematical model are defined over $\Omega \times (0, \infty)$.

The behaviour of the body is described by the equations of diffusive microstretch thermoelasticity with microtemperatures and microconcentrations, see [19]

$$t_{ji,j} + \rho f_i = \rho \ddot{u}_i, \tag{1}$$

$$h_{k,k} + g + \rho l = J\ddot{\varphi}, \tag{2}$$

$$m_{ji,j} + \varepsilon_{irs} t_{rs} + \rho g_i = I_{ij}\ddot{\varphi}_j, \tag{3}$$

$$\dot{C} = \eta_{i,i}, \tag{4}$$

$$\rho T_0 \dot{S} = q_{i,i} + \rho s, \tag{5}$$

$$\rho \dot{\varepsilon}_i = q_{ji,j} + q_i - Q_i + \rho G_i, \tag{6}$$

$$\rho \dot{\omega}_i = \eta_{ji,j} + \eta_i - \tilde{\sigma}_i. \tag{7}$$

The strain tensors are

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\varphi_k, \qquad \kappa_{ij} = \varphi_{j,i}, \qquad \zeta_i = \varphi_{,i}. \tag{8}$$

The following relation between the concentration and the chemical potential [19] holds true

$$\varrho C = P - d_{ij}e_{ij} - f_{ij}\kappa_{ij} - \tilde{f}_i\zeta_i - \tilde{g}_1\varphi + \varpi\theta + m_iT_i + m_i'C_i. \tag{9}$$

For the equations above we need the following initial conditions:

$$\begin{aligned}
&u_i(x,0) = u_i^0(x), && \dot{u}_i(x,0) = u_i^1(x), && P(x,0) = P^0(x),\\
&\varphi(x,0) = \varphi^0(x), && \dot{\varphi}(x,0) = \varphi^1(x), && C_i(x,0) = C_i^0(x),\\
&\varphi_i(x,0) = \varphi_i^0(x), && \dot{\varphi}_i(x,0) = \varphi_i^1(x), && \theta(x,0) = \theta^0(x),\\
&T_i(x,0) = T_i^0(x), && && x \in \bar{\Omega},
\end{aligned} \tag{10}$$

where u_i^0, u_i^1, φ_i^0, φ_i^1, φ^0, φ^1, θ^0, T_i^0, P^0 and C_i^0 are given. Let S_r, $r = 1, ..., 14$ be subsets of $\partial\Omega$ such that $\bar{S}_1 \cup S_2 = \bar{S}_3 \cup S_4 = \bar{S}_5 \cup S_6 = \bar{S}_7 \cup S_8 = \bar{S}_9 \cup S_{10} = \bar{S}_{11} \cup S_{12} = \bar{S}_{13} \cup S_{14} = \partial\Omega$ and $S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = S_7 \cap S_8 = S_9 \cap S_{10} = S_{11} \cap S_{12} = S_{13} \cap S_{14} = \varnothing$. In the case of the mixed problem we need the following boundary conditions:

$$\begin{aligned}
&u_i = \bar{u}_i \text{ on } \bar{S}_1 \times I, && \varphi_i = \bar{\varphi}_i \text{ on } \bar{S}_3 \times I, && \theta = \bar{\theta} \text{ on } \bar{S}_7 \times I,\\
&\varphi = \bar{\varphi} \text{ on } \bar{S}_5 \times I, && P = \bar{P} \text{ on } \bar{S}_{11} \times I, && T_i = \bar{T}_i \text{ on } \bar{S}_9 \times I,\\
&C_i = \bar{C}_i \text{ on } \bar{S}_{13} \times I, && t_{ji}n_j = \bar{t}_i \text{ on } S_2 \times I, && h_kn_k = \bar{h} \text{ on } S_6 \times I,\\
&m_{ji}n_j = \bar{m}_i \text{ on } S_4 \times I, && q_{ki}n_k = \bar{q}_i \text{ on } S_{10} \times I, && q_jn_j = \bar{q} \text{ on } S_8 \times I,\\
&\eta_in_i = \bar{\eta} \text{ on } S_{12} \times I, && \eta_{ki}n_k = \bar{\eta}_i \text{ on } S_{14} \times I,
\end{aligned} \tag{11}$$

where $\bar{u}_i$, $\bar{\varphi}_i$, $\bar{\varphi}$, $\bar{\theta}$, $\bar{T}_i$, $\bar{P}$, $\bar{C}_i$, $\bar{t}_i$, $\bar{m}_i$, $\bar{h}$, $\bar{q}$, $\bar{q}_i$, $\bar{\eta}$, $\bar{\eta}_i$ are given functions and $I = (0, \infty)$.

The constitutive equations of diffusive microstretch thermoelasticity with microtemperatures and microconcentrations in the homogeneous, anisotropic and non-centro-symmetric case are

$$\begin{aligned}
t_{ij} &= A^*_{ijrs}e_{rs} + B^*_{ijrs}\kappa_{rs} + D^*_{ij}\varphi + F^*_{ijk}\zeta_k + t'^*_{ij},\\
m_{ij} &= B^*_{rsij}e_{rs} + C^*_{ijrs}\kappa_{rs} + E^*_{ij}\varphi + G^*_{ijk}\zeta_k + m'^*_{ij},\\
h_i &= F^*_{rsi}e_{rs} + G^*_{rsi}\kappa_{rs} + A^*_{ij}\zeta_j + B^*_i\varphi + h'^*_i,\\
g &= -D^*_{ij}e_{ij} - E^*_{ij}\kappa_{ij} - B^*_i\zeta_i - \xi^*\varphi + g'^*,\\
\rho S &= a^*_{ij}e_{ij} + b^*_{ij}\kappa_{ij} + d^*_i\zeta_i + F^*\varphi + S'^*,\\
\rho\varepsilon_i &= L^*_{rsi}e_{rs} + M^*_{rsi}\kappa_{rs} - N^*_{ji}\zeta_j + R^*_i\varphi + \varepsilon'^*_i,\\
\rho\omega_i &= L'^*_{rsi}e_{rs} + M'^*_{rsi}\kappa_{rs} - N'^*_{ji}\zeta_j + R'^*_i\varphi + \omega'^*_i,
\end{aligned} \tag{12}$$

where the thermal and diffusive effects are described by

$$
\begin{aligned}
&t'^*_{ij} = L^*_{ijk}T_k - a^*_{ij}\theta + \frac{d_{ij}}{\varrho}P + L'^*_{ijk}C_k,\\
&m'^*_{ij} = M^*_{ijk}T_k - b^*_{ij}\theta + \frac{f_{ij}}{\varrho}P + M'^*_{ijk}C_k,\\
&h'^*_i = -N^*_{ij}T_j - d^*_i\theta + \frac{\tilde{f}_i}{\varrho}P - N'^*_{ij}C_j,\\
&g'^* = -R^*_iT_i + F^*\theta - \frac{\tilde{g}_1}{\varrho}P - R'^*_iC_i,\\
&S'^* = b^*_iT_i + a^*\theta + \frac{\varpi}{\varrho}P + b'^*_iC_i,\\
&\varepsilon'^*_i = -B^*_{ij}T_j - b^*_i\theta - R^*_{ij}C_j - \frac{m_i}{\varrho}P,\\
&\omega'^*_i = -C^*_{ij}C_j - R^*_{ji}T_j - b'^*_i\theta - \frac{m'_i}{\varrho}P
\end{aligned}
\tag{13}
$$

with the following symmetries for the coefficients:

$$
A^*_{ijrs} = A^*_{rsij},\ C^*_{ijrs} = C^*_{rsij},\ A^*_{ij} = A^*_{ji},\ B^*_{ij} = B^*_{ji},\ C^*_{ij} = C^*_{ji}. \tag{14}
$$

We have

$$
\begin{aligned}
&q_i = k_{ij}\theta_{,j} + K_{ij}T_j, \qquad \eta_i = h_{ij}P_{,j} + H_{ij}C_j,\\
&q_{ij} = -P_{ijkl}T_{l,k}, \qquad \eta_{ij} = -F_{ijkl}C_{l,k},\\
&Q_i = (k_{ij} - \tilde{k}_{ij})\theta_{,j} + (K_{ij} - \tilde{K}_{ij})T_j,\\
&\tilde{\sigma}_i = (h_{ij} - \tilde{h}_{ij})P_{,j} + (H_{ij} - \tilde{H}_{ij})C_j,
\end{aligned}
\tag{15}
$$

where the coefficients satisfy the following symmetry relations:

$$
\begin{aligned}
&P_{ijkl} = P_{klij}, \quad F_{ijkl} = F_{klij}, \quad K_{ij} = K_{ji},\\
&h_{ij} = h_{ji}, \quad \tilde{K}_{ij} = \tilde{K}_{ji}, \quad \tilde{H}_{ij} = \tilde{H}_{ji}, \quad \tilde{k}_{ij} = \tilde{k}_{ji},\\
&H_{ij} = H_{ji}, \quad k_{ij} = k_{ji}, \quad \tilde{h}_{ij} = \tilde{h}_{ji}
\end{aligned}
\tag{16}
$$

and $\theta = T - T_0$.

In the following we will introduce some useful notations. Below, W is expressed in terms of the strain tensors

$$
\begin{aligned}
W(s) = {}& \frac{1}{2}A^*_{ijrs}e_{rs}(s)e_{ij}(s) + B^*_{ijrs}e_{ij}(s)\kappa_{rs}(s) + \frac{1}{2}C^*_{ijrs}\kappa_{rs}(s)\kappa_{ij}(s)\\
&+ D^*_{ij}e_{ij}(s)\varphi(s) + E^*_{ij}\kappa_{ij}(s)\varphi(s) + F^*_{ijk}e_{ij}(s)\zeta_k(s)\\
&+ G^*_{ijk}\kappa_{ij}(s)\zeta_k(s) + B^*_i\varphi(s)\zeta_i(s) + \frac{1}{2}\xi^*\varphi(s)^2 + \frac{1}{2}A^*_{ij}\zeta_j(s)\zeta_i(s)
\end{aligned}
\tag{17}
$$

and is used in the formulation of $\bar{\Gamma}$. The thermal and diffusive effects are accounted for by the expressions below

$$\bar{\Gamma}(s) = \frac{1}{2}\rho\dot{u}_i(s)\dot{u}_i(s) + \frac{1}{2}J\dot{\varphi}(s)^2 + \frac{1}{2}I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s) + W(s) + \Gamma_p(s), \tag{18}$$

where

$$\begin{aligned} \Gamma_p(s) = {} & a^*\frac{1}{2}\theta(s)^2 + \frac{1}{\varrho}\frac{1}{2}P(s)^2 + \frac{1}{2}B^*_{ij}T_i(s)T_j(s) \\ & + \frac{1}{2}C^*_{ij}C_i(s)C_j(s) + b^*_iT_i(s)\theta(s) + \frac{\varpi}{\varrho}\theta(s)P(s) \\ & + R^*_{ij}C_j(s)T_i(s) + b'^*_i\theta(s)C_i(s) + \frac{m_i}{\varrho}P(s)T_i(s) + \frac{m'_i}{\varrho}P(s)C_i(s) \end{aligned} \tag{19}$$

and

$$\begin{aligned} \bar{K}(s) = {} & \frac{1}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s) + \left(\frac{1}{T_0}K_{ij} + \tilde{k}_{ij}\right)\theta_{,i}(s)T_j(s) \\ & + P_{ijkl}T_{l,k}(s)T_{j,i}(s) + \tilde{K}_{ij}T_i(s)T_j(s) + \tilde{H}_{ij}C_i(s)C_j(s) \\ & + \left(\tilde{h}_{ij} + H_{ij}\right)P_{,i}(s)C_j(s) + h_{ij}P_{,i}(s)P_{,j}(s) - F_{jikl}C_{l,kj}(s)C_i(s). \end{aligned} \tag{20}$$

We will need the following assumptions in the sequel, see [13]:

(H1) ρ and J are strictly positive;

(H2) I_{ij} is positive definite;

(H3) W is a positive semidefinite quadratic form;

(H4) Γ_p is positive definite for any admissible process p;

(H5) W is a positive definite quadratic form, so there exist the positive constants μ_m and μ_M such that

$$\begin{aligned} & \mu_m\left(e_{ij}e_{ij} + \varphi^2 + \frac{I_0}{\varrho}\kappa_{ij}\kappa_{ij} + \frac{J_0}{\varrho}\zeta_i\zeta_i\right) \\ & \qquad \le 2W \le \mu_M\left(e_{ij}e_{ij} + \varphi^2 + \frac{I_0}{\varrho}\kappa_{ij}\kappa_{ij} + \frac{J_0}{\varrho}\zeta_i\zeta_i\right). \end{aligned} \tag{21}$$

Furthermore, we assume that $\bar{K}(s)$ is a positive definite quadratic form, that is

$$\bar{K}(s) \leq \frac{1}{T_0} k_M \theta_{,i}(s)\theta_{,i}(s) + P_M T_{i,j}(s)T_{i,j}(s) + \tilde{K}_M T_i(s)T_i(s) + \tilde{H}_M C_i(s)C_i(s) + h_M P_{,i}(s)P_{,i}(s) + F_M C_{i,j}(s)C_{i,j}(s) \tag{22}$$

and

$$\bar{K}(s) \geq \frac{1}{T_0} k_m \theta_{,i}(s)\theta_{,i}(s) + P_m T_{i,j}(s)T_{i,j}(s) + \tilde{K}_m T_i(s)T_i(s) + \tilde{H}_m C_i(s)C_i(s) + h_m P_{,i}(s)P_{,i}(s) + F_m C_{i,j}(s)C_{i,j}(s). \tag{23}$$

3 Uniqueness

In this section, we prove uniqueness of the solution for the problem of anisotropic microstretch thermoelasticity with microtemperatures and microconcentrations in the non-centro-symmetric case. To this end, we examine the admissible process

$$p = \{u_i, \varphi_i, \varphi, \theta, T_i, P, C_i, t_{ij}, m_{ij}, h_i, g, S, \varepsilon_i, q_i, Q_i, q_{ij}, \omega_i, \eta_i, \eta_{ij}\}. \tag{24}$$

We follow the strategy from [13].

Theorem 3.1. *We consider that the assumptions (H1), (H2), (H3) and (H4) hold true. Then the problem of microstretch thermoelasticity with microtemperatures and microconcentrations has at most one solution.*

Proof The relation below follows from the constitutive equations (12) and from (17), (19)

$$t_{ij}\dot{e}_{ij} + m_{ij}\dot{\kappa}_{ij} + h_i\dot{\zeta}_i - g\dot{\varphi} + \rho\dot{S}\theta - \rho\dot{\varepsilon}_i T_i - \rho\dot{\omega}_i C_i + P\dot{C} = \dot{W} + \dot{\Gamma}_p. \tag{25}$$

Moreover, relations (8), (5), (6), (7) in the first step and then, in the second step, relations (1) multiplied by $\dot{u}_i$, (3) multiplied by $\dot{\varphi}_i$, (2) multiplied by $\dot{\varphi}$ and (15) lead to

$$\begin{aligned} t_{ij}\dot{e}_{ij} + m_{ij}\dot{\kappa}_{ij} + h_i\dot{\zeta}_i - g\dot{\varphi} + \rho\dot{S}\theta - \rho\dot{\varepsilon}_i T_i - \rho\dot{\omega}_i C_i + P\dot{C} = \\ = \left(t_{ji}\dot{u}_i + m_{ji}\dot{\varphi}_i + h_j\dot{\varphi} + \frac{1}{T_0}\theta q_j - T_i q_{ji} - C_i\eta_{ji} + h_{ij}P_{,i}P + \right. \\ \left. + H_{ij}C_iP - F_{jikl}C_{l,k}C_i \right)_{,j} + \rho\left(f_i\dot{u}_i + g_i\dot{\varphi}_i + l\dot{\varphi} + \frac{1}{T_0}\theta s - T_iG_i \right) \\ - \rho\ddot{u}_i\dot{u}_i - I_{ij}\ddot{\varphi}_j\dot{\varphi}_i - J\ddot{\varphi}\dot{\varphi} - \bar{K}. \end{aligned} \tag{26}$$

We obtain the following equality from (25) and (26)

$$\frac{1}{2}\frac{\partial}{\partial t}\left(2W+2\Gamma_p+\rho\dot{\mathbf{u}}^2+I_{ij}\dot{\varphi}_i\dot{\varphi}_j+J\dot{\varphi}^2\right)= \\ =\left(t_{ji}\dot{u}_i+m_{ji}\dot{\varphi}_i+h_j\dot{\varphi}+\frac{1}{T_0}\theta q_j-T_iq_{ji}-C_i\eta_{ji}+h_{ij}P_{,i}P+H_{ij}C_iP- \right. \\ \left. -F_{jikl}C_{l,k}C_i\right)_{,j}+\rho\left(f_i\dot{u}_i+g_i\dot{\varphi}_i+l\dot{\varphi}+\frac{1}{T_0}\theta s-T_iG_i\right)-\bar{K}. \tag{27}$$

For the process p, we define the following function on I

$$E_p=\frac{1}{2}\int_\Omega\left(2W+2\Gamma_p+\rho\dot{\mathbf{u}}^2+I_{ij}\dot{\varphi}_i\dot{\varphi}_j+J\dot{\varphi}^2\right)dV. \tag{28}$$

Hence, we obtain

$$\dot{E}_p=\int_\Omega\rho\left(f_i\dot{u}_i+g_i\dot{\varphi}_i+l\dot{\varphi}+\frac{1}{T_0}\theta s-T_iG_i\right)dV \\ +\int_{\partial\Omega}\left(t_{ji}\dot{u}_i+m_{ji}\dot{\varphi}_i+h_j\dot{\varphi}+\frac{1}{T_0}\theta q_j-T_iq_{ji}-C_i\eta_{ji}+h_{ij}P_{,i}P+ \right. \\ \left. +H_{ij}C_iP-F_{jikl}C_{l,k}C_i\right)n_jdA-\int_\Omega\bar{K}dV. \tag{29}$$

In the following, we take two solutions of the problem, namely

$$p_\alpha=\{u_i^{(\alpha)},\varphi_i^{(\alpha)},\varphi^{(\alpha)},\theta^{(\alpha)},T_i^{(\alpha)},P^{(\alpha)},C_i^{(\alpha)},t_{ij}^{(\alpha)},m_{ij}^{(\alpha)},h_i^{(\alpha)}, \\ g^{(\alpha)},S^{(\alpha)},\varepsilon_i^{(\alpha)},q_i^{(\alpha)},Q_i^{(\alpha)},q_{ij}^{(\alpha)},\omega_i^{(\alpha)},\eta_i^{(\alpha)},\eta_{ij}^{(\alpha)}\}. \tag{30}$$

Then we consider the process

$$\pi=\{u_i^*,\varphi_i^*,\varphi^*,\theta^*,T_i^*,P^*,C_i^*,t_{ij}^*,m_{ij}^*,h_i^*,g^*,S^*,\varepsilon_i^*,q_i^*, \\ Q_i^*,q_{ij}^*,\omega_i^*,\eta_i^*,\eta_{ij}^*\} \tag{31}$$

by
$u_i^*=u_i^{(1)}-u_i^{(2)}$, $\varphi_i^*=\varphi_i^{(1)}-\varphi_i^{(2)}$, $\varphi^*=\varphi^{(1)}-\varphi^{(2)}$, $\theta^*=\theta^{(1)}-\theta^{(2)}$, $T_i^*=T_i^{(1)}-T_i^{(2)}$, $P^*=P^{(1)}-P^{(2)}$, $C_i^*=C_i^{(1)}-C_i^{(2)}$, $t_{ij}^*=t_{ij}^{(1)}-t_{ij}^{(2)}$, $m_{ij}^*=m_{ij}^{(1)}-m_{ij}^{(2)}$, $h_i^*=h_i^{(1)}-h_i^{(2)}$, $g^*=g^{(1)}-g^{(2)}$, $S^*=S^{(1)}-S^{(2)}$, $\varepsilon_i^*=\varepsilon_i^{(1)}-\varepsilon_i^{(2)}$, $q_i^*=q_i^{(1)}-q_i^{(2)}$, $Q_i^*=Q_i^{(1)}-Q_i^{(2)}$, $q_{ij}^*=q_{ij}^{(1)}-q_{ij}^{(2)}$, $\omega_i^*=\omega_i^{(1)}-\omega_i^{(2)}$, $\eta_i^*=\eta_i^{(1)}-\eta_i^{(2)}$, $\eta_{ij}^*=\eta_{ij}^{(1)}-\eta_{ij}^{(2)}$. We assume that $S_1=S_3=S_5=S_7=S_9=S_{11}=S_{13}=\partial\Omega$.
Then the process π is associated to zero data. Therefore, by relation (29) we are led to

$$\dot{E}_\pi\le 0 \quad \text{on} \quad I. \tag{32}$$

Using the initial conditions (10), we are led to $E_\pi \leq 0$ on I. By considering the assumptions (H1), (H2), (H3), (H4) and (28), we obtain $E_\pi \geq 0$. Hence $E_\pi = 0$ and $\dot{u}_i^* = 0$, $\dot{\varphi}_i^* = 0$, $\dot{\varphi}^* = 0$, $\theta^* = 0$, $T_i^* = 0$, $P^* = 0$, $C_i^* = 0$ on I. From the initial conditions (10) we obtain $u_i^* = 0$, $\varphi_i^* = 0$, $\varphi^* = 0$ on I. Hence our problem has at most one solution. ■

4 Spatial Behaviour

As in [15], we denote by $\hat{D}_T$ the support of the initial and boundary data and the body actions in the problem $\mathcal{P}$. We use the convention that $\hat{D}_T$ is a bounded regular region. Then we define the set D_R, $R \geq 0$, by

$$D_R = \{\boldsymbol{x} \in \bar{\Omega} \,|\, \hat{D}_T \cap \overline{\Sigma(\boldsymbol{x}, R)} \neq \emptyset\}, \tag{33}$$

where $\Sigma(\boldsymbol{x}, R)$ is the open ball with radius R and center at $\boldsymbol{x}$. As in [15], we assume that B_R is the part of Ω contained in $\Omega \setminus D_R$ and that $B(R_1, R_2)$ is the set $B_{R_2} \setminus B_{R_1}$, $R_1 > R_2$. Furthermore, we introduce the notation S_R for the subsurface of ∂B_R contained inside Ω and whose outward unit normal vector is forwarded to the exterior of D_R. Note that B_R is a part of Ω where the initial and boundary data and the body actions are zero.

Moreover, let us consider $\lambda > 0$ to be a given real parameter. Then we define

$$\bar{\Lambda}_1(t) = \int_0^t \int_{\partial\Omega} e^{-\lambda s} \left[t_{ji}(s)n_j(s)\dot{u}_i(s) + h_i(s)n_i(s)\dot{\varphi}(s) + m_{ji}(s)n_j(s) \right. \\ \left. \times \dot{\varphi}_i(s) + \frac{1}{T_0} q_i(s)n_i(s)\theta(s) + \eta_i(s)n_i(s)P(s) - q_{ji}(s)n_j(s)T_i(s)\right] \mathrm{d}a\mathrm{d}s, \tag{34}$$

a function that will be used in the evaluation of the spatial behaviour by considering the interior integral over the subset S_R. The main properties of this function will follow from the lemma below.

Lemma 4.1. *We assume that the behaviour of a microstretch elastic body with thermal and mass diffusion at the micro- and macrolevel is described by the governing equations (1)-(7) and by the constitutive equations (12) with the geometrical relations (8). Under suitable regularity assumptions and with the notations above, the following identity holds true*

$$\int_\Omega e^{-\lambda t}\bar{\Gamma}(t)\mathrm{d}v + \int_0^t \int_\Omega \left[e^{-\lambda s}\lambda\bar{\Gamma}(s) + e^{-\lambda s}\bar{K}(s)\right] \mathrm{d}v\mathrm{d}s = \\ = \int_\Omega \bar{\Gamma}(0)\mathrm{d}v + \int_0^t \int_\Omega e^{-\lambda s}\left[\rho f_i(s)\dot{u}_i(s) + \rho l(s)\dot{\varphi}(s) \right. \\ \left. + \rho g_i(s)\dot{\varphi}_i(s) + \frac{1}{T_0}\rho s(s)\theta(s) - \rho G_i(s)T_i(s)\right] \mathrm{d}v\mathrm{d}s + \bar{\Lambda}_1(t). \tag{35}$$

Proof We multiply equation (1) by $\dot{u}_i$, then use the constitutive equations $(12)_1$ and multiply the resulting equation by $e^{-\lambda s}$ in order to obtain

$$
\begin{aligned}
(36)\quad & \frac{\partial}{\partial s}\left[e^{-\lambda s}\frac{1}{2}\rho\dot{u}_i(s)\dot{u}_i(s)\right]+\lambda e^{-\lambda s}\frac{1}{2}\rho\dot{u}_i(s)\dot{u}_i(s)=e^{-\lambda s}\rho f_i(s)\dot{u}_i(s)\\
&+e^{-\lambda s}\left[t_{ji}(s)\dot{u}_i(s)\right]_{,j}+e^{-\lambda s}\Big[-A^*_{ijrs}e_{rs}(s)\dot{u}_{j,i}(s)-B^*_{ijrs}\kappa_{rs}(s)\dot{u}_{j,i}(s)\\
&-D^*_{ij}\varphi(s)\dot{u}_{j,i}(s)-F^*_{ijk}\zeta_k(s)\dot{u}_{j,i}(s)-L^*_{ijk}T_k(s)\dot{u}_{j,i}(s)\\
&+a^*_{ij}\theta(s)\dot{u}_{j,i}(s)-\frac{d_{ij}}{\varrho}P(s)\dot{u}_{j,i}(s)-L'^*_{ijk}C_k(s)\dot{u}_{j,i}(s)\Big].
\end{aligned}
$$

Equation (2) is multiplied by $\dot{\varphi}$. Then we replace the constitutive equations $(12)_3$ and $(12)_4$, then multiply by $e^{-\lambda s}$ in order to obtain

$$
\begin{aligned}
(37)\quad & \frac{\partial}{\partial s}\left[e^{-\lambda s}\frac{1}{2}J\dot{\varphi}(s)^2\right]+\lambda e^{-\lambda s}\frac{1}{2}J\dot{\varphi}(s)^2=e^{-\lambda s}\rho l(s)\dot{\varphi}(s)\\
&+e^{-\lambda s}\left[h_i(s)\dot{\varphi}(s)\right]_{,i}+e^{-\lambda s}\Big[-F^*_{rsi}e_{rs}(s)\dot{\varphi}_{,i}(s)-G^*_{rsi}\kappa_{rs}(s)\dot{\varphi}_{,i}(s)\\
&-A^*_{ij}\zeta_j(s)\dot{\varphi}_{,i}(s)-B^*_i\varphi(s)\dot{\varphi}_{,i}(s)+N^*_{ij}T_j(s)\dot{\varphi}_{,i}(s)\\
&+d^*_i\theta(s)\dot{\varphi}_{,i}(s)-\frac{\tilde{f}_i}{\varrho}P(s)\dot{\varphi}_{,i}(s)+N'^*_{ij}C_j(s)\dot{\varphi}_{,i}(s)-D^*_{ij}e_{ij}(s)\dot{\varphi}(s)\\
&-E^*_{ij}\kappa_{ij}(s)\dot{\varphi}(s)-B^*_i\zeta_i(s)\dot{\varphi}(s)-\xi^*\varphi(s)\dot{\varphi}(s)-R^*_iT_i(s)\dot{\varphi}(s)\\
&+F^*\theta(s)\dot{\varphi}(s)-\frac{\tilde{g}_1}{\varrho}P(s)\dot{\varphi}(s)-R'^*_iC_i(s)\dot{\varphi}(s)\Big].
\end{aligned}
$$

We use the same technique with equation (3) multiplied by $\dot{\varphi}_i$ and the constitutive equations $(12)_1$ and $(12)_2$. Hence, we obtain

$$
\begin{aligned}
(38)\quad & \frac{\partial}{\partial s}\left[e^{-\lambda s}\frac{1}{2}I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s)\right]+\lambda e^{-\lambda s}\frac{1}{2}I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s)=e^{-\lambda s}\rho g_i(s)\dot{\varphi}_i(s)\\
&+e^{-\lambda s}\left[m_{ji}(s)\dot{\varphi}_i(s)\right]_{,j}+e^{-\lambda s}\Big[-B^*_{rsij}e_{rs}(s)\dot{\varphi}_{j,i}(s)-C^*_{ijrs}\kappa_{rs}(s)\dot{\varphi}_{j,i}(s)\\
&-E^*_{ij}\varphi(s)\dot{\varphi}_{j,i}(s)-G^*_{ijk}\zeta_k(s)\dot{\varphi}_{j,i}(s)-M^*_{ijk}T_k(s)\dot{\varphi}_{j,i}(s)\\
&+b^*_{ij}\theta(s)\dot{\varphi}_{j,i}(s)-\frac{f_{ij}}{\varrho}P(s)\dot{\varphi}_{j,i}(s)-M'^*_{ijk}C_k(s)\dot{\varphi}_{j,i}(s)\\
&+\varepsilon_{ijk}A^*_{jkrs}e_{rs}(s)\dot{\varphi}_i(s)+\varepsilon_{ijk}B^*_{jkrs}\kappa_{rs}(s)\dot{\varphi}_i(s)+\varepsilon_{ijk}D^*_{jk}\varphi(s)\dot{\varphi}_i(s)\\
&+\varepsilon_{ijk}F^*_{jkl}\zeta_l(s)\dot{\varphi}_i(s)+\varepsilon_{ijk}L^*_{jkl}T_l(s)\dot{\varphi}_i(s)-\varepsilon_{ijk}a^*_{jk}\theta(s)\dot{\varphi}_i(s)\\
&+\varepsilon_{ijk}\frac{d_{jk}}{\varrho}P(s)\dot{\varphi}_i(s)+\varepsilon_{ijk}L'^*_{jkl}C_l(s)\dot{\varphi}_i(s)\Big].
\end{aligned}
$$

After multiplying equation (5) by θ, we replace the constitutive equation $(12)_5$ and multiply by $e^{-\lambda s}$ to obtain

$$\begin{aligned}(39)\quad \frac{\partial}{\partial s}\left[e^{-\lambda s}a^*\frac{1}{2}\theta(s)^2\right] + \lambda e^{-\lambda s}a^*\frac{1}{2}\theta(s)^2 = e^{-\lambda s}\Big[&- a^*_{ij}\dot{e}_{ij}(s)\theta(s) \\ - b^*_{ij}\dot{\kappa}_{ij}(s)\theta(s) - d^*_i\dot{\zeta}_i(s)\theta(s) - F^*\dot{\varphi}(s)\theta(s) &- b^*_i\dot{T}_i(s)\theta(s) \\ - \frac{\varpi}{\varrho}\dot{P}(s)\theta(s) - b'^*_i\dot{C}_i(s)\theta(s)\Big] + e^{-\lambda s}\Big\{&\frac{1}{T_0}[q_i(s)\theta(s)]_{,i} \\ - \frac{1}{T_0}k_{ij}\theta_{,j}(s)\theta_{,i}(s) - \frac{1}{T_0}K_{ij}T_j(s)\theta_{,i}(s) &+ \frac{1}{T_0}\rho s(s)\theta(s)\Big\}.\end{aligned}$$

The equation for the concentration (4) is multiplied by the chemical potential P. Then the equations (9) and $(15)_2$ are replaced and the factor $e^{-\lambda s}$ is employed.

$$\begin{aligned}(40)\quad \frac{\partial}{\partial s}\left[e^{-\lambda s}\frac{1}{\varrho}\frac{1}{2}P(s)^2\right] + \lambda e^{-\lambda s}\frac{1}{\varrho}\frac{1}{2}P(s)^2 = e^{-\lambda s}\Big[&\frac{d_{ij}}{\varrho}\dot{e}_{ij}(s)P(s) \\ + \frac{f_{ij}}{\varrho}\dot{\kappa}_{ij}(s)P(s) + \frac{\tilde{f}_i}{\varrho}\dot{\zeta}_i(s)P(s) + \frac{\tilde{g}_1}{\varrho}\dot{\varphi}(s)P(s) &- \frac{\varpi}{\varrho}\dot{\theta}(s)P(s) \\ - \frac{m_i}{\varrho}\dot{T}_i(s)P(s) - \frac{m'_i}{\varrho}\dot{C}_i(s)P(s)\Big] &+ e^{-\lambda s}\left[\eta_i(s)P(s)\right]_{,i} \\ + e^{-\lambda s}\left[-h_{ij}P_{,j}(s)P_{,i}(s) - H_{ij}C_j(s)P_{,i}(s)\right]&.\end{aligned}$$

The proof proceeds in the same way with equation (6) multiplied by T_i and the use of relations $(12)_6$, $(15)_1$, $(15)_3$, $(15)_5$ and the factor $e^{-\lambda s}$.

$$\begin{aligned}(41)\quad \frac{\partial}{\partial s}\left[e^{-\lambda s}\frac{1}{2}B^*_{ij}T_i(s)T_j(s)\right] &+ \lambda e^{-\lambda s}\frac{1}{2}B^*_{ij}T_i(s)T_j(s) = \\ = e^{-\lambda s}\Big[L^*_{rsi}\dot{e}_{rs}(s)T_i(s) &+ M^*_{rsi}\dot{\kappa}_{rs}(s)T_i(s) - N^*_{ji}\dot{\zeta}_j(s)T_i(s) \\ + R^*_i\dot{\varphi}(s)T_i(s) - b^*_i\dot{\theta}(s)T_i(s) &- R^*_{ij}\dot{C}_j(s)T_i(s) - \frac{m_i}{\varrho}\dot{P}(s)T_i(s)\Big] \\ + e^{-\lambda s}\Big[-\tilde{k}_{ij}\theta_{,j}(s)T_i(s) &- \tilde{K}_{ij}T_j(s)T_i(s) - \rho G_i(s)T_i(s)\Big] \\ + e^{-\lambda s}\left[-q_{ji}(s)T_i(s)\right]_{,j} &- e^{-\lambda s}P_{ijkl}T_{l,k}(s)T_{j,i}(s).\end{aligned}$$

Equation (7) is multiplied by C_i. Then, the relations $(12)_7$, $(15)_2$, $(15)_4$, $(15)_6$ and the factor $e^{-\lambda s}$ lead to the following result:

$$(42)\quad \frac{\partial}{\partial s}\left[e^{-\lambda s}\frac{1}{2}C^*_{ij}C_i(s)C_j(s)\right] + \lambda e^{-\lambda s}\frac{1}{2}C^*_{ij}C_i(s)C_j(s) =$$

$$
\begin{aligned}
&= e^{-\lambda s}\Big[- R^*_{ji}\dot{T}_j(s)C_i(s) + L'^*_{rsi}\dot{e}_{rs}(s)C_i(s) \\
&+ M'^*_{rsi}\dot{\kappa}_{rs}(s)C_i(s) - N'^*_{ji}\dot{\zeta}_j(s)C_i(s) + R'^*_i\dot{\varphi}(s)C_i(s) - b'^*_i\dot{\theta}(s)C_i(s) \\
&- \frac{m'_i}{\varrho}\dot{P}(s)C_i(s) + F_{jikl}C_{l,kj}(s)C_i(s) - \tilde{h}_{ij}P_{,j}(s)C_i(s) - \tilde{H}_{ij}C_j(s)C_i(s)\Big].
\end{aligned}
$$

By adding up the relations above, we obtain the required result. ■

In the following lemma we will prove an estimate in order to derive in the next theorem a differential inequality for a time-weighted function that we will employ for the characterization of the spatial behaviour.

Lemma 4.2. *Under the assumptions of Lemma 4.1 and for each $\varepsilon > 0$ constant, the following inequality holds true*

$$
\begin{aligned}
(43)\quad & t_{ij}t_{ij} + \frac{\rho_0}{J_0}h_ih_i + \frac{\rho_0}{I_0}m_{ij}m_{ij} \le 2\mu_M(1+\varepsilon)W \\
&+ 4\left(1+\frac{1}{\varepsilon}\right)\Big\{ \left[(L^*_{ijk})^2 + (N^*_{ik})^2 + (M^*_{ijk})^2\right]T_k^2 \\
&+ \left[(a^*_{ij})^2 + (d^*_i)^2 + (b^*_{ij})^2\right]\theta^2 + \left[\left(\frac{d_{ij}}{\varrho}\right)^2 + \left(\frac{\tilde{f}_i}{\varrho}\right)^2 + \left(\frac{f_{ij}}{\varrho}\right)^2\right]P^2 \\
&+ \left[(L'^*_{ijk})^2 + (N'^*_{ik})^2 + (M'^*_{ijk})^2\right]C_k^2\Big\}.
\end{aligned}
$$

Proof Let us define

$$
J_1 = \sqrt{\frac{J_0}{\rho_0}}, \quad I_1 = \sqrt{\frac{I_0}{\rho_0}} \tag{44}
$$

and the vector

$$
\boldsymbol{U} := \{u_i, J_1\varphi, I_1\varphi_i\} \tag{45}
$$

in order to consider the state of strain $\boldsymbol{E}(\boldsymbol{U})$ in the form

$$
\boldsymbol{E}(\boldsymbol{U}) := \{e_{ij}(\boldsymbol{U}), \varphi, J_1\zeta_i(\boldsymbol{U}), I_1\kappa_{ij}(\boldsymbol{U})\}. \tag{46}
$$

We define

$$
\begin{aligned}
(47)\qquad s_{ij}(\boldsymbol{E}) &= t_{ij}(\boldsymbol{E}) - L^*_{ijk}T_k + a^*_{ij}\theta - \frac{d_{ij}}{\varrho}P - L'^*_{ijk}C_k \\
&= A^*_{ijrs}e_{rs} + B^*_{ijrs}\kappa_{rs} + D^*_{ij}\varphi + F^*_{ijk}\zeta_k,
\end{aligned}
$$

$$\begin{aligned} M_{ij}(\boldsymbol{E}) &= m_{ij}(\boldsymbol{E}) - M^*_{ijk}T_k + b^*_{ij}\theta - \frac{f_{ij}}{\varrho}P - M'^*_{ijk}C_k \\ &= B^*_{rsij}e_{rs} + C^*_{ijrs}\kappa_{rs} + E^*_{ij}\varphi + G^*_{ijk}\zeta_k, \\ \Gamma_i(\boldsymbol{E}) &= h_i(\boldsymbol{E}) + N^*_{ij}T_j + d^*_i\theta - \frac{\tilde{f}_i}{\varrho}P + N'^*_{ij}C_j \\ &= F^*_{rsi}e_{rs} + G^*_{rsi}\kappa_{rs} + A^*_{ij}\zeta_j + B^*_i\varphi, \\ G(\boldsymbol{E}) &= g(\boldsymbol{E}) + R^*_iT_i - F^*\theta + \frac{\tilde{g}_1}{\varrho}P + R'^*_iC_i \\ &= -D^*_{ij}e_{ij} - E^*_{ij}\kappa_{ij} - B^*_i\zeta_i - \xi^*\varphi. \end{aligned}$$

Moreover, for every $\boldsymbol{E}$, we define

$$M(\boldsymbol{E}) = \{s_{ij}(\boldsymbol{E}), G(\boldsymbol{E}), \frac{1}{J_1}\Gamma_i(\boldsymbol{E}), \frac{1}{I_1}M_{ij}(\boldsymbol{E})\}. \tag{48}$$

The magnitude of M is given by

$$|M(\boldsymbol{E})| = \left\{ s_{ij}(\boldsymbol{E})s_{ij}(\boldsymbol{E}) + G(\boldsymbol{E})^2 + \frac{\rho_0}{J_0}\Gamma_i(\boldsymbol{E})\Gamma_i(\boldsymbol{E}) + \frac{\rho_0}{I_0}M_{ij}(\boldsymbol{E})M_{ij}(\boldsymbol{E}) \right\}^{\frac{1}{2}}. \tag{49}$$

Note that

$$\boldsymbol{E}^{(\alpha)}(\boldsymbol{U}) := \{e_{ij}\left(\boldsymbol{U}^{(\alpha)}\right), \varphi^{(\alpha)}, J_1\zeta_i\left(\boldsymbol{U}^{(\alpha)}\right), I_1\kappa_{ij}\left(\boldsymbol{U}^{(\alpha)}\right)\}. \tag{50}$$

With this notation, we define the following bilinear form:

$$\begin{aligned} \mathcal{L}\left(\boldsymbol{E}^{(1)}, \boldsymbol{E}^{(2)}\right) := \frac{1}{2}\Big[& A^*_{ijrs}e^{(1)}_{ij}e^{(2)}_{rs} + \xi^*\varphi^{(1)}\varphi^{(2)} \\ & + C^*_{ijrs}\kappa^{(1)}_{ij}\kappa^{(2)}_{rs} + A^*_{ij}\zeta^{(1)}_i\zeta^{(2)}_j \\ & + F^*_{ijk}\left(e^{(1)}_{ij}\zeta^{(2)}_k + e^{(2)}_{ij}\zeta^{(1)}_k\right) + D^*_{ij}\left(e^{(1)}_{ij}\varphi^{(2)} + e^{(2)}_{ij}\varphi^{(1)}\right) \\ & + B^*_{ijrs}\left(e^{(1)}_{ij}\kappa^{(2)}_{rs} + e^{(2)}_{ij}\kappa^{(1)}_{rs}\right) + G^*_{ijk}\left(\kappa^{(1)}_{ij}\zeta^{(2)}_k + \kappa^{(2)}_{ij}\zeta^{(1)}_k\right) \\ & + E^*_{ij}\left(\kappa^{(1)}_{ij}\varphi^{(2)} + \kappa^{(2)}_{ij}\varphi^{(1)}\right) + B^*_i\left(\zeta^{(1)}_i\varphi^{(2)} + \zeta^{(2)}_i\varphi^{(1)}\right)\Big] \end{aligned} \tag{51}$$

and by the symmetry conditions (14), we are led to

$$\mathcal{L}\left(\boldsymbol{E}^{(1)}, \boldsymbol{E}^{(2)}\right) = \mathcal{L}\left(\boldsymbol{E}^{(2)}, \boldsymbol{E}^{(1)}\right). \tag{52}$$

Hence

$$\mathcal{L}(\boldsymbol{E}, \boldsymbol{E}) = W(\boldsymbol{E}) \tag{53}$$

and by the assumption (H5) and the Cauchy-Schwarz inequality we are led to

$$\mathcal{L}\left(\boldsymbol{E}^{(1)},\boldsymbol{E}^{(2)}\right)\leq\left[W\left(\boldsymbol{E}^{(1)}\right)\right]^{\frac{1}{2}}\left[W\left(\boldsymbol{E}^{(2)}\right)\right]^{\frac{1}{2}}. \tag{54}$$

By using (47), we are led to

$$|M\left(\boldsymbol{E}\right)|^{2}=2\mathcal{L}\left(\boldsymbol{E},\tilde{M}\left(\boldsymbol{E}\right)\right), \tag{55}$$

with

$$\tilde{M}\left(\boldsymbol{E}\right)=\{s_{ij}\left(\boldsymbol{E}\right),G\left(\boldsymbol{E}\right),J_{1}\frac{\rho_{0}}{J_{0}}\Gamma_{i}\left(\boldsymbol{E}\right),I_{1}\frac{\rho_{0}}{I_{0}}M_{ij}\left(\boldsymbol{E}\right)\}. \tag{56}$$

Hence

$$|M\left(\boldsymbol{E}\right)|^{2}\leq 2\mu_{M}W\left(\boldsymbol{E}\right) \tag{57}$$

and

$$\begin{aligned} s_{ij}(\boldsymbol{E})s_{ij}(\boldsymbol{E})+G(\boldsymbol{E})^{2}+\frac{\rho_{0}}{J_{0}}\Gamma_{i}(\boldsymbol{E})\Gamma_{i}(\boldsymbol{E})&\\ +\frac{\rho_{0}}{I_{0}}M_{ij}(\boldsymbol{E})M_{ij}(\boldsymbol{E})&\leq 2\mu_{M}W(\boldsymbol{E}). \end{aligned} \tag{58}$$

By using (47), we evaluate

$$\begin{aligned} &t_{ij}t_{ij}+\frac{\rho_{0}}{J_{0}}h_{i}h_{i}+\frac{\rho_{0}}{I_{0}}m_{ij}m_{ij}\\ &\leq(1+\varepsilon)\Big(s_{ij}s_{ij}+\frac{\rho_{0}}{J_{0}}\Gamma_{i}\Gamma_{i}+\frac{\rho_{0}}{I_{0}}M_{ij}M_{ij}\Big)\\ &+\left(1+\frac{1}{\varepsilon}\right)\Bigg[(L^{*}_{ijk}T_{k}-a^{*}_{ij}\theta+\frac{d_{ij}}{\varrho}P+L'^{*}_{ijk}C_{k})^{2}\\ &+\Big(-N^{*}_{ij}T_{j}-d^{*}_{i}\theta+\frac{\tilde{f}_{i}}{\varrho}P-N'^{*}_{ij}C_{j}\Big)^{2}+\Big(M^{*}_{ijk}T_{k}-b^{*}_{ij}\theta+\frac{f_{ij}}{\varrho}P+M'^{*}_{ijk}C_{k}\Big)^{2}\Bigg]\\ &\leq(1+\varepsilon)\Big(s_{ij}s_{ij}+\frac{\rho_{0}}{J_{0}}\Gamma_{i}\Gamma_{i}+\frac{\rho_{0}}{I_{0}}M_{ij}M_{ij}\Big)\\ &+4\Big(1+\frac{1}{\varepsilon}\Big)\Bigg\{[(L^{*}_{ijk})^{2}+(N^{*}_{ik})^{2}+(M^{*}_{ijk})^{2}]T_{k}^{2}\\ &+[(a^{*}_{ij})^{2}+(d^{*}_{i})^{2}+(b^{*}_{ij})^{2}]\theta^{2}+\Big[\Big(\frac{d_{ij}}{\varrho}\Big)^{2}+\Big(\frac{\tilde{f}_{i}}{\varrho}\Big)^{2}+\Big(\frac{f_{ij}}{\varrho}\Big)^{2}\Big]P^{2}\\ &+[(L'^{*}_{ijk})^{2}+(N'^{*}_{ik})^{2}+(M'^{*}_{ijk})^{2}]C_{k}^{2}\Bigg\} \end{aligned} \tag{59}$$

■

We consider λ to be a given positive parameter. Let us define the following function:

$$\begin{aligned}(60)\quad \Lambda_1(R,t) = -\int_0^t\int_{S_R} e^{-\lambda s}\Big[&t_{ji}(s)n_j(s)\dot{u}_i(s) + h_i(s)n_i(s)\dot{\varphi}(s)\\ &+ m_{ji}(s)n_j(s)\dot{\varphi}_i(s) + \frac{1}{T_0}q_i(s)n_i(s)\theta(s)\\ &+ \eta_i(s)n_i(s)P(s) - q_{ji}(s)n_j(s)T_i(s)\Big]\mathrm{d}a\mathrm{d}s, \quad R \geq 0, t \in [0,T],\end{aligned}$$

which will be useful in analysing the spatial behaviour. In the theorem below we study its properties.

Theorem 4.1 (Properties of the function $\Lambda_1(R,t)$)**.** *Under the assumptions of Lemma 4.1, under the assumptions (H1), (H2) and (H4) and by considering that the external prescribed data of the problem $\mathcal{P}$ have the bounded support $\hat{D}_T$ on the time interval $[0,T]$, we derive the following properties for $\Lambda_1(R,t)$ for each $t \in [0,T]$:*

(i) for $0 \leq R_2 < R_1$, we have

$$\begin{aligned}(61)\quad \Lambda_1(R_1,t) - \Lambda_1(R_2,t) = &-\int_{B(R_1,R_2)} e^{-\lambda t}\bar{\Gamma}(t)\mathrm{d}v\\ &-\int_0^t\int_{B(R_1,R_2)}\left[e^{-\lambda s}\lambda\bar{\Gamma}(s) + e^{-\lambda s}\bar{K}(s)\right]\mathrm{d}v\mathrm{d}s;\end{aligned}$$

(ii) $\Lambda_1(R,t)$ is a continuous differentiable function with respect to $R \geq 0$ and

$$(62)\quad \frac{\partial\Lambda_1}{\partial R}(R,t) = -\int_{S_R} e^{-\lambda t}\bar{\Gamma}(t)\mathrm{d}v - \int_0^t\int_{S_R}\left[e^{-\lambda s}\lambda\bar{\Gamma}(s) + e^{-\lambda s}\bar{K}(s)\right]\mathrm{d}v\mathrm{d}s;$$

(iii) under the assumption (H3), $\Lambda_1(R,t)$ is a nonincreasing function with respect to $R \geq 0$;

(iv) under the assumption (H5), $\Lambda_1(R,t)$ satisfies the following differential inequality

$$(63)\quad \frac{\lambda}{k}|\Lambda_1(R,t)| + \frac{\partial\Lambda_1}{\partial R}(R,t) \leq 0, \quad R \geq 0,$$

with

$$(64)\quad k = \sqrt{\frac{\mu_M(1+\varepsilon_0)}{\rho_0}},$$

where ε_0 is the positive root of the equation

(65)
$$\varepsilon^2 + \varepsilon\left(1 - \frac{4m_2}{a_0\mu_M} - \frac{\rho_0\lambda T_0 k_m^2}{a_0\underline{k}_m\mu_M}\right) - \frac{4m_2}{a_0\mu_M} = 0.$$

Proof (i) We derive a formulation for $\bar{\Lambda}_1$ from formula (35), then consider B_{R_1} and B_{R_2} instead of Ω in this expression with $0 \leq R_2 \leq R_1$ and take the difference.

(ii) This property follows easily from (i).

(iii) This follows immediately from formula (62) since $\bar{\Gamma}$ and $\bar{K}$ are positive by the assumptions (H1), (H2), (H3) and (H4).

(iv) The goal is to choose k such that

(66)
$$|\Lambda_1(R,t)| \leq -\frac{k}{\lambda}\frac{\partial\Lambda_1}{\partial R}(R,t).$$

We apply the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality in order to obtain

(67)
$$\begin{aligned}
|\Lambda_1(R,t)| \leq\; & \frac{\varepsilon_1}{2\rho_0}\int_0^t\int_{S_R} e^{-\lambda s} t_{ji}(s)t_{ji}(s)\mathrm{d}a\mathrm{d}s + \frac{\rho_0}{2\varepsilon_1}\int_0^t\int_{S_R} e^{-\lambda s}\dot{u}_i(s)\dot{u}_i(s)\mathrm{d}a\mathrm{d}s \\
&+ \frac{\varepsilon_1}{2J_0}\int_0^t\int_{S_R} e^{-\lambda s} h_i(s)h_i(s)\mathrm{d}a\mathrm{d}s + \frac{J_0}{2\varepsilon_1}\int_0^t\int_{S_R} e^{-\lambda s}\dot{\varphi}(s)\dot{\varphi}(s)\mathrm{d}a\mathrm{d}s \\
&+ \frac{\varepsilon_1}{2I_0}\int_0^t\int_{S_R} e^{-\lambda s} m_{ji}(s)m_{ji}(s)\mathrm{d}a\mathrm{d}s + \frac{I_0}{2\varepsilon_1}\int_0^t\int_{S_R} e^{-\lambda s}\dot{\varphi}_i(s)\dot{\varphi}_i(s)\mathrm{d}a\mathrm{d}s \\
&+ \frac{\varepsilon_2}{2a_0}\int_0^t\int_{S_R} e^{-\lambda s} q_i(s)q_i(s)\mathrm{d}a\mathrm{d}s + \frac{a_0}{2\varepsilon_2}\int_0^t\int_{S_R} e^{-\lambda s}\theta(s)^2\mathrm{d}a\mathrm{d}s \\
&+ \frac{\varrho\varepsilon_3}{2}\int_0^t\int_{S_R} e^{-\lambda s}\eta_i(s)\eta_i(s)\mathrm{d}a\mathrm{d}s + \frac{1}{2\varrho\varepsilon_3}\int_0^t\int_{S_R} e^{-\lambda s}P(s)^2\mathrm{d}a\mathrm{d}s \\
&+ \frac{\varepsilon_4}{2B_0}\int_0^t\int_{S_R} e^{-\lambda s} q_{ji}(s)q_{ji}(s)\mathrm{d}a\mathrm{d}s + \frac{B_0}{2\varepsilon_4}\int_0^t\int_{S_R} e^{-\lambda s}T_i(s)T_i(s)\mathrm{d}a\mathrm{d}s.
\end{aligned}$$

In the following, we will estimate all these terms separately. To this end, let us set

(68)
$$\begin{aligned}
m_1 &= L^*_{ijk}L^*_{ijk} + N^*_{ik}N^*_{ik} + M^*_{ijk}M^*_{ijk}, \\
m_2 &= a^*_{ij}a^*_{ij} + d^*_i d^*_i + b^*_{ij}b^*_{ij}, \\
m_3 &= \frac{d_{ij}}{\varrho}\frac{d_{ij}}{\varrho} + \frac{\tilde{f}_i}{\varrho}\frac{\tilde{f}_i}{\varrho} + \frac{f_{ij}}{\varrho}\frac{f_{ij}}{\varrho}, \\
m_4 &= L'^*_{ijk}L'^*_{ijk} + N'^*_{ik}N'^*_{ik} + M'^*_{ijk}M'^*_{ijk}
\end{aligned}$$

in the inequality from Lemma 4.2. Finally, we estimate the following quantities

$$q_i q_i = (k_{ij}\theta_{,j} + K_{ij}T_j)\, q_i \leq \Big[(k_{mn}k_{mn})^{\frac{1}{2}}(\theta_{,j}\theta_{,j})^{\frac{1}{2}} + (K_{mn}K_{mn})^{\frac{1}{2}}(T_jT_j)^{\frac{1}{2}}\Big]\,(q_iq_i)^{\frac{1}{2}}. \tag{69}$$

By setting $k_m = (k_{mn}k_{mn})^{\frac{1}{2}}, \tilde{K}_m = (K_{mn}K_{mn})^{\frac{1}{2}}$, we obtain $q_iq_i \leq 2(k_m^2\theta_{,j}\theta_{,j} + \tilde{K}_m^2 T_jT_j)$. We have

$$\eta_i\eta_i = (h_{ij}P_{,j} + H_{ij}C_j)\,\eta_i \leq \Big[(h_{mn}h_{mn})^{\frac{1}{2}}(P_{,j}P_{,j})^{\frac{1}{2}} + (H_{mn}H_{mn})^{\frac{1}{2}}(C_jC_j)^{\frac{1}{2}}\Big]\,(\eta_i\eta_i)^{\frac{1}{2}}, \tag{70}$$

$$\eta_i\eta_i \leq 2\left(h_m^2 P_{,j}P_{,j} + \tilde{H}_m^2 C_jC_j\right), \tag{71}$$

where $h_m = (h_{mn}h_{mn})^{\frac{1}{2}}, \tilde{H}_m = (H_{mn}H_{mn})^{\frac{1}{2}}$.

We have $q_{ij}q_{ij} \leq (P_{ijkl}P_{ijkl})^{\frac{1}{2}}(T_{l,k}T_{l,k})^{\frac{1}{2}}(q_{ij}q_{ij})^{\frac{1}{2}}$, hence $q_{ij}q_{ij} \leq P_m T_{i,j}T_{i,j}$.

By the estimate (67) and with the bounds that we derived for each term separately, we choose a suitable value for k such that

$$|\Lambda_1(R,t)| \leq \frac{k}{\lambda}\left[\int_{S_R} e^{-\lambda t}\bar{\Gamma}(t)dv + \int_0^t \int_{S_R}\left(e^{-\lambda s}\lambda\bar{\Gamma}(s) + e^{-\lambda s}\bar{K}(s)\right)dvds\right] \tag{72}$$

and then derive the corresponding equations for ε_0. ■

Based on the properties presented in the previous theorem, we establish some decay estimates that describe the spatial behaviour for unbounded bodies. Note that the decay of the exponential is independent of time.

Theorem 4.2 (Spatial behaviour for unbounded bodies)**.** *Under the assumptions of Theorem 4.1 and for each* $t \in [0,T]$*, we have*

(i) if $\Lambda_1(R,t) \geq 0$*, for all* $R \geq 0$*, we have*

$$\Lambda_1(R,t) \leq \Lambda_1(0,t)e^{-\frac{\lambda}{k}R}, \quad R \geq 0; \tag{73}$$

(ii) if there exists a value $R_1 \geq 0$ *such that* $\Lambda_1(R_1,t) < 0$ *and* $\Lambda_1(R,t) < 0$ *for all* $R \geq R_1$*, then we have*

$$-\Lambda_1(R,t) \geq -\Lambda_1(R_1,t)e^{\frac{\lambda}{k}(R-R_1)}, \quad R \geq R_1. \tag{74}$$

Proof Let t be a fixed value in $[0,T]$. Since by Theorem 4.1 (iii), we know that $\Lambda_1(R,t)$ is a nonincreasing function with respect to R, we have two alternatives: (i) $\Lambda_1(R,t) \geq 0$ for all $R \geq 0$ or (ii) there exists $R_1 \geq 0$ such that $\Lambda_1(R_1,t) < 0$.

By part (i), the dfferential inequality (63) leads to

$$\frac{\partial}{\partial R}\left[e^{\frac{\lambda}{k}R}\Lambda_1(R,t)\right] \leq 0, \quad R \geq 0 \tag{75}$$

and the conclusion follows easily by integration.

By part (ii), the differential inequality (63) leads to

$$\frac{\partial}{\partial R}\left[e^{-\frac{\lambda}{k}R}\Lambda_1(R,t)\right] \leq 0, \quad R \geq R_1 \tag{76}$$

and the result is obtained by integration. ■

5 Conclusion

We analysed the spatial behaviour for the mathematical model of diffusive microstretch thermoelasticity with microtemperatures and microconcentrations in the linear, anisotropic and non-centro-symmetric case. We considered a suitable function with a weight depending on time and including the main characteristics of the problem. We studied its properties and proved in two cases that there is a spatial decay of exponential type, with a rate that does not depend on time.

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